Classification of quasitoric manifolds with codimension one extended actions

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Abstract. The goal of this paper is to classify quasitoric manifolds $(M^{2n},\ T^n)$ with codimension 1 extended G-actions up to essential isomorphism, where G is a compact, connected Lie group whose maximal torus is T^n. For technical reasons, we classify more general class which consists of simply connected torus manifolds $(M^{2n},\ T^n)$ with codimension 1 extended G-actions such that two singular orbits of extended G-actions are also simply connected torus manifolds. The main result of this paper is as follows: such M^{2n} is a fibre bundle over a product of some complex projective spaces $\mathbb{C}P(l)$ and even dimensional spheres S^{2m} whose fibre is a complex projective space or an even dimensional sphere. As a result, if M^{2n} is a quasitoric manifold with codimension 1 extended G-actions, then M^{2n} is a complex projective bundle over a product of complex projective spaces.

1. Introduction

One of the essential problems in geometry is to find the most natural group action on the given space. A *torus manifold* is an even dimensional oriented manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set. Since the previous papers [**17**], we have studied the extended G-actions $(M^{2n},\ G)$ on torus manifolds $(M^{2n},\ T^n)$, where G is a compact, connected Lie group whose maximal torus is T^n. To study extended actions of torus manifolds is to find the natural group action on M which induces the given T^n-action on M.

In the previous paper [**17**], we classified homogeneous torus manifolds, i.e., torus manifolds with transitive extended G-actions (also see Theorem 2.3 in this paper). If G acts on M transitively, then its principal orbit G/K is M itself, i.e., $M = G/K$. In other words, the codimension of the principal orbit in the transitive action is zero, i.e., $\dim M - \dim G/K = 0$. Therefore, we can regard the classification in [**17**] as the classification of codimension 0 extended actions. So we are naturally led to study codimension 1 extended actions, i.e., extended G-actions on torus manifolds with codimension 1 principal orbits.

A torus manifold is called a *quasitoric manifold* over a simple polytope P^n if the following two conditions are satisfied (see [**5, 6**]):

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(1) the T^n-action is *locally standard*, that is, locally modelled by the standard T^n-action on $\mathbb{C}^n$;
(2) there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0, \cdots, n$,

where an n-dimensional convex polytope P^n is *simple* if precisely n codimension-one faces (*facets*) meet at each vertex. The goal of the present paper is to classify codimension 1 extended actions of T^n-actions on quasitoric manifolds. In order to classify such actions, we classify more general classes as in the following main theorem (Theorem 8.4, 8.6 and 8.8 for detail):

THEOREM 1. Suppose a simply connected torus manifold M has a codimension one extended G-action. If two singular orbits are also simply connected torus manifolds, then (M, G) is essentially isomorphic to

$$\left(\prod_{j=1}^{b} S^{2m_j} \times N, \quad \prod_{j=1}^{b} SO(2m_j+1) \times H\right),$$

where (N, H) is one of the followings:

N	H
$\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k)$
$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$
$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$

where $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$ (or $\prod \mathbb{C}P(l_i)$) respectively; and $U(k)$, $S(U(k_1) \times U(k_2))$ and $SO(2k)$ also act naturally on $\mathbb{C}^k_{\mathfrak{a}}$, $\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}$ and $\mathbb{R}^{2k}$ respectively.

Using Theorem 1 and basic results for quasitoric manifolds, we reach our goal as the following corollary:

COROLLARY 2. If a quasitoric manifold M has a codimension one extended G-action, then (M, G) is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}), \quad \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))\right).$$

For the classification of codimension 0 extended actions on torus manifolds, we only used classical Lie theory, in the previous paper [**17**]. However, for the classification of codimension 1 extended actions on torus manifolds, we need to use not only classical Lie theory but also transformation group theory. In history of studying transformation group theory, there are so many classification results of actions with codimension 1 principal orbits, (e.g., [**1, 9, 14, 16, 28, 29**]). In particular, the Uchida's method in [**28**], developed the Wang's method in [**29**], is very powerful method in the case that we classify vast classes for compact transformation groups on compact manifolds with codimension 1 principal orbits. So we apply the Uchida's method to classify our case.

When we use the Uchida's method, we often need to divide our proof into two cases (see [**16, 28**]); specifically, in our case, the case that two singular orbits of G-actions are torus manifolds, and the other case that one of two singular orbits of G-actions is not a torus manifold (see Lemma 3.2). Proofs of these two cases are quite different; thus, we divide our paper into two papers: this paper and the next paper [**18**]. In this paper, we focus on the first case, i.e., we classify the class which consists of simply connected torus manifolds $(M^{2n},\ T^n)$ with codimension 1 extended G-actions such that two singular orbits of extended G-actions are also simply connected torus manifolds (see Theorem 1). In particular, quasitoric manifolds with codimension 1 extended actions are contained in this class by Lemma 3.3 (also see Corollary 2).

Due to [**17**, Corollary 2], a homogeneous quasitoric manifold is a product of $\mathbb{C}P(l)$'s only. On the other hand, by making use of the above Corollary 2, a quasitoric manifold which has codimension 1 extended actions is a complex projective bundle over a homogeneous quasitoric manifold. As we mentioned in the previous paper [**17**], our extension problem is reminiscent of the study of automorphism groups of *toric varieties* by Demazure in [**7**], where here a toric variety is a normal algebraic variety V on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit (see [**8**]). In fact, quasitoric manifolds were defined by Davis and Januszkiewicz as a topological counterpart of projective, non-singular toric varieties in [**6**]. In this paper, the term "dimension" means a real dimension. Therefore, to classify codimension 1 extended G-actions belongs to purely topological problems. However, the resulting manifolds in Corollary 2 are projective, non-singular toric varieties, that is, these objects also belong to algebraic geometry. Remark that $\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ is equivariantly diffeomorphic to $\prod_{i=1}^{a-1} \mathbb{C}_0^{l_i+1} \times_{(\mathbb{C}^*)^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$, where $\mathbb{C}_0^{l_i+1} = \mathbb{C}^{l_i+1} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Finally, we note that the classification results for low dimensional torus manifolds are known, e.g., [**25**] for simply connected, 4-dimensional torus manifolds or [**22**] for simply connected, spin, 6-dimensional torus manifolds; however, in [**22**], there is a mistake (see [**19**]).

The organization of this paper is as follows. In Section 2, we prepare some fundamental results from toric topology, classical Lie theory, the previous paper, and transformation group theory. In Section 3, we show that a G-orbit through T-fixed points is a singular orbit, and give some difference between quasitoric manifolds and torus manifolds with codimension 1 extended G-actions. From Section 4, we assume that our torus manifold is simply connected, our extended G-action of torus manifold has two singular orbits and they are simply connected torus manifolds. Moreover, from this section, we start the Uchida's method; thus, we first show a topological type of two singular orbits. In Section 5, we next classify two tubular neighborhoods of two singular orbits by computing slice representations. In Section 6, we get possibilities for G and two singular isotropy subgroups by comparing two slice representations. In Section 7, we study more precise structure of slice representations and get possibilities for two tubular neighborhoods in $(M,\ G)$. The key lemmas of this classification are Lemma 7.1, 7.2 and 7.3, that is, two slice representations of two singular isotropy groups must coincide up to sign. We also have principal isotropy subgroups in this section. Finally, in Section 8, we compute the attaching maps between boundaries of two tubular neighborhoods. Then we can construct our manifolds explicitly as in Theorem 1. In final Section 9, as an application, we show the following fact: codimension 1 (resp. 0) extended actions of quasitoric manifolds can be lifted to actions on moment-angle manifolds with codimension 1 (resp. 0) principal orbits. In other wards, codimension 1 (resp.

0) extended actions of quasitoric manifolds are always induced from actions on moment-angle manifolds with codimension 1 (resp. 0) principal orbits.

2. Preliminary

In this first section, we recall some basic notations and facts from toric topology [**11, 20**], classical Lie theory [**26**], the previous paper [**17**], and transformation group theory [**4, 13**]. Throughout of this paper, T^n is always an n-dimensional, compact, abelian group, i.e., T^n is a product of n circles $(S^1)^n$, we often call it an n-dimensional torus or just a torus.

2.1. Torus manifold. A torus manifold introduced in [**11, 20**] is the main research object in this paper. First we recall this definition. A *torus manifold* is a pair $(M^{2n},\ T^n)$ of a smooth, $2n$-dimensional, compact manifold M^{2n} and an n-dimensional (half dimensional) torus T^n which satisfies the following two properties:

(1) M^{2n} is an oriented manifold equipped with an almost effective T^n-action (also see Remark 2.1);
(2) its fixed point set is non-empty, i.e., $M^T \neq \emptyset$,

and it is often denoted by $(M,\ T)$ or M simply. Automatically, the fixed point set M^T is finite and the principal orbit (defined in the next Section 2.2) is T^n itself.

A compact, connected, codimension two submanifold of M without boundary is called *characteristic* if it is a connected component of the set fixed pointwise by a certain circle subgroup of T and contains at least one T-fixed point. There are only finitely many characteristic submanifolds and they are orientable, because M is compact and orientable.

Remark 2.1. The concept of a torus manifold is an ultimate generalization which can develop a topological generalization of toric theory. However, in this paper we do not use this theory, that is, we do not use a *multi-fan*, so our definition of a torus manifold becomes rather briefer than the definitions in [**11, 20**]. For example, we do not need to define an *omniorientation* of the torus manifold and characteristic submanifolds. For the torus manifold in [**11, 20**], we also need an omniorientation. One of the reason to introduce this concept is to define a multi-fan. However, in this paper we do not use a multi-fan. So we do not need an omniorientation.

Furthermore, for technical reasons, we assume a T-action on M is an *almost effective*, that is, the intersection of all isotropy subgroups $\cap_{x\in M}T_x$ is finite set (if this set only consists of the identity element, then this action is *effective*).

2.2. Review of classical Lie theory and the previous paper. In order to study extended actions of torus actions on torus manifolds, we next introduce some terminology and fact from classical Lie theory, transformation group theory and our previous paper [**17**]. We start with recalling general terminology.

Let $(M,\ G)$ be a pair of a manifold M and its smooth G-action. A *principal orbit* in $(M,\ G)$ is a maximal orbit in $(M,\ G)$. A *singular orbit* in $(M,\ G)$ is an orbit whose dimension is strictly less than the dimension of principal orbits. An *exceptional orbit* in $(M,\ G)$ is an orbit which is not maximal but whose dimension is the same as that of a principal orbit (see Example 3.6). Let G_x be an isotropy subgroup of $x \in M$, i.e., $G_x = \{g \in G \mid gx = x\}$.

In this paper and the next paper, we will classify $(M,\ G)$ up to essential isomorphism. So we next define an essential isomorphism. Let N be the intersection of all isotropy subgroups $\cap_{x \in M} G_x$ of $(M,\ G)$. Then the induced action $(M,\ G/N)$ is always effective, and we call it the *induced effective action*. If two induced effective actions of $(M,\ G)$ and $(M',\ G')$ are weak equivariantly diffeomorphic, that is, there are an isomorphism $\rho : G/N \to G'/N'$ and a diffeomorphism $f : M \to M'$ such that $f(\varphi(g, x)) = \psi(\rho(g), f(x))$ for $(g, x) \in G/N \times M$ (where $\varphi : G/N \times M \to M$ and $\psi : G'/N' \times M' \to M'$ are two induced effective actions), then we call that $(M,\ G)$ and $(M',\ G')$ are *essentially isomorphic*. Now we may recall the following facts (see [**26**, Section 5]). For any compact, connected Lie group G, there is a finite covering, homomorphic map:

$$p : \widetilde{G} = G_1 \times \cdots \times G_k \to G, \tag{2.1}$$

where G_i is a compact, (simply) connected, simple Lie group, or a torus for $i = 1,\ \cdots, k$. Let the kernel of p be denoted as N. Because p is a surjective homomorphism, we have

$$G \simeq (G_1 \times \cdots \times G_k)/N,$$

where N is some finite, central, normal subgroup in $G_1 \times \cdots \times G_k$. By the definition of p, we have that $(M,\ G)$ can be lifted to $(M,\ \widetilde{G})$ and they are essentially isomorphic.

A *rank* of G is the dimension of a maximal torus subgroup of G. The following lemma is known for a maximal rank subgroup H^o of G (see [**26**, Theorem 7.2]).

Lemma 2.2. *Let G_i $(i = 1,\ \cdots,\ k)$ be compact, connected Lie groups and let G be their product. Assume H^o is a compact, connected, maximal rank subgroup in G. Then $H^o = H_1 \times \cdots \times H_k$, where H_i is a maximal rank subgroup in G_i.*

We next recall the results of the previous paper [**17**]. In order to recall it, we introduce the extended action. Let $(M,\ T)$ be a torus manifold, and $\varphi : T \times M \to M$ be its T-action. Suppose that a compact, connected Lie group G has T as its maximal torus subgroup. If there exists an action $\Phi : G \times M \to M$ such that the restricted T-action $\Phi|_{T \times M}$ is the given φ, then we call Φ is an *extended G-action* of $(M,\ T)$, and Φ also denotes $(M,\ G)$. If there is a principal G-orbit $G(x)$ such that $\dim G(x) = \dim M^{2n} - k = 2n - k$ in the extended G-action, then we call $(M^{2n},\ G)$ a *codimension k extended G-action* of $(M,\ T)$, where an integer k satisfies $0 \leq k \leq n$. In this paper and the next paper, we study the case $k = 1$.

Before we state the results in [**17**], we need to introduce some actions. Let $\mathbb{Z}_2$ be defined as $\{I_{2m_j+1},\ -I_{2m_j+1}\} \subset O(2m_j + 1)$, where $O(2m_j + 1)$ is the orthogonal group. We remark that $\mathbb{Z}_2$ acts on the $2m_j$-dimensional sphere $S^{2m_j} \subset \mathbb{R}^{2m_j+1}$ canonically (we call this action the *antipodal action* on sphere). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$. Then $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ through the canonical $\prod_{j=1}^{b} \mathbb{Z}_2$-action on $\prod_{j=1}^{b} S^{2m_j}$ (the product of antipodal actions). For codimension 0 extended G-actions, we proved the following two classification results (see [**17**, Theorem 1, Corollary 3.7 and Remark 3.8]):

Theorem 2.3. *Let $(M^{2n},\ T^n)$ be a $2n$-dimensional, compact, connected manifold M equipped with an almost effective, half dimensional torus T-action with fixed points. Suppose $(M^{2n},\ T^n)$ extends to a codimension 0 extended G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$*

is essentially isomorphic to

$$\left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \frac{\prod_{j=1}^{b} S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where the above group acts on M^{2n} *in the natural way, and* $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$.

Corollary 2.4. If a simply connected torus manifold has a codimension 0 extended G-action, $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j},\ \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$ and $m_j \geq 2$ for all $j = 1,\ \cdots,\ b$.

Remark 2.5. In [**17**, Theorem 1 and Corollary 3.7], we did not use $SU(l+1)$. Instead, we used $PU(l+1)$ as the transformation group, where $PU(l+1)$ is defined as the quotient of $SU(l+1)$ by its center $Z(SU(l+1))$. However, $(\prod \mathbb{C}P(l),\ \prod SU(l+1))$ is essentially isomorphic to $(\prod \mathbb{C}P(l),\ \prod PU(l+1))$ (see [**17**, Example 2.7]). So we may change $PU(l+1)$'s into $SU(l+1)$'s of [**17**, Theorem 1 and Corollary 3.7]. For technical reasons, in this paper, we use $SU(l+1)$.

In order to classify codimension 1 extended G-actions, we need to use the above results.

2.3. Review of transformation group theory. Next we recall some fact from transformation group theory. For the general codimension 1 actions, the following theorem is known (see [**4**, 8.2 Theorem in Chapter IV] or [**28**, Lemma 1.2.1]):

Theorem 2.6. *Let* G *be a compact, connected Lie group and* M *a compact, connected manifold without boundary such that* $H^1(M;\ \mathbb{Z}_2) = 0$. *Assume* G *acts smoothly on* M *with codimension one orbits* G/K. *Then all codimension one orbits* G/K *are principal orbits, and there are just two singular orbits* G/K_1 *and* G/K_2. *Furthermore, there exists a closed, invariant tubular neighborhood* X_s *(of* G/K_s *for* $s = 1,\ 2$*) such that*

$$M = X_1 \cup X_2 \text{ and } X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

The Figure 1 shows the image of $(M,\ G)$ with codimension 1 principal orbits.

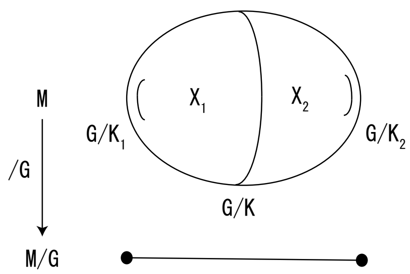

Figure 1. The orbit structure of $(M,\ G)$ with codimension 1 orbits.

Once we know the singular orbits G/K_1 and G/K_2, then their tubular neighborhoods X_1 and X_2 are also known by the following *differentiable slice theorem*, the *slice theorem* for short (see, e.g., [**4, 13**]).

THEOREM 2.7 (differentiable slice theorem). *Let G be a compact Lie group and M a smooth G-manifold, and G_x an isotropy subgroup of $x \in M$. Then, for all $x \in M$, there is a closed G-invariant neighborhood X (of the orbit $G(x) \cong G/G_x$) such that $X \cong G \times_{G_x} D_x$ as a G-diffeomorphism, where $D_x \subset \mathbb{R}^N$ is some closed disk, $G \times_{G_x} D_x$ is the quotient space $(G \times D_x)/G_x$ induced by the G_x-action on $G \times D_x$. Here, this G_x-action is defined as follows: G_x canonically acts on G as a subgroup of G; and on a closed disk D_x through an orthogonal representation $\sigma : G_x \to O(D_x)$, where $O(D_x)$ is an orthogonal group of $D_x \subset \mathbb{R}^N$ ($N = \dim D_x = \dim M - \dim G(x)$).*

In Theorem 2.7, σ is called a *slice representation* of G_x, and D_x a *slice* of X on x. We identify a tubular neighborhood X of $G(x)$ with $G \times_{G_x} D_x$. The slice theorem will be often used throughout this paper.

Suppose a compact Lie group G acts on a manifold M smoothly and it has a fixed point $p \in M^G$. Using the slice theorem, the tangent space $T_p(M)$ of $p \in M^G$ is an orthogonal G-representation space. We call it a *tangential representation space*, or simply a *tangential representation* on p. Let α_i be a representation from T to $S^1 \simeq SO(2)$, i.e., $\alpha_i : T \to S^1 \simeq SO(2) (\in Hom(T,\ S^1) \simeq \mathbb{Z}^n)$, and let $V(\alpha_i) \simeq \mathbb{R}^2$ be a representation space of α_i. For tangential representation spaces of torus manifolds, the following proposition holds.

PROPOSITION 2.8. Let $(M,\ T)$ be a torus manifold and $p \in M$ a fixed point. Then there is the following decomposition for the tangential representation space on p:

$$T_p(M) \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$$

such that $\{\alpha_1,\ \cdots,\ \alpha_n\}$ is linearly independent in $Hom(T,\ S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$.

PROOF. Because $p \in M^T$, we may regard $T_p(M)$ as an orthogonal T-representation space by the slice theorem. From the definition of the torus manifold, the T-action on M is an almost effective. It follows that there is a non-degenerate representation ρ from T^n to the orthogonal group $O(T_p(M)) \simeq O(2n)$, i.e., the image of ρ is also an n-dimensional torus. Moreover, the image of ρ is in the special orthogonal group $SO(2n)$ because T^n is connected. Therefore, the image of ρ and the diagonal maximal torus $SO(2) \times \cdots \times SO(2) \subset SO(2n)$ are conjugate in $SO(2n)$. This gives an equivalence between ρ and $\alpha_1 \oplus \cdots \oplus \alpha_n$ for some $\alpha_i = k_i \circ \pi_i$, where π_i is the i-th projection $\pi_i : SO(2) \times \cdots \times SO(2) \to SO(2)$ and k_i is a non-degenerate representation $k_i : SO(2) \to SO(2)$, i.e., $k_i \neq 1 : SO(2) \to \{I_2\} \subset SO(2)$ ($i = 1,\ \cdots,\ n$). Moreover, we can easily see that $\{\alpha_1,\ \cdots,\ \alpha_n\}$ is linearly independent in $Hom(T,\ S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$ by their definitions. □

3. Orbits in extended actions

In this section, we study orbits in extended actions $(M,\ G)$ of torus manifolds $(M,\ T)$.

3.1. G-orbits on T-fixed points. By the definition of the torus manifold $(M,\ T)$, there are non-empty isolated fixed points M^T. We first consider a G-orbit on a fixed point $p \in M^T$.

If p is in the T-fixed point set M^T, then we have

$$T \subset G_p \subset G, \tag{3.1}$$

where G_p is the isotropy subgroup of p. We have the following lemma.

LEMMA 3.1. *Suppose that a torus manifold $(M^{2n},\ T^n)$ has a codimension 1 extended G-action. Then a G-orbit G/K_1 of a T-fixed point is a singular orbit in $(M,\ G)$, i.e., $\dim G/K_1 < 2n-1$. In other words, there is at least one singular orbit in $(M,\ G)$.*

PROOF. Let p be a fixed point ($p \in M^T$). We shall prove that the orbit $G(p)$ is a singular orbit, i.e., $\dim G(p) < 2n-1$. By the above relation (3.1), rank G_p^o coincides with rank $G = \dim T$. It follows that the Euler character of G/G_p^o is not zero (e.g. [**10**, Theorem 1.1 (2), (3)]), and thus $\dim G/G_p = \dim G/G_p^o$ is even. Because $(M^{2n},\ G)$ is a codimension 1 extended action, the dimension of a principal orbit is $2n-1$. Therefore, we have $\dim G(p) = \dim G/G_p = \dim G/G_p^o < 2n-1$. $\square$

Moreover, we see that a G-orbit on a fixed point $p \in M^T$ is a torus manifold by the following lemma.

LEMMA 3.2. *Let G/K_1 be a singular orbit of $(M,\ G)$ which contains a fixed point of $(M,\ T)$, i.e., $G/K_1 \cap M^T \neq \emptyset$. Then there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold.*

PROOF. Let $p : \widetilde{G} \to G$ be a covering of G in (2.1), and let $\widetilde{K_1}$ (resp. $\widetilde{T}$) be a connected component of $p^{-1}(K_1)$ (resp. $p^{-1}(T)$). By (3.1), we have $\widetilde{T}$ is a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{K_1}$. By the argument of Section 2.2 and Lemma 2.2, there is the following decomposition:

$$\widetilde{G} = G_1' \times G_1'',\ \widetilde{K_1} = K_1' \times G_1'',\ \widetilde{T} = T_1' \times T_1'',$$

where G_1' and G_1'' are products of compact, simply connected, simple Lie groups and tori, and G_1'' is the same factor in $\widetilde{G}$ and $\widetilde{K_1}$, that is, the connected component of the intersection of all isotropy subgroup $\cap_{x \in G/K_1} \widetilde{G}_x$ (i.e., the *kernel* of the $\widetilde{G}$ action on G/K_1). Here, G_1' and G_1'' satisfy that rank $G_1' =$ rank $K_1' = \dim T_1'$ and rank $G_1'' = \dim T_1''$.

Because $\widetilde{K_1} \subset p^{-1}(K_1) \subset \widetilde{G}$, we also have the following decomposition:

$$p^{-1}(K_1) = H_1' \times G_1'',$$

where $K_1' \subset H_1' \subset G_1'$ and the connected component of H_1' is K_1'. Then the projection p induces the diffeomorphism between G_1'/H_1' and G_1/K_1. We remark that T_1'' is a connected component of the kernel of $\widetilde{T}$-action on G_1'/H_1'. Now we may prove that $(G_1'/H_1',\ T_1')$ is an *unoriented* torus manifold, i.e., a torus manifold which does not assume orientability. Because T_1' is a maximal torus of G_1' and $K_1' = (H_1')^o$, we have that this T_1'-action on G_1'/H_1' is almost effective and there are fixed points. Moreover, we have the following decomposition on the fixed point $q \in G/K_1 \cap M^T$:

$$T_qM = T_qG/K_1 \oplus N_qG/K_1.$$

Now T'_1 acts on T_qG/K_1 and T''_1 acts on N_qG/K_1 through the finite covering p. Moreover, using a T-action on T_qM, we have an irreducible decomposition of $T_qM = V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$ (by Proposition 2.8). Therefore, we have $\dim G/K_1 = 2n-2k_1$ and $2k_1 = \dim N_qG/K_1 = 2\dim T''_1$. Hence, $(G'_1/H'_1,\ T'_1)$ is an unoriented torus manifold.

We define T' as $p(T'_1)$ and T'' as $p(T''_1)$. Because $(G'_1/H'_1,\ T'_1)$ is an unoriented torus manifold, $(G_1/K_1,\ T')$ is also an unoriented torus manifold. Furthermore, we see that G_1/K_1 is one of the fixed pointwise connected component of $M^{T''}$. Therefore, G_1/K_1 can be written as some connected component of an intersection of characteristic manifolds. Because all characteristic manifolds are orientable, G_1/K_1 is an oriented manifold. Hence, $(G_1/K_1,\ T')$ is a torus manifold. $\square$

In the next subsection, we recall quasitoric manifolds briefly, and give some differences between quasitoric manifolds and torus manifolds with codimension 1 extended G-actions.

3.2. Differences between quasitoric manifolds and torus manifolds. The concept of a quasitoric manifold introduced in [**6**] is a topological counterpart of the projective toric variety. Let P^n be a simple convex polytope, i.e., a convex hull of some vertices such that neighborhoods of all vertices look like simplexes. If the torus manifold $(M^{2n},\ T^n)$ satisfies the following two properties:

(1) T^n-action is *locally standard*, that is, locally looks like the standard torus representation in $\mathbb{C}^n$;
(2) there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0,\ \cdots,\ n$,

then we call $(M^{2n},\ T^n)$ a *quasitoric manifold.* For quasitoric manifolds with codimension 1 extended G-actions, we have the following lemma.

Lemma 3.3. *Suppose that the quasitoric manifold* $(M^{2n},\ T^n)$ *has a codimension* 1 *extended* G*-action. Then there are just two singular orbits in* $(M,\ G)$*, and these two singular orbits are quasitoric manifolds.*

Proof. Let $p \in M^T$. We first prove that an orbit $G(p) = G/K_1$ is a quasitoric manifolds. By Lemma 3.2, there is a subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold. Because $(M,\ T)$ satisfies the locally standard property, $(G/K_1,\ T')$ is also locally standard.

Moreover, G/K_1 is a fixed pointwise connected component of $M^{T''}$ (see the proof of Lemmal 3.2). In other wards, G/K_1 can be constructed by an intersection of some characteristic submanifolds M_i, i.e., $G/K_1 = \cap M_i$. Because $(M,\ T)$ is a quasitoric manifold, an orbit space M_i/T of a characteristic submanifold M_i corresponds with a *facet* in M/T, i.e., a codimension 1 subface in a simple convex polytope M/T. Hence, $(G/K_1)/T$ is a subface of M/T, and it becomes a simple convex polytope. Thus, G/K_1 is a quasitoric manifold.

If all $p \in M^T$ are included in G/K_1, then the convex hull of M^T and $(G/K_1)^T$ must be same, that is, $M/T = (G/K_1)/T$. This gives a contradiction to dimensions of M/T and $(G/K_1)/T$. Hence, both two singular orbits G/K_1 and G/K_2 contain fixed points of $(M,\ T)$, and we have that G/K_2 is also a quasitoric manifold by the same argument for G/K_1. $\square$

This Lemma 3.3 does not hold for all torus manifolds, the following two examples give a simple difference between quasitoric manifolds and torus manifolds.

EXAMPLE 3.4. Let $(M,\ T) = (\mathbb{C}P(2),\ T^2)$ be the torus manifold defined by the standard multiplication of T^2 on the last two coordinates in $[z_0 : z_1 : z_2] \in \mathbb{C}P(2)$ (also see [**17**, Example 2.2]). This torus manifold has a codimension 1 extended $G = PU(2) \times T^1$-action: for $[z_0 : z_1 : z_2] \in \mathbb{C}P(2)$, $PU(2) = U(2)/Z(U(2))$ acts on the first two coordinates $(z_0,\ z_1)$ by the canonical multiplication where $Z(U(2))$ is the center of $U(2)$; and T^1 also acts on the third coordinate z_2 by the canonical multiplication. Now we can easily check $(M,\ T)$ is a quasitoric manifold (also see the left "triangle" in Figure 2), and $(M,\ G)$ has codimension 1 orbits $G([1 : 0 : 1]) \cong \mathbb{C}P(1) \times S^1$ and two singular orbits $G([1 : 0 : 0]) \cong \mathbb{C}P(1)$ and $G([0 : 0 : 1]) \cong \{*\}$ (one point). We can also check both singular orbits are quasitoric manifolds (also see Lemma 3.3).

EXAMPLE 3.5. Let $(M,\ T) = (S^4,\ T^2)$ be the torus manifold defined by the standard multiplication of $T^2 = SO(2) \times SO(2)$ on $S^4 \cap \mathbb{R}^4$, where $S^4 \subset \mathbb{R} \oplus \mathbb{R}^4$ (also see [**17**, Example 2.3]). This torus manifold has a codimension 1 extended $G = SO(3) \times T^1$-action: for $(x, y) \in S^4 \subset \mathbb{R}^3 \oplus \mathbb{R}^2$, $SO(3)$ canonically acts on $x \in \mathbb{R}^3$; and $T^1 \simeq SO(2)$ also acts on $y \in \mathbb{R}^2$ canonically. Now we can check $(M,\ T)$ is not a quasitoric manifold because its orbit space is not a convex polytope (see the right "half-moon" in Figure 2, this half-moon is not a convex polytope), and $(M,\ G)$ has codimension 1 orbits $G(e_1, f_1) \cong S^2 \times S^1$ and two singular orbits $G(e_1,\ 0) \cong S^2$ and $G(0,\ f_1) \cong S^1$, where $e_1 = (1,\ 0,\ 0) \in \mathbb{R}^3$ and $f_1 = (1,\ 0) \in \mathbb{R}^2$. We can easily check S^1 is not a torus manifold because its dimension is not even. Hence, Lemma 3.3 does not hold in this case.

The Figure 2 shows the difference of the quasitoric manifold and the torus manifold in the above Examples 3.4 and 3.5.

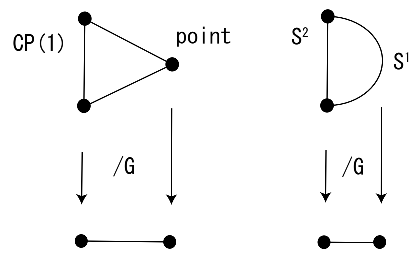

FIGURE 2. This is an image of a reason why Lemma 3.3 does not hold for all torus manifolds, and which submanifolds correspond to singular orbits. The left triangle shows the orbit space $\mathbb{C}P(2)/T^2$, the right half-moon also shows the orbit space S^4/T^2, and interval shows the orbit space of $\mathbb{C}P(2)/(PU(2)\times T^1)$ and $S^4/(SO(3)\times T^1)$.

We also remark the following example:

EXAMPLE 3.6. Let $(S^4,\ T^2)$ be a torus manifold defined in Example 3.5. Then we may naturally define the product of two copies $(S^4 \times S^4,\ T^2 \times T^2)$, and this is a torus manifold with 4 fixed points. If N and S denote the 2 fixed points in

$(S^4,\ T^2)$, then the 4 fixed points in $(S^4 \times S^4,\ T^2 \times T^2)$ can be denoted as $(N,\ N)$, $(N,\ S)$, $(S,\ N)$ and $(S,\ S)$.

Let $\mathbb{Z}_2$ be the group generated by $(-I_5,\ -I_5)$, where $-I_5$ is the antipodal involution on $S^4 \subset \mathbb{R}^5$ and I_5 is the identity map on $\mathbb{R}^5$. We remark that $-I_5$ does not preserve an orientation on S^4; however, $(-I_5,\ -I_5)$ preserves an orientation on $S^4 \times S^4$. Now we may consider the following manifold

$$(S^4 \times S^4)/\mathbb{Z}_2 = S^4 \times_{\mathbb{Z}_2} S^4.$$

Since $(-I_5,\ -I_5)$ preserves an orientation of $S^4 \times S^4$ and $(-I_5,\ -I_5)$ commutes with $T^2 \times T^2$-action on $S^4 \times S^4$, we have that $S^4 \times_{\mathbb{Z}_2} S^4$ is an oriented manifold equipped with $T^2 \times T^2$-action induced from $(S^4 \times S^4,\ T^2 \times T^2)$. Moreover, there are 2 fixed points denoted by $[N:\ N] = [S:\ S]$ and $[N:\ S] = [S:\ N]$. Therefore, $(S^4 \times_{\mathbb{Z}_2} S^4,\ T^2 \times T^2)$ is a torus manifold (also see Theorem 2.3).

This action extends to the canonical $G = SO(5) \times SO(4)$-action on $S^4 \times_{\mathbb{Z}_2} S^4$. Then we have the following three orbit types: $G([e_1 :\ e_1]) = (SO(5) \times SO(4))/(SO(4) \times SO(4)) = S^4$; $G([e_1 :\ e_2]) \cong (SO(5) \times SO(4))/(SO(4) \times SO(3) \times \mathbb{Z}_2) \cong S^4 \times_{\mathbb{Z}_2} S^3$; and $G([e_1 :\ e_1 + e_2]) = (SO(5) \times SO(4))/(SO(4) \times SO(3)) = S^4 \times S^3$. Here, $e_1,\ \cdots,\ e_5$ are the canonical basis of $\mathbb{R}^5$. Therefore, in this case there are one singular orbit S^4, principal orbits $S^4 \times S^3$, and the exceptional orbit $S^4 \times_{\mathbb{Z}_2} S^3$.

From the above Example 3.6, we know that there is a case which has an exceptional orbit in $(M,\ G)$. However, in this paper, we do not need to deal with such kind of actions (we will discuss with such actions in the next paper [**18**]) by the assumption that our manifold M is simply connected and Theorem 2.6 (see Section 4).

3.3. Singular orbit G/K_1. Next we study more precise structures of a singular orbit by making use of Theorem 2.3, i.e., the classification result for homogeneous torus manifolds. Let $(M,\ G)$ be a codimension 1 extended G-action of a torus manifold $(M,\ T)$, and let $(G/K_1,\ T)$ be one of the singular orbits through T-fixed points on M. By (3.1), we see that G and K_1 have the same maximal torus T. Moreover, by Lemma 3.2, there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold.

By the argument of Section 2.2, $(M,\ G)$ is essentially isomorphic to $(M,\ \widetilde{G})$, where $\widetilde{G}$ is a product of connected, compact, simple Lie groups. Let $p : \widetilde{G} \to G$ be a finite projection in (2.1), and let $\widetilde{K_1}$ (resp. $\widetilde{T}$) be the identity component of $p^{-1}(K_1)$ (resp. $p^{-1}(T)$). We note that $p(\widetilde{K_1}) = K_1^o$, $p(\widetilde{T}) = T$ and $\widetilde{G}/\widetilde{K_1} \cong G/K_1^o$ induced by p, where K_1^o is the identity component of K_1. Let G_1'' be the same factors in $\widetilde{G}$ and $\widetilde{K_1}$. By the proof of Lemma 3.2, there are the following decompositions:

$$\widetilde{T} = T_1' \times T_1'' \subset \widetilde{K_1} = K_1' \times G_1'' \subset p^{-1}(K_1) = H_1' \times G_1'' \subset \widetilde{G} = G_1' \times G_1''$$

such that the identity component of H_1' is K_1'.

Now $(G_1'/H_1',\ T_1')$ is a torus manifold. Therefore, we have from Theorem 2.3, $G_1' = \prod_{i=1}^a SU(l_i+1) \times \prod_{j=1}^b SO(2m_j+1)$ and $H_1' = \prod_{i=1}^a S(U(1) \times U(l_i)) \times \mathcal{S}$, where we may take $\mathcal{S}$ as the following subgroup:

$$\prod_{j=1}^b SO(2m_j) \subset \mathcal{S} \subset \Big(\prod_{j=1}^b S(O(1) \times O(2m_j))\Big),$$

such that the manifold $\prod_{j=1}^{b} SO(2m_j+1)/\mathcal{S}$ has an orientation.

In summary, we have the following proposition by Theorem 2.3.

Proposition 3.7. Suppose that a torus manifold $(M,\ T)$ extends to a codimension 1 extended action. Then this codimension 1 extended action is essentially isomorphic to $(M,\ G)$, such that its singular orbit G/K_1 which contains one of the fixed points M^T can be denoted as follows:

$$G = G_1' \times G_1'', \quad K_1^o = K_1' \times G_1'' \quad \text{and} \quad K_1 = H_1' \times G_1'',$$

where G_1'' is a product of compact, connected, simple Lie groups or tori, and $(G_1',\ K_1',\ H_1')$ is

$$G_1' = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1),$$

$$K_1' = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \prod_{j=1}^{b} SO(2m_j),$$

$$H_1' = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \mathcal{S},$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n - \text{rank}\ G_1''$.

4. Assumptions and structures of two singular orbits

In the remainder of the paper, we assume $(M,\ T)$ is a torus manifold, where M is simply connected. Because M is simply connected, we have $H^1(M;\ \mathbb{Z}_2) = 0$. Hence, by Theorem 2.6, a codimension 1 extended action has just two singular orbits. Moreover, by the argument of Section 2.2, we may only consider that a codimension 1 extended action $(M,\ G)$ such that G is a product of compact, connected, simple Lie groups, up to essential isomorphism.

Because of Lemma 3.1, one of the singular orbit has T-fixed points. Thus, we need to discuss the following two cases:

- both of two singular orbits contain T^n-fixed points (see e.g. Example 3.4);
- one singular orbit contains all T^n-fixed points, but the other singular orbit has no T^n-fixed points (see e.g. Example 3.5).

In this paper, we only discuss the first case, i.e., both singular orbits contain T^n-fixed points. Moreover, we assume two singular orbits are simply connected. In order to classify quasitoric manifolds with codimension 1 extended G-actions, it is enough to classify such torus manifolds, by using Lemma 3.3 in this paper and [**6**, Corollary 3.9], i.e., the fact that quasitoric manifolds are simply connected.

In this case, Proposition 3.7 holds for two singular orbits. Because G/K_1 and G/K_2 are simply connected, we also have $K_1 = K_1^o$ and $K_2 = K_2^o$. Since $SU(2)/S(U(1)\times U(1)) \cong SO(3)/SO(2) \cong S^2$, we can regard $S^2 = SU(2)/S(U(1)\times U(1))$ if there is an S^2 factor in G/K_s. It follows that we can assume $m_j \geq 2$ for all $j = 1,\ \cdots,\ b$ in Proposition 3.7. In summary, we have the following proposition.

Proposition 4.1. If two singular orbits G/K_1 and G/K_2 contain T^n-fixed points and they are simply connected, then there are the following decompositions: for $s = 1,\ 2$,

$$G = G_s' \times G_s'', \quad K_s = K_s^o = K_s' \times G_s'',$$

where G''_s is a product of simply connected, compact, simple Lie groups or tori, and

$$G'_1 = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1), \quad K'_1 = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \prod_{j=1}^{b} SO(2m_j);$$

$$G'_2 = \prod_{i=1}^{c} SU(l'_i+1) \times \prod_{j=1}^{d} SO(2m'_j+1), \quad K'_2 = \prod_{i=1}^{c} S(U(l'_i)\times U(1)) \times \prod_{j=1}^{d} SO(2m'_j),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n - \text{rank } G''_1$, and $\sum_{i=1}^{c} l'_i + \sum_{j=1}^{d} m'_j = n - \text{rank } G''_2$, and $m_j,\ m'_j \geq 2$. Furthermore, for the maximal torus subgroup $T'_s \subset T$ of G'_s and K'_s, $(G'_s/K'_s,\ T'_s)$ is a simply connected torus manifold.

From the next section, we start with classifying codimension 1 extended G-actions of torus manifolds which satisfy the above assumptions.

5. Tubular neighborhoods of singular orbits

From the previous section, we assume that M is a simply connected torus manifold and two singular orbits G/K_1 and G/K_2 are also simply connected torus manifolds: we have already known structures of G/K_s in Proposition 4.1 ($s = 1,\ 2$). As the next step of the classification, we will investigate tubular neighborhoods of singular orbits by computing slice representations (see Theorem 2.7). In order to compute slice representations, we first prepare some fact.

5.1. Preparation. Because a singular orbit G/K_s is a torus manifold, we can define its dimension as $2k_s = \dim M - \dim G/K_s$, i.e.,

$$\dim G/K_s = 2n - 2k_s$$

for $s = 1,\ 2$ and $k_s \geq 1$.

By the slice theorem, we also know that a tubular neighborhood X_s of G/K_s is as follows:

$$X_s \cong G \times_{K_s} D^{2k_s}$$

such that K_s acts on D^{2k_s} by the slice representation

$$\sigma_s : K_s \to O(2k_s).$$

Since $(M,\ G)$ has a codimension 1 principal orbits, we also have that K_s acts transitively on a sphere $S^{2k_s-1} \subset D^{2k_s}$ through this slice representation $\sigma_s : K_s \to O(2k_s)$, and a principal isotropy subgroup of K_s-action on D^{2k_s} is isomorphic to a principal isotropy subgroup of $(M,\ G)$. Hence, the slice representation can be computed using the classification of transitive actions on spheres.

5.2. Transitive actions on sphere. The transitive actions on spheres were studied by Borel, Montgomery and Samelson, and Poncet, and the following results are proved in a series of the papers [**2, 23, 27**].

Theorem 5.1. *Let G be a compact, connected Lie group acting effectively and transitively on a homotopy sphere Σ^k and H be the isotropy subgroup, namely, $G/H \cong \Sigma^k$. Then there always exists a simple normal subgroup G_1 of G that is already transitive on Σ^k, i.e., $G_1/(G_1 \cap H) = \Sigma^k$.*

Theorem 5.2. *Let G_1 be a one of the simple groups in G such that $G_1/H_1 \cong \Sigma^k$. Then we have that:*

- *if k is even, $G_1 = SO(k+1)$ or the exceptional Lie group G_2 in the case $k = 6$;*
- *if $k = 2l-1$ and l odd, $G_1 = SO(k+1)$ or $SU(l)$;*
- *if $k = 2l-1$ and l even, $G_1 = SO(k+1)$, $SU(l)$, $Sp(l/2)$, $Spin(9)$ (in the case $k = 15$, $l = 8$), or $Spin(7)$ (in the case $k = 7$, $l = 4$).*

As we may check easily in each of the above cases, we have a unique embedding of G_1 into $SO(k+1)$ such that $G_1 \cap SO(k) = H_1$. Hence, Σ^k is diffeomorphic to the standard sphere S^k.

THEOREM 5.3. *Let $G_1(\subset G)$ be the simple subgroup in Theorem 5.2. Then $G_1 \subset G \subset N(G_1)^o \subset SO(k+1)$, where $N(G_1)^o$ is the identity component of the normalizer of G_1 in $SO(k+1)$, and the following holds.*
- *In cases $G_1 = SO(k+1)$, G_2 $(k = 6)$, $Spin(7)$ $(k = 7,\ l = 4)$, or $Spin(9)$ $(k = 15,\ l = 8)$, we have that $N(G_1)^o = G_1$, hence $G_1 = G$.*
- *In the case $G_1 = SU(l)$, we see that $N(G_1)^o = U(l)$, hence we may have either $G = G_1$ or $U(l)$.*
- *In the case $G_1 = Sp(\frac{l}{2})$, $N(G_1)^o$ is the subgroup of $SO(k+1)$ generated by $Sp(\frac{l}{2})$, and the S^3-subgroup realized as right multiplications of unit quaternions, As a group, $N(Sp(\frac{l}{2}))^o$ is isomorphic to $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$, where $\mathbb{Z}_2$ is the subgroup generated by $(-Id,\ -1)$. Hence, G is either $Sp(\frac{l}{2})$ or $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^1$ or $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$.*

The above results are also referred in the paper [**12**].
In particular, we see the following results from the above theorems.

COROLLARY 5.4. Assume the connected subgroup H in $O(2l)$ acts on S^{2l-1} transitively and its rank is l, i.e., rank $H = l$. Then $H \simeq U(l)$ or $SO(2l)$ in $O(2l)$.

In the next Section 5.3, we will discuss about slice representations.

5.3. Slice representations (rough structures). By Lemma 3.2 and the assumptions in Section 4, two singular isotropy subgroups can be denoted as $K_s = G_p$ $(s = 1,\ 2)$ for some fixed point $p \in M^T$. From the argument in Section 3.3, there are the following decompositions for this given $p \in M^T$:

$$G = G'_s \times G''_s \supset K_s = K_s^o = K'_s \times G''_s \supset T = T'_s \times T''_s, \tag{5.1}$$

where G'_s and G''_s are products of compact, connected, simple Lie groups and tori, and T'_s and T''_s are their maximal tori. Then a slice representation is as follows:

$$\sigma_s : K_s = K'_s \times G''_s \to SO(2k_s) \subset O(2k_s).$$

Remark that we can take the target of the slice representation σ_s as $SO(2k_s)$ because K_s is connected.

By the decomposition (5.1), we have that T''_s is in the kernel of the T-action on $G/K_s \cong G'/K'_s$. It follows that $T''_s \subset G''_s$ acts almost effectively on the normal space of $p \in G/K_s \cap M^T$. Moreover by Lemma 3.2 and Section 5.1, we have $\dim T''_s = k_s$. Therefore, we have $\sigma_s(T''_s) = T^{k_s} \subset SO(2k_s)$, i.e., a maximal torus of $SO(2k_s)$.

Because of the decomposition of $K_s = K'_s \times G''_s$, we know that K'_s commutes with G''_s in K_s. In particular, K'_s also commutes with $T''_s(\subset G''_s)$. It follows that the image $\sigma_s(K'_s)$ of K'_s is in the group $\{g \in SO(2k_s) \mid gt = tg \text{ for all } t \in T^{k_s}\}$: we remark that this group (the centralizer of T^{k_s} in $SO(2k_s)$) is T^{k_s} itself. Therefore, we have $\sigma_s(K'_s) \subset T^{k_s}$. Hence, G''_s acts transitively on S^{2k_s-1} through the slice representation $\sigma_s : K'_s \times G''_s \to SO(2k_s)$. Because rank $\sigma_s(G''_s) = k_s = \dim T''_s =$ rank G''_s, we have from Corollary 5.4:

$$\begin{aligned} G''_s &\approx \sigma_s(G''_s) \simeq U(k_s) \quad \text{or} \\ & SO(2k_s), \end{aligned}$$

where $X \approx Y$ means that they have the same Lie algebra. Now we may consider that if $k_s = 1$ then $SO(2) = U(1)$; therefore, we may assume

$$\begin{aligned} G''_s &= SU(k_s) \times T^1 \quad \text{or} \\ & SO(2k_s) \text{ and } k_s \geq 2. \end{aligned}$$

Because $\sigma_s(K'_s)$ is also in the centralizer of $\sigma_s(G''_s)$, we see

$$\begin{aligned} &\sigma_s(K'_s) \subset Z(U(k_s)) \quad (\text{if } \sigma_s(G''_s) \simeq U(k_s)) \quad \text{or} \\ &\sigma_s(K'_s) \simeq \{e\} \quad (\text{if } \sigma_s(G''_s) \simeq SO(2k_s),\ k_s \geq 2), \end{aligned}$$

where $Z(U(k_s)) \simeq T^1$ is the center of $U(k_s)$, i.e., the diagonal subgroup of $U(k_s)$.

In summary, we have the following two lemmas by the above arguments and Proposition 4.1.

Lemma 5.5. *Two singular orbits G/K_s $(s = 1,\ 2)$ satisfy the followings:*

- $(G,\ K_s) = (G'_s \times G''_s,\ K'_s \times G''_s)$;
- $G'_1 = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1)$ *and* $G'_2 = \prod_{i=1}^{c} SU(l'_i + 1) \times \prod_{j=1}^{d} SO(2m'_j + 1)$;
- $K'_1 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \prod_{j=1}^{b} SO(2m_j)$ *and* $K'_2 = \prod_{i=1}^{c} S(U(l'_i) \times U(1)) \times \prod_{j=1}^{d} SO(2m'_j)$;
- $G''_s \approx SU(k_s) \times T^1$ *or* $SO(2k_s)$ *and* $k_s \geq 2$.

Furthermore, $m_j,\ m'_j \geq 2$.

Lemma 5.6. *Two slice representations $\sigma_s : K_s = K'_s \times G''_s \to SO(2k_s)$ $(s = 1,\ 2)$ satisfy the following list:*

	G''_s	$\sigma_s(G''_s)$	$\sigma_s(K'_s)$
(1)	$SU(k_s) \times T^1$	$U(k_s) \subset SO(2k_s)$	$Z(U(k_s)) \simeq T^1 \subset U(k_s) \subset SO(2k_s)$
(2)	$SO(2k_s)$	$SO(2k_s)$	$\{e\} \subset SO(2k_s)$

where the right list means $\sigma_s(K'_s) \subset Z(U(k_s))$ for (1) and $\sigma_s(K'_s) = \{e\}$ (identity) for (2).

6. Possibility for G, K_1, K_2

In this section we shall give a possibility for a transformation Lie group G and two singular isotropy subgroups K_1, K_2.

6.1. Decomposition of G. We first remark that the decompositions of G in Lemma 5.5 depends on $s = 1,\ 2$, i.e., two decompositions $G'_1 \times G''_1$ and $G'_2 \times G''_2$ may not be the same decomposition. However, for all compact Lie groups G, there is an unique covering $\widetilde{G}$ consists of the product of simply connected, simple Lie group and tori. Therefore, the coverings of two G's for $s = 1,\ 2$ are the same coverings.

Using Lemma 5.5 and the relation $\widetilde{G} = \widetilde{G_1'} \times \widetilde{G_1''} = \widetilde{G_2'} \times \widetilde{G_2''}$, we know that $\widetilde{G}$ satisfies

$$\prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} Spin(2m_j+1) \times X = \prod_{i=1}^{c} SU(l_i'+1) \times \prod_{j=1}^{d} Spin(2m_j'+1) \times Y,$$

where $(X,\ Y)$ is one of the followings:

(1) $(X,\ Y) = (SU(k_1) \times T^1,\ SU(k_2) \times T^1)$;
(2) $(X,\ Y) = (Spin(2k_1),\ Spin(2k_2))$, and $k_1,\ k_2 \geq 2$;
(3) $(X,\ Y) = (SU(k_1) \times T^1,\ Spin(2k_2))$, and $k_2 \geq 2$;
(4) $(X,\ Y) = (Spin(2k_1),\ SU(k_2) \times T^1)$, and $k_1 \geq 2$.

Note that $m_j,\ m_j' \geq 2$.

Comparing left and right sides of the above equations from (1) to (4), we will find a possibility for G.

6.2. The cases (3) and (4). In this subsection, we shall prove that the above two cases (3) and (4) do not occur.

In the case (4), if we change the role of k_1 and k_2 then it is the same as the case (3). Hence, we can regard the last two cases (3) and (4) as the same cases.

First, we remark the following well-known lemma.

LEMMA 6.1. *The following three statements hold:*

- *if* $Spin(2m) \simeq SU(m+1)$, *then* $m = 3$ *and* $Spin(6) \simeq SU(4)$;
- *if* $Spin(2m+1) \simeq SU(m+1)$, *then* $m = 1$ *and* $Spin(3) \simeq SU(2)$;
- $Spin(4) \simeq SU(2) \times SU(2)$.

For the other m in Lemma 6.1, there are no isomorphisms between $Spin(2m)$ (or $Spin(2m+1)$) and $SU(m+1)$, because their dimensions are different. Therefore, for each case from (1) to (4), we can regard

$$b = d \quad \text{and} \quad Spin(2m_j+1) = Spin(2m_j'+1),$$

because $m_j,\ m_j' \geq 2$. In particular, we can regard $SO(2m_j+1) = SO(2m_j'+1)$ for $j = 1,\ \cdots,\ b$ in Lemma 5.5.

Hence, for the equation (3), we have

$$\prod_{i=1}^{a} SU(l_i+1) \times SU(k_1) \times T^1 = \prod_{i=1}^{c} SU(l_i+1) \times Spin(2k_2).$$

It follows that $Spin(2k_2) = T^1$, i.e., $k_2 = 1$. However, we can assume $k_s \geq 2$ for $Spin(2k_s)$. Hence, this case does not occur, and the equation (4) also does not occur because we can regard (3) and (4) are the same cases.

6.3. The case (2). Suppose the equation (2). If $Spin(2k_1) = Spin(2k_2)$, then we have that $k_1 = k_2$, $a = c$ and $l_i = l_i'$ for all $i = 1,\ \cdots,\ a$ (we call this case the case $(2)-(a)$).

If $Spin(2k_1) \neq Spin(2k_2)$, then there are the following four cases using Lemma 6.1:

(b): if $Spin(2k_1) = SU(l_c'+1)$, $(k_1,\ l_c') = (3,\ 3)$ and $Spin(2k_2) = SU(l_a+1)$, $(k_2,\ l_a) = (3,\ 3)$, then we have that $a = c$ and $l_i = l_i'$ for all $i = 1,\ \cdots,\ a-1$ (the case $(2)-(b)$);

(c): if $Spin(2k_1) = SU(l'_c+1)$, $(k_1,\ l'_c) = (3,\ 3)$ and $Spin(2k_2) = SU(l_{a-1}+1) \times SU(l_a+1)$, $(k_2,\ l_{a-1},\ l_a) = (2,\ 1,\ 1)$, then we have that $a = c+1$ and $l_i = l'_i$ for all $i = 1, \cdots, a-2 (= c-1)$ (the case $(2)-(c)$);

(d): if $Spin(2k_1) = SU(l'_{c-1}+1) \times SU(l'_c+1)$, $(k_1,\ l'_{c-1},\ l'_c) = (2,\ 1,\ 1)$ and $Spin(2k_2) = SU(l_{a-1}+1) \times SU(l_a+1)$, $(k_2,\ l_{a-1},\ l_a) = (2,\ 1,\ 1)$, then we have that $a = c$ and $l_i = l'_i$ for all $i = 1, \cdots, a-2$ (the case $(2)-(d)$);

(e): if $Spin(2k_1) \cap Spin(2k_2) \neq \{e\}$, that is, $Spin(2k_1) = SU(l'_c+1) \times SU(2)$, $(k_1,\ l'_c) = (2,\ 1)$ and $Spin(2k_2) = SU(l_a+1) \times SU(2)$, $(k_2,\ l_a) = (2,\ 1)$ $(Spin(2k_1) \cap Spin(2k_2) = SU(2))$, then we have that $a = c$ and $l_i = l'_i$ for all $i = 1, \cdots, a-1$ also hold (the case $(2)-(e)$).

In this subsection, we shall prove that these cases (from $(2)-(b)$ to $(2)-(e)$) do not occur. Before we prove this fact, we give the following remark.

Remark 6.2. Because G''_s acts transitively on the boundary of the slice, we can take the principal isotropy subgroup as $K = \sigma_s^{-1}(SO(2k_s-1)) \subset K_1 \cap K_2$. Hence, we can regard the subgroups $SO(2m_j)$, $S(U(l_i) \times U(1)) \subset K_1,\ K_2$ in Lemma 5.5 as the same subgroup in G by conjugating K_1 and K_2 in G, that is, they are in $K_1 \cap K_2$.

Suppose that the case $(2)-(b)$ occurs. Using Lemma 5.5, the above arguments and Remark 6.2, we have that

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(4) \times SU(4)$,
- $K_1 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(4) \times S(U(3) \times U(1))$,
- $K_2 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(3) \times U(1)) \times SU(4)$,

where $k_s = 3$, $\dim G/K_s = 2n - 2k_s = 2n-6$ and we identify $Spin(2k_s)$ as $SU(4)$. By Lemma 5.6, for the slice representation $\sigma_s : K_s \to SO(2k_s) = SO(6)$, we have that

$$\sigma_s(Spin(2k_s)) = \sigma_s(SU(4)) = SO(6).$$

Because $\sigma_s(K_s) = SO(6)$ acts transitively on the sphere $S^5 \cong SO(6)/SO(5) \cong K_s/K$, the restricted representation $\sigma_s|_{Spin(2k_s)}$ is the double covering $Spin(6)$ to $SO(6)$ by Theorem 5.2. Hence, we have that the following conjugation in G by $K \equiv \sigma_s^{-1}(SO(5))$:

$$\sigma_1^{-1}(SO(5)) = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times Spin(5) \times S(U(3) \times U(1))$$

$$\equiv \sigma_2^{-1}(SO(5)) = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(3) \times U(1)) \times Spin(5).$$

Moreover, we see that $Spin(5) \times S(U(3) \times U(1)) \subset \sigma_1^{-1}(SO(5))$ and $S(U(3) \times U(1)) \times Spin(5) \subset \sigma_2^{-1}(SO(5))$ are conjugate in $SU(4) \times SU(4) \subset G$. In particular, $Spin(5) \times \{I_4\}$ and $S(U(3) \times U(1)) \times \{I_4\}$ are conjugate in $SU(4) \times \{I_4\}$. This gives a contradiction, because $\dim Spin(5) = 10 \neq \dim S(U(3) \times U(1)) = 9$. Hence, the case $(2)-(b)$ does not occur.

For the other cases $(2)-(c)$ to $(2)-(e)$, we can similarly prove that these cases do not occur.

Hence, the case $(2)-(a)$ only occurs for the equation (2).

6.4. The case (1) and the summary. Suppose the equation (1). If $SU(k_1) = SU(k_2)$, then we have that $k_1 = k_2$, $a = c$ and $l_i = l'_i$ for all $i = 1, \cdots, a$ (we call this case the case $(1)-(a)$).

If $SU(k_1) \neq SU(k_2)$, then we can put $SU(l_a+1) = SU(k_2)$, $SU(l'_a+1) = SU(k_1)$, $a = c$ and $l_i = l'_i$ for all $i = 1, \cdots, a-1$ (we call this case the case $(1)-(b)$). Therefore, there are two cases $(1)-(a)$ and $(1)-(b)$ for the equation (1).

Hence, we have the following lemma by the above arguments, Lemma 5.5 and Remark 6.2.

LEMMA 6.3. *If (M, G) is a codimension one extended action of a simply connected torus manifold and its two singular orbits are simply connected torus manifolds, then the compact connected Lie group $G = \prod_{j=1}^{b} SO(2m_j+1) \times \widehat{G}$, and two singular isotropy subgroups $K_1 = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K_1}$ and $K_2 = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K_2}$, where $\widehat{G}$, $\widehat{K_1}$ and $\widehat{K_2}$ are one of the followings:*

(1) (a) $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k) \times T^1$,
where $\dim G/K_s = 2n - 2k$ *and* $k \geq 1$*;*

(b) $\widehat{G} = \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$,
$\widehat{K_1} = \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(k_1) \times S(U(k_2-1) \times U(1)) \times T^1$,
$\widehat{K_2} = \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(k_1-1) \times U(1)) \times SU(k_2) \times T^1$,
where $\dim G/K_s = 2n - 2k_s$ *and* $k_s \geq 1$*;*

(2) (a) $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,
where $\dim G/K_s = 2n - 2k$ *and* $k \geq 2$,

where $SO(2m_j),\ S(U(l_i) \times U(1)) \subset K_1 \cap K_2$.

In order to give a complete classification of G, K_1 and K_2, we study precise structures of slice representations for all cases $(1)-(a)$, $(1)-(b)$ and $(2)-(a)$ in Lemma 6.3, in the next Section 7 and 8.

7. Possibility for K and slice representations

In this section, we shall compute the slice representations for each cases in Lemma 6.3.

First we prepare the following notation:

$$(t_1, \cdots, t_a) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & t_1 \end{pmatrix}, \cdots, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix} \right) \in \prod_{i=1}^{a} S(U(l_i) \times U(1)),$$

where $A_i \in U(l_i)$ and $\det A_i^{-1} = t_i$.

7.1. Slice representations and K for the case $(1)-(a)$. In this subsection, we study the slice representations for the case $(1)-(a)$ in Lemma 6.3, that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
- $K_1 = K_2 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k) \times T^1$,

where $\dim G/K_s = 2n-2k$ and $k \geq 1$. By Lemma 5.6, for the slice representation $\sigma_s : K_s \to U(k) \subset SO(2k)$, we have that

$$\sigma_s(S(U(l_i)\times U(1))) = \{I_{2k}\} \text{ or } Z(U(k)) \subset U(k) \text{ for all } i = 1, \cdots, a,$$
$$\sigma_s(SU(k)\times T^1) = U(k).$$

Because $\sigma_s(S(U(l_i)\times U(1))) \subset Z(U(k))$ for all $i = 1, \cdots, a$ and the diagonal subgroup $Z(U(k)) \subset U(k)$ is the abelian group, we have that

$$\sigma_s \begin{pmatrix} A_i & 0 \\ 0 & t_i \end{pmatrix} = t_i^{\omega} I_k \in Z(U(k)) \subset U(k),$$

where $A_i \in U(l_i)$, $t_i = \det A_i^{-1} \in U(1)$, $\omega \in \mathbb{Z}$ and $I_k \in U(k)$ is the identity matrix. Hence, the image of $\prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k) \times T^1$ is as follows:

$$\sigma_1\left((t_1, \cdots, t_a), A, t\right) = t_1^{\alpha_1}\cdots t_a^{\alpha_a} t^{\alpha} A \in U(k), \tag{7.1}$$
$$\sigma_2\left((t_1, \cdots, t_a), A, t\right) = t_1^{\beta_1}\cdots t_a^{\beta_a} t^{\beta} A \in U(k), \tag{7.2}$$

where $(A, t) \in SU(k)\times T^1$ and $(\alpha_1, \cdots, \alpha_a, \alpha), (\beta_1, \cdots, \beta_a, \beta) \in \mathbb{Z}^{a+1}$.

We next show a principal isotropy subgroup K. Because $\sigma_s(K_s) = U(k)$ acts on the sphere $S^{2k-1} \cong U(k)/U(k-1) \cong K_s/K$ transitively, we have $\sigma_s^{-1}(U(k-1)) \equiv K$ (conjugation in G) for $s = 1, 2$. Hence, the following conjugation holds:

$$\begin{aligned} & \sigma_1^{-1}(U(k-1)) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; x^{-1} = \det X = t_1^{\alpha_1}\cdots t_a^{\alpha_a} t^{\alpha} \right\} \\ \equiv\ & \sigma_2^{-1}(U(k-1)) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix}, t \right) \;\middle|\; y^{-1} = \det Y = t_1^{\beta_1}\cdots t_a^{\beta_a} t^{\beta} \right\}, \end{aligned}$$

where $X, Y \in U(k-1)$ and we omit the subgroup $\prod_{j=1}^{b} SO(2m_j) \subset \ker \sigma_s \subset K$.

In order to analyse relations between $(\alpha_1, \cdots, \alpha_a, \alpha)$ and $(\beta_1, \cdots, \beta_a, \beta)$, we first consider the following subgroups:

$$\sigma_1^{-1}(U(k-1)) \cap T^1 = \{(e, t) \mid t \in T^1, t^{\alpha} = 1\} \simeq \mathbb{Z}_{\alpha} \text{ and}$$
$$\sigma_2^{-1}(U(k-1)) \cap T^1 = \{(e, t) \mid t \in T^1, t^{\beta} = 1\} \simeq \mathbb{Z}_{\beta},$$

where $e \in \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SU(k)$ is the identity element and $\mathbb{Z}_{\omega} = \{t \in T^1 \mid t^{\omega} = 1\}$ ($\omega \in \mathbb{Z}$). Since $\sigma_1^{-1}(U(k-1)) \equiv \sigma_2^{-1}(U(k-1))$ in G, we have $\sigma_1^{-1}(U(k-1)) \cap T^1 = \sigma_2^{-1}(U(k-1)) \cap T^1$. Therefore, we have $\mathbb{Z}_{\alpha} \simeq \mathbb{Z}_{\beta}$, i.e., $|\alpha| = |\beta|$. Because the slice representation $\sigma_s : K_s \to U(k) \subset O(2k)$ is equivalent up to conjugate in the target group $O(2k)$, we can put $\alpha \geq 0$ and $\beta \geq 0$. Hence, we have $\alpha = \beta$. Because $\sigma_s(SU(k)\times T^1) = U(k)$, we also have $\alpha = \beta \neq 0$, i.e., $\alpha = \beta \in \mathbb{N}$ (i.e., natural number).

Next we consider the following subgroups:

$$\begin{aligned} & \sigma_1^{-1}(U(k-1)) \cap (S(U(l_a)\times U(1)) \times T^1) \\ = & \left\{ \left(e, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix}, t \right) \;\middle|\; \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix} \in S(U(l_a)\times U(1)),\ t \in T^1,\ t_a^{\alpha_a} t^{\alpha} = 1 \right\} \end{aligned}$$

and

$$\sigma_2^{-1}(U(k-1))\cap(S(U(l_a)\times U(1))\times T^1)$$
$$= \left\{\left(e,\ \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix},\ t\right)\ \middle|\ \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix}\in S(U(l_a)\times U(1)),\ t\in T^1,\ t_a^{\beta_a}t^{\beta}=1\right\},$$

where $e\in\prod_{j=1}^b SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(k)$ is the identity element. Therefore, by using $\alpha=\beta$ and the above two subgroups (they are conjugate), we have that $t_a^{\alpha_a-\beta_a}=1$ for all $t_a\in U(1)$. Hence, we have $\alpha_a=\beta_a$. Similarly we can prove the equation $\alpha_i=\beta_i$ for all $i=1,\ \cdots,\ a$. Thus, we have the following lemma.

Lemma 7.1. *Let $\sigma_s : K_s\to SO(2k)\subset O(2k)$ be a slice representation of the case $(1)-(a)$. Then $\sigma_1=\sigma_2$, and they are defined by $\sigma_s(SO(2m_j))=\{e\}$ for all $j=1,\ \cdots,\ b$ and*

$$\sigma_s\left((t_1,\ \cdots,\ t_a),\ A,\ t\right)=t_1^{\alpha_1}\cdots t_a^{\alpha_a}t^{\alpha}A\in U(k)\subset SO(2k),$$

where $((t_1,\ \cdots,\ t_a),\ A,\ t)\in\prod_{i=1}^a S(U(l_i)\times U(1))\times SU(k)\times T^1$, $(\alpha_1,\ \cdots,\ \alpha_a)\in\mathbb{Z}^a$, $\alpha\in\mathbb{N}$ and $s=1,\ 2$.

Furthermore, we have that the principal isotropy group is $K=\prod_{j=1}^b SO(2m_j)\times\widehat{K}$ in the case $(1)-(a)$, where $\widehat{K}\subset\prod_{i=1}^a S(U(l_i)\times U(1))\times S(U(k-1)\times U(1))\times T^1$ is as follows:

$$\left\{\left((t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t\right)\ \middle|\ x^{-1}=\det X=t_1^{\alpha_1}\cdots t_a^{\alpha_a}t^{\alpha}\right\}.$$

7.2. Slice representations and K for the case $(1)-(b)$. In this subsection, we study the slice representations for the case $(1)-(b)$ in Lemma 6.3, that is,

- $G=\prod_{j=1}^b SO(2m_j+1)\times\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_1)\times SU(k_2)\times T^1$,
- $K_s=\prod_{j=1}^b SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(k_s)\times S(U(k_r-1)\times U(1))\times T^1$,

where $s+r=3$ $(s,\ r\geq 1)$, $\dim G/K_s=2n-2k_s$ and $k_s\geq 1$. By Lemma 5.6, we have for the slice representation $\sigma_s : K_s\to U(k_s)\subset SO(2k_s)$:

$$\sigma_s(S(U(l_i)\times U(1)))=\{I_{2k_s}\}\text{ or }Z(U(k_s))\subset U(k_s)\text{ for all }i=1,\ \cdots,\ a-1;$$
$$\sigma_s(S(U(k_r-1)\times U(1)))=\{I_{2k_s}\}\text{ or }Z(U(k_s))\subset U(k_s);$$
$$\sigma_s(SU(k_s)\times T^1)=U(k_s).$$

Because of a similar reason for (7.1) and (7.2) in Section 7.1, the images of $\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(k_s)\times S(U(k_r-1)\times U(1))\times T^1$ of slice representations are as follows:

$$\sigma_1\left((t_1,\ \cdots,\ t_{a-1}),\ A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t\right)=t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}}x^{\alpha_a}t^{\alpha}A\in U(k_1);$$
$$\sigma_2\left((t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ B,\ t\right)=t_1^{\beta_1}\cdots t_{a-1}^{\beta_{a-1}}y^{\beta_a}t^{\beta}B\in U(k_2),$$

where $(A,\ t)\in SU(k_1)\times T^1$, $X\in U(k_2-1)$, $\det X^{-1}=x\in U(1)$ and $(B,\ t)\in SU(k_2)\times T^1$, $Y\in U(k_1-1)$, $\det Y^{-1}=y\in U(1)$.

We next show a principal isotropy subgroup K. Because $\sigma_s(K_s) = U(k_s)$ acts on the sphere $S^{2k_s-1} \cong U(k_s)/U(k_s-1) \cong K_s/K$ transitively, we have $\sigma_s^{-1}(U(k_s-1)) \equiv K$ (conjugation in G) for $s = 1,\ 2$. Hence, the following conjugation holds:

$$\begin{aligned}
& \sigma_1^{-1}(U(k_1-1)) \\
= \ & \left\{ \left((t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} C & 0 \\ 0 & c \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ c^{-1} = \det C = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} \right\} \\
\equiv \ & \sigma_2^{-1}(U(k_2-1)) \\
= \ & \left\{ \left((t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} D & 0 \\ 0 & d \end{pmatrix},\ t \right) \ \middle|\ d^{-1} = \det D = t_1^{\beta_1} \cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} \right\},
\end{aligned}$$

where $C \in U(k_1-1),\ D \in U(k_2-1)$ and we omit the subgroup $\prod_{j=1}^{b} SO(2m_j) \subset \ker \sigma_s \subset K$. Therefore, we may assume the following equations:

$$c = y;\ d = x, \tag{7.3}$$

because the subgroup $S(U(k_1-1) \times U(1)) \times S(U(k_2-1) \times U(1)) \subset \sigma_1^{-1}(U(k_1-1))$ coincides with the subgroup $S(U(k_1-1) \times U(1)) \times S(U(k_2-1) \times U(1)) \subset \sigma_2^{-1}(U(k_2-1))$ up to conjugate in $SU(k_1) \times SU(k_2) \subset G$.

We can similarly prove $\alpha = \beta > 0$ and $\alpha_i = \beta_i$ for all $i = 1,\ \cdots,\ a-1$ like the case $(1)-(a)$ in Section 7.1, by analysing $\sigma_s^{-1}(U(k_s-1)) \cap T^1$ and $\sigma_s^{-1}(U(k_s-1)) \cap (S(U(l_i) \times U(1)) \times T^1)$.

Moreover, we have the following equation using the above $\sigma_1^{-1}(U(k_1-1)) \equiv \sigma_2^{-1}(U(k_2-1))$:

$$t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} c = t_1^{\beta_1} \cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} d.$$

Therefore, we have the equation $x^{\alpha_a - 1} = y^{\beta_a - 1}$ by using $\alpha_i = \beta_i$ $(i = 1,\ \cdots,\ a-1)$, $\alpha = \beta$ and Eq. (7.3). This equation holds for all $x,\ y \in U(1)$; therefore, we have $\alpha_a = \beta_a = 1$. Hence, we have the following lemma.

LEMMA 7.2. *Let $\sigma_s : K_s \to SO(2k_s) \subset O(2k_s)$ be a slice representation of the case $(1)-(b)$. Then σ_s is defined by $\sigma_s(SO(2m_j)) = \{e\}$ for all $j = 1,\ \cdots,\ b$ and*

$$\sigma_s \left((t_1,\ \cdots,\ t_{a-1}),\ A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x t^{\alpha} A \in U(k_s) \subset SO(2k_s),$$

where $(t_1,\ \cdots,\ t_{a-1}) \in \prod_{i=1}^{a-1} S(U(l_i) \times U(1))$, $A \in SU(k_s)$, $t \in T^1$, $X \in U(k_r-1)$, $x = \det X^{-1}$, $(\alpha_1,\ \cdots,\ \alpha_{a-1}) \in \mathbb{Z}^{a-1}$, $\alpha \in \mathbb{N}$ and $s + r = 3$ $(s,\ r \geq 1)$.

Furthermore, we have that the principal isotropy group is $K = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K}$ in the case $(1)-(b)$, where $\widehat{K} \subset \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(k_1-1) \times U(1)) \times S(U(k_2-1) \times U(1)) \times T^1$ is as follows:

$$\left\{ \left((t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x y t^{\alpha} = 1 \right\}.$$

7.3. Slice representations and K for the case $(2)-(a)$. In this subsection, we study the slice representations and a principal isotropy subgroup K for the case $(2)-(a)$ in Lemma 6.3, that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$,
- $K_1 = K_2 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,

where $\dim G/K_s = 2n - 2k$ and $k \geq 2$. By Lemma 5.6, we have that the slice representation $\sigma_s : K_s \to SO(2k)$ satisfies

$$\sigma_s(SO(2k)) = SO(2k)$$

and other factors are trivial. Therefore we have the following lemma.

LEMMA 7.3. *Let $\sigma_s : K_s \to SO(2k) \subset O(2k)$ be a slice representation of the case $(2) - (a)$. Then $\sigma_1 = \sigma_2$, and σ_s are defined by*

$$\sigma_s(SO(2m_j)) = \sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\} \text{ for all } j \text{ and } i,$$
$$\sigma_s|_{SO(2k)} = id_{SO(2k)},$$

where $\sigma_s|_{SO(2k)}$ is a restricted representation of σ_s to $SO(2k)$, and $id_{SO(2k)}$ is the identity representation, where $k \geq 2$.

Furthermore, we have that the principal isotropy group in the case $(2) - (a)$ is as follows:

$$K = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k-1).$$

PROOF. Because $\sigma_s(K_s) = SO(2k)$ acts transitively on the sphere $S^{2k-1} \cong SO(2k)/SO(2k-1) \cong K_s/K$, we see that the restricted representation $\sigma_s|_{SO(2k)}$ is the identity. Because $SO(2m_j)$ and $S(U(l_i) \times U(1))$ (for all $j = 1, \cdots, b$ and $i = 1, \cdots, a$) commute with $SO(2k)$ in K_s and $k \geq 2$, we also have $\sigma_s(SO(2m_j)) = \sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\}$. Hence, the slice representation σ_s of $(2)-(a)$ is unique for $s = 1, 2$. It follows the first statement of this lemma.

Because $K \equiv \sigma_s^{-1}(SO(2k-1))$, $S^{2k-1} \cong SO(2k)/SO(2k-1) \cong K_s/K$ and $\sigma_s|_{SO(2k)}$ is the identity, we also have the second statement of this lemma. □

8. Attaching maps from ∂X_1 to ∂X_2 and constructions of G-manifolds

In this final section for our classification, we devote to study attaching maps from ∂X_1 to ∂X_2, and construct the G-manifolds for each cases $(1) - (a)$, $(1) - (b)$ and $(2) - (a)$ in Lemma 6.3.

8.1. Preparation. Because of Theorem 2.6, we see $\partial X_1 = \partial X_2 = G/K$, that is, ∂X_s is a codimension 1 principal orbit. Hence, the attaching map f can be taken from the G-equivariant automorphism group $Aut_G(G/K)$ on G/K. As is well known, there is the following isomorphism:

$$Aut_G(G/K) \simeq N(K;G)/K, \tag{8.1}$$

where $N(K;G)$ is the normalizer of K in G (see [**13**]).

Attaching two boundaries ∂X_1 and ∂X_2 by $f \in N(K;G)/K$, we can construct a G-manifold, and such manifold is denoted by

$$M(f) = X_1 \cup_f X_2. \tag{8.2}$$

In order to check that $M(f)$ and $M(f')$ are equivariantly diffeomorphic or not for two attaching maps f and f', the following lemma is useful (see [**28**, Lemma 5.3.1]).

LEMMA 8.1. *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms, where ∂X_i means a boundary of X_i. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions are satisfied:*

(1) *f is G-diffeotopic to f';*
(2) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1;*
(3) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

As in [**9**], we call this lemma the *Uchida's criterion*.

Because of the Uchida's criterion (1), we do not need to compute (8.1). Instead, for attaching maps, we may compute

$$N(K;G)/N(K;G)^o, \tag{8.3}$$

where $N(K;G)^o$ is a connected component of $N(K;G)$.

After the criterion, we construct the G-manifolds explicitly. The following lemma gives one of the construction methods of such G-manifolds.

LEMMA 8.2. *Suppose that a compact, connected Lie group H acts on a compact manifold N with codimension one orbits H/K and two singular isotropy subgroups K_1 and K_2. Then, for a compact, connected Lie group G such that $G \supset H$, a compact, connected manifold*

$$M = G \times_H N$$

has the natural G-action on the first factor G by the left multiplication, and this action has codimension one orbits G/K and two singular isotropy subgroups K_1 and K_2.

From the next subsection 8.2, we analyse the attaching maps and construct G-manifolds, for each cases $(1)-(a)$, $(1)-(b)$ and $(2)-(a)$ in Lemma 6.3.

8.2. The case $(1)-(a)$**.** First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(1)-(a)$ (see (8.3)), and prove this case has only one attaching map up to G-diffeomorphism. In this case, $G = \prod_{j=1}^b SO(2m_j+1) \times \prod_{i=1}^a SU(l_i+1) \times SU(k) \times T^1$, and $K = \prod_{j=1}^b SO(2m_j) \times \widehat{K}$ where $\widehat{K}$ is as follows by Lemma 7.1:

$$\left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \middle| x^{-1} = \det X = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} \right\}.$$

Hence, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^b \mathbb{Z}_2 \times \prod_{i=1}^a W_{l_i+1}$$

because of $\alpha \in \mathbb{N}$, where $\prod_{j=1}^b S(O(2m_j) \times O(1))/SO(2m_j) \simeq \prod_{j=1}^b \mathbb{Z}_2$ and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

Remark the above $S_2(\simeq \mathbb{Z}_2)$ is the Weyl group of $SU(2)$.

In order to check the Uchida's criterion (2) or (3), we need to study a tubular neighborhood. By the slice theorem (Theorem 2.7), the tubular neighborhood X_s

and its boundary ∂X_s in this case is as follows:

$$
\begin{aligned}
(8.4) \qquad & X_s = G \times_{K_s} D^{2k} \\
&\cong \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} D(\mathbb{C}^k) \right); \\
(8.5) \qquad & \partial X_s \cong G \times_{K_s} (K_s/K) \\
&\cong \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} S(\mathbb{C}^k) \right),
\end{aligned}
$$

where $D(\mathbb{C}^k)(\cong D^{2k})$ is a disk and $S(\mathbb{C}^k)(\cong S^{2k-1})$ is a sphere in $\mathbb{C}^k$, and $\prod_{i=1}^{a} S(U(l_i)\times U(1))$ acts on $\mathbb{C}^k$ by the scalar multiplication defined by $\sigma'_s(t_1, \cdots, t_a) = t_1^{\alpha_1}\cdots t_a^{\alpha_a}$. Note that the above manifold has the following G-action:

(1) $\prod_{j=1}^{b} S^{2m_j}$ has the canonical transitive $\prod_{j=1}^{b} SO(2m_j+1)$-action;
(2) $\prod_{i=1}^{a} SU(l_i+1)$ has also the canonical transitive $\prod_{i=1}^{a} SU(l_i+1)$-action on itself;
(3) $\mathbb{C}^k$ has an action of $SU(k)\times T^1$ by $\sigma''_s(A,\ t) = At^{\alpha} \in U(k)$ for $(A,\ t) \in SU(k)\times T^1$.

Now we may check the Uchida's criterion (2) or (3).

First, we take the attaching map $f_j(\neq I_{2m_j+1})$ in $\mathbb{Z}_2 \simeq S(O(2m_j)\times O(1))/SO(2m_j) \subset \prod_{j=1}^{b}\mathbb{Z}_2$. Then this attaching map $f_j : \partial X_1 = G/K \to G/K = \partial X_2$ can be regarded as $f_j(gK) = gf_jK$, i.e., the non-trivial $SO(2m_j+1)$-equivariant diffeomorphism on the S^{2m_j} factor in $\partial X_s = G/K$ for $s = 1,\ 2$. Let $I : \partial X_2 = G/K \to G/K = \partial X_1$ be the identity attaching map. We shall prove $I\circ f_j : \partial X_1 = G/K \to G/K = \partial X_1$ is extendable to the equivariant diffeomorphism $X_1 \to X_1$. Remark that we can identify $I\circ f_j : G/K \to G/K$ as $f_j : G/K \to G/K$. Because $f_j = I\circ f_j$ induces an identity map on the $S(\mathbb{C}^k)$ factor in $\partial X_1 = G/K$ (see (8.5)), we have the following commutative diagram:

$$
\begin{array}{ccccc}
\partial X_1 & = & G\times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \\
 & & R_{f_j}\times id \downarrow & & \downarrow f_j \\
\partial X_1 & = & G\times_{K_1} K_1/K & \xrightarrow{\pi} & G/K
\end{array}
$$

where $\pi([g,\ kK]) = gkK$ (equivariantly diffeomorphic), $id : K_1/K = S(\mathbb{C}^k) \to K_1/K = S(\mathbb{C}^k)$ is the identity, R_{f_j} is the product of the map $S^{2m_j} \to S^{2m_j}$ (involution) and identities for the other factors. Now $id : K_1/K \cong S(\mathbb{C}^k) \to S(\mathbb{C}^k) \cong K_1/K$ is extendable to $id : D(\mathbb{C}^k) \to D(\mathbb{C}^k)$. Hence, $R_{f_j}\times id$ is extendable to the equivariant diffeomorphism $G\times_{K_1} D(\mathbb{C}^k) \to G\times_{K_1} D(\mathbb{C}^k)$, i.e., $I\circ f_j : \partial X_1 \to \partial X_1$ is extendable to $X_1 \to X_1$. Hence, we have that $M(f_j)\cong M(I)$ for all $j = 1, \cdots, b$ by the Uchida's criterion (Lemma 8.1 (2)).

Next, we take the attaching map $f_i(\neq I_{l_i+1})$ in W_{l_i+1} (when $l_i = 1$ and $\alpha_i = 0$). Because $\alpha_i = 0$, the $S(U(l_i)\times U(1))$ factor acts trivially on $S(\mathbb{C}^k)$ in ∂X_s. Hence, there is a trivial product factor $SU(l_i+1)/S(U(l_i)\times U(1)) \cong \mathbb{C}P(1) \cong S^2$ in ∂X_s ($l_i = 1$), that is, $\partial X_s \cong \mathbb{C}P(1)\times N$ for some manifold N, and the composition of two attaching maps $I\circ f_i : \partial X_s(\cong \mathbb{C}P(1)\times N) \to \partial X_s(\cong \mathbb{C}P(1)\times N)$ can be regarded as an $SU(l_i+1)$-equivariant diffeomorphism on this $\mathbb{C}P(1)\cong S^2$ factor in ∂X_s and trivially on the other factors N. Therefore, we can similarly show that $I \circ f_i$ is extendable to the G-equivariant diffeomorphism map $X_s \to X_s$, and $M(f_i) \cong M(I)$ by the Uchida's criterion (2) or (3).

Hence, the following proposition holds by the above argument and Lemma 7.1.

LEMMA 8.3. *For all the attaching map $f : \partial X_1 \to \partial X_2$ in the case $(1)-(a)$, we have $M(f) \cong M(I)$ where $I : \partial X_2 \to \partial X_1$ is the identity attaching map. In particular, for fixed integers $\alpha_1, \cdots, \alpha_a$ and a natural number α, a G-manifold M which satisfies the case $(1)-(a)$ is unique up to essential isomorphism.*

Hence, the case $(1)-(a)$ is only determined by the integers $\alpha_1, \cdots, \alpha_a$ and the natural number α which appear in the slice representation, up to essential isomorphism. $M(\mathfrak{a}) = M(\alpha_1, \cdots, \alpha_a, \alpha)$ denotes such manifold.

Finally we construct G-manifolds of this case $(1)-(a)$. Using the above X_s in (8.4) and Lemma 8.2 and 8.3, we can easily show the following manifold corresponds with $M(\mathfrak{a})$ (by considering orbits of the G-action on $M(\mathfrak{a})$):

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \right),$$

where $S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \simeq S^{2k}$ is a sphere in $\mathbb{C}^k \oplus \mathbb{R}$, $\mathbb{C}^k_{\mathfrak{a}}$ has an $\prod_{i=1}^{a} S(U(l_i) \times U(1))$-action by the representation $\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$, and the G-action on this manifold is induced from the G-action on X_s (see the lists (1), (2) and (3) below just (8.4)).

Because we classify $(M(\mathfrak{a}), G)$ up to essential isomorphism, we can regard G as $G/\ker$ where ker is a kernel of $(M(\mathfrak{a}), G)$. Hence, we can put $\alpha = 1$. Moreover, we can regard $SU(k) \times T^1$ as $U(k)$ through the finite covering $SU(k) \times T^1 \to SU(k) \times_{\mathbb{Z}_k} T^1 \simeq U(k)$, where $\mathbb{Z}_k$ is the center of $SU(k)$. Hence, we can put our transformation group G as $\prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k)$.

Now we can easily show this manifold is equivariantly diffeomorphic to

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \right),$$

where T^a acts on $\prod_{i=1}^{a} S^{2l_i+1} (\subset \prod_{i=1}^{a} \mathbb{C}^{l_i+1})$ canonically and on $S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \cap \mathbb{C}^k_{\mathfrak{a}}$ through the representation $\mathfrak{a} : T^a \to S^1$ such that $\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$. Hence, we have the following theorem.

THEOREM 8.4. *If the torus manifold (M, T) has codimension one extended G-action of the case $(1)-(a)$, then (M, G) is essential isomorphic to as follows:*

- $M(\mathfrak{a}) = \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})\right)$;
- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k)$,

where $(t_1, \cdots, t_a) \in T^a$ acts on $\mathbb{C}^k_{\mathfrak{a}}$ by the following scalar multiplication:

$$\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a},$$

and G acts on $M(\mathfrak{a})$ canonically as follows:

(1) $\prod_{j=1}^{b} SO(2m_j+1)$ *acts on the* $\prod_{j=1}^{b} S^{2m_j}$ *factor in* $M(\mathfrak{a})$;
(2) $\prod_{i=1}^{a} SU(l_i+1)$ *acts on the* $\prod_{i=1}^{a} S^{2l_i}$ *factor in* $M(\mathfrak{a})$;
(3) $U(k)$ *acts on the* $S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \cap \mathbb{C}^k$ *factor in* $M(\mathfrak{a})$.

8.3. The case $(1)-(b)$**.** First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(1)-(b)$ (see (8.3)). In this case, $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_1)\times SU(k_2)\times T^1$ and $K=\prod_{j=1}^{b}SO(2m_j)\times\widehat{K}$ where $\widehat{K}$ is as follows by Lemma 7.2:

$$\left\{\left((t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix} Y & 0\\ 0 & y\end{pmatrix},\ \begin{pmatrix} X & 0\\ 0 & x\end{pmatrix},\ t\right)\ \middle|\ t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}}xyt^{\alpha}=1\right\}.$$

Because $\alpha\in\mathbb{N}$, we also have

$$N(K;G)/N(K;G)^o\simeq\prod_{j=1}^{b}\mathbb{Z}_2\times\prod_{i=1}^{a-1}W_{l_i+1},$$

where

$$W_{l_i+1}=\begin{cases}\{I_{l_i+1}\} & \text{if } l_i\geq 2 \text{ or } \alpha_i\neq 0\\ S_2 & \text{if } l_i=1 \text{ and } \alpha_i=0\end{cases}$$

By a similar arguments in Section 8.2 for the case $(1)-(a)$, we have the following lemma by Lemma 7.2.

Lemma 8.5. *For all the attaching map* $f:\partial X_s\to\partial X_s$ *in the case* $(1)-(b)$*, we have* $M(f)\cong M(I)$ *where* $I:\partial X_s\to\partial X_s$ *is the identity attaching map. In particular, for fixed integers* $\alpha_1,\ \cdots,\ \alpha_{a-1}$ *and a natural number* α*, a* G*-manifold* M *which satisfies the case* $(1)-(b)$ *is unique up to essential isomorphism.*

Hence, the case $(1)-(b)$ is also determined by the integers $\alpha_1,\ \cdots,\ \alpha_{a-1}$ and the natural number α which appear in the slice representation, up to essential isomorphism. Let $M(\mathfrak{b})$ denote such manifold.

Finally, we construct the G-manifolds $M(\mathfrak{b})$ of this case $(1)-(b)$. Remark, in this case, tubular neighborhoods X_s are G-diffeomorphic to the following:

$$\prod_{j=1}^{b}S^{2m_j}\times\left\{\left(\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_r)\right)\times_{\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(k_r-1)\times U(1))}D(\mathbb{C}^{k_s}_{\mathfrak{b}_s})\right\},$$

where $s+r=3$ $(s,\ r\geq 1)$, $\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(k_r-1)\times U(1))$ acts on $D(\mathbb{C}^{k_s}_{\mathfrak{b}_s})$ by the representation $\sigma'_s(t_1,\ \cdots t_{a-1},\ x)=t_1^{\alpha_a}\cdots t_{a-1}^{\alpha_{a-1}}x$. The G-action on X_s is as follows: $\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_r)$ acts canonically; and $SU(k_s)\times T^1$ acts on $D(\mathbb{C}^{k_s}_{\mathfrak{b}_s})$ by $\sigma''_s(A,\ t)=At^{\alpha}$. By using Lemma 8.2, 8.5 and the above X_s, the following manifold corresponds with $M(\mathfrak{b})$:

$$\prod_{j=1}^{b}S^{2m_j}\times\left(\prod_{i=1}^{a-1}SU(l_i+1)\times_{\prod_{i=1}^{a-1}S(U(l_i)\times U(1))}P(\mathbb{C}^{k_1}_{\mathfrak{b}}\oplus\mathbb{C}^{k_2}))\right),$$

where $P(\mathbb{C}^{k_1}_{\mathfrak{b}}\oplus\mathbb{C}^{k_2}))\cong\mathbb{C}P(k_1+k_2-1)$ is a complex projective space, $\mathbb{C}^{k_1}_{\mathfrak{b}}$ has an $\prod_{i=1}^{a-1}S(U(l_i)\times U(1))$-action by the representation $\mathfrak{b}(t_1,\ \cdots,\ t_{a-1})=t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}}$, and the G-action on this manifold is induced from the G-action on the above X_s.

By the same reasons in the case $(1)-(a)$, we can put $\alpha=1$. Moreover, we can regard $SU(k_1)\times SU(k_2)\times T^1$ as $S(U(k_1)\times U(k_2))$ through the following finite covering $SU(k_1)\times SU(k_2)\times T^1\to S(U(k_1)\times U(k_2))$:

$$(A,\ B,\ t)\mapsto\begin{pmatrix} At^{k_2} & 0\\ 0 & Bt^{-k_1}\end{pmatrix}.$$

By this covering, we also have $(SU(k_1) \times SU(k_2)) \times_{\mathbb{Z}_{k_1k_2}} T^1 \simeq S(U(k_1) \times U(k_2))$. Hence, we can regard G as $\prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$.

Now we can easily show this manifold is equivariantly diffeomorphic to

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) \right),$$

where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ canonically and $P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ through the representation $\mathfrak{b}$. Hence, we have the following theorem.

THEOREM 8.6. *If the torus manifold $(M,\ T)$ has codimension one extended G-action of the case $(1)-(b)$, then $(M,\ G)$ is essential isomorphic to as follows:*

- $M(\mathfrak{b}) = \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})) \right)$;
- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$,

where $(t_1,\ \cdots,\ t_{a-1}) \in T^{a-1}$ acts on $\mathbb{C}^{k_1}_{\mathfrak{b}}$ by

$$\mathfrak{b}(t_1,\ \cdots,\ t_{a-1}) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}},$$

and G-action on $M(\mathfrak{b})$ is as follows:

(1) $\prod_{j=1}^{b} SO(2m_j+1)$ *acts on the* $\prod_{j=1}^{b} S^{2m_j}$ *factor in* $M(\mathfrak{b})$*;*
(2) $\prod_{i=1}^{a-1} SU(l_i+1)$ *acts on the* $\prod_{i=1}^{a-1} S^{2l_i+1}$ *factor in* $M(\mathfrak{b})$*;*
(3) $S(U(k_1) \times U(k_2))$ *acts on the* $P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ *factor in* $M(\mathfrak{b})$.

8.4. The case $(2)-(a)$. First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(2)-(a)$ (see (8.3)). Because $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$ and $K = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k-1)$ by Lemma 7.3, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a} W_{l_i} \times W,$$

where $W = S(O(2k-1) \times O(1))/SO(2k-1) = \{I_{2k},\ -I_{2k}\} \simeq \mathbb{Z}_2$. Because of the similar reason of the case $(1)-(a)$ and $(1)-(b)$, we have that $M(f) \cong M(I)$ for all attaching maps $f \in \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a} W_{l_i}$. Hence, we need to consider the attaching map $f = -I_{2k}$ in W. Remark this attaching map can be taken from the center of $SO(2k) \subset K_s$. Therefore, the following map is well-defined and commute:

$$\begin{array}{ccccc} \partial X_s & = & G \times_{K_s} K_s/K & \xrightarrow{\pi} & G/K \\ & & R_f \times id \downarrow & & \downarrow f \\ \partial X_s & = & G \times_{K_s} K_s/K & \xrightarrow{\pi} & G/K \end{array}$$

where $\pi([g,\ kK]) = gkK$, $f(kK) = kfK$ and $(R_f \times id)([g,\ k_1K]) = [gf,\ k_sK]$. Here, $id : K_s/K \cong S^{2k-1} \to S^{2k-1} \cong K_s/K$ is extendable to $id : D^{2k} \to D^{2k}$. Therefore, $R_f \times id$ is extendable to the equivariant diffeomorphism $G \times_{K_s} D^{2k} = X_s \to X_s = G \times_{K_s} D^{2k}$. Hence, $M(f) \cong M(I)$ by the Uchida's criterion. Thus, we have the following lemma by Lemma 7.3.

LEMMA 8.7. *For all the attaching map $f : \partial X_s \to \partial X_s$ in the case $(2)-(a)$, we have $M(f) \cong M(I)$ where I is the identity attaching map. In particular, a G-manifold M which satisfies the case $(2)-(a)$ is unique up to essential isomorphism.*

By Lemma 7.3, the tubular neighborhood X_s is as follows:

$$\prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} \mathbb{C}P(l_i) \times D(\mathbb{R}^{2k}).$$

Therefore, we have the following theorem by Lemma 8.2 and 8.7.

THEOREM 8.8. *If the torus manifold $(M,\ T)$ has codimension one extended G-action of the case $(2)-(a)$, then $(M,\ G)$ is essential isomorphic to as follows:*

- $M(\mathfrak{c}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$*;*
- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$*,*

where G acts on $M(\mathfrak{c})$ canonically.

As the result, we get the main result Theorem 1.

Because cohomology rings of quasitoric manifolds are generated by the degree 2 elements (see [**5, 6**]), we can easily show the following corollary:

COROLLARY 8.9. If a quasitoric manifold M has a codimension one extended G-action, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}),\quad \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1)\times U(k_2))\right).$$

9. On moment-angle manifolds

Finally, we observe some relation between a *moment-angle manifold* and our classification results in Corollary 8.9. A moment-angle manifold is defined as follows (see [**3, 5, 6**]). Let P be a simple convex polytope with the set of facets $\mathcal{F} = \{F_1,\ \cdots,\ F_m\}$. For each facet $F_i \in \mathcal{F}$, the 1-dimensional coordinate subgroup of the m-torus $T^{\mathcal{F}} \simeq T^m$ corresponding to F_i is denoted by T_{F_i}. Then assign to every face L the coordinate subtorus

$$T_L = \prod_{F_i \supset L} T_{F_i} \subset T^{\mathcal{F}}.$$

For every point $q \in P$, $L(q)$ denotes the unique face containing q in its relative interior.

Then the *moment-angle manifold* $\mathcal{Z}_P$ is the identification space

$$\mathcal{Z}_P = (T^{\mathcal{F}} \times P)/\sim,$$

where $(t_1,\ p) \sim (t_2,\ q)$ if and only if $p = q$ and $t_1^{-1}t_2 \in T_{L(p)}$.

We remark that moment-angle manifolds $\mathcal{Z}_P$ have natural T^m-actions on their $T^{\mathcal{F}}$ factors. Moreover, there is the following relations between quasitoric manifolds M over P and the moment-angle manifold $\mathcal{Z}_P$ over P (see [**5**, Proposition 6.5]):

PROPOSITION 9.1. There is the subtorus $H \subset T^{\mathcal{F}}$ such that $H \simeq T^{m-n}$ and H acts freely on $\mathcal{Z}_P$, where m is a number of facets in P and $2n$ is a dimension of M. Furthermore, this freely action defies a principal T^{m-n}-bundle $\mathcal{Z}_P \to M$.

In our case, the orbit space of the quasitoric manifold $M(\mathfrak{b}) = \prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ becomes a product of simplices $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$. For the moment-angle manifold $\mathcal{Z}_{P_1\times P_2}$ satisfies the following relation (see [**5**, Proposition 6.4]):

$$\mathcal{Z}_{P_1\times P_2} = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}.$$

Moreover, the moment-angle manifold over the simplex Δ^n becomes an odd dimensional sphere S^{2n+1} (see [**5**, Example 6.7]). Therefore, the moment angle manifold over $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ is as follows:

$$\mathcal{Z}(\mathfrak{b}) = \prod_{i=1}^{a-1} S^{2l_i+1} \times S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}),$$

where $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) \cong S^{2k_1+2k_2-1}$.

Remark that the number of facets in $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ and the half of dimension of $M(\mathfrak{b})$ are

$$m = \sum_{i=1}^{a-1}(l_i+1) + k_1 + k_2 \quad \text{and} \quad n = \sum_{i=1}^{a-1} l_i + k_1 + k_2 - 1.$$

Therefore, in our case, $H = T^{a-1} \times S^1$ (see Proposition 9.1). By the definition of $M(\mathfrak{b})$ and $\mathcal{Z}(\mathfrak{b})$, this group H acts on $\mathcal{Z}(\mathfrak{b})$ as follows:

(1) $T^{a-1} \subset H$ acts naturally on the $\prod_{i=1}^{a-1} S^{2l_i+1}$ factor, and acts on the $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) \cap \mathbb{C}^{k_1}_{\mathfrak{b}}$ factor through the representation $\mathfrak{b}$;
(2) $S^1 \subset H$ acts only on the $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) = S^{2k_1+2k_2-1} \subset \mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}$ factor naturally.

Moreover, $\mathcal{Z}(\mathfrak{b})$ has the natural action of $G = \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$, with codimension one principal orbits $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_1-1} \times S^{2k_2-1}$, and two singular orbits $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_1-1}$ and $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_2-1}$. Furthermore, we see that this G-action on $\mathcal{Z}(\mathfrak{b})$ is commute with the $H = T^{a-1} \times S^1$-action and it induces the codimension one action on $M(\mathfrak{b})$. We can also show the above fact for quasitoric manifolds with codimension 0 extended G-actions (such quasitoric manifolds are only products of complex projective spaces, see Corollary 2.4). Hence, we have the following theorem from our classification results.

Theorem 9.2. *If a quasitoric manifold M^{2n} has a codimension 0 or 1 extended G-actions, then its orbit space of T^n-action becomes a product of simplices $\prod_{i=1}^{a} \Delta^{l_i}$, where $\sum_{i=1}^{a} l_i = n$, and there is the following principal T^a-bundle for such quasitoric manifolds:*

$$\mathcal{Z} = \prod_{i=1}^{a} S^{2l_i+1} \to M^{2n}.$$

Furthermore, a codimension 0 and 1 extended G-actions on M can be lifted to G-actions on $\mathcal{Z}$ with codimension 0 and 1 principal orbits respectively. In other wards, all of codimension 0 and 1 extended G-actions on M can be induced from G-actions on $\mathcal{Z}$ with codimension 0 and 1 principal orbits respectively.

Remark 9.3. We can easily see that two singular orbits of $(\mathcal{Z}(\mathfrak{b}),\ G)$ are moment-angle manifolds of two singular orbits of $(M,\ G)$ respectively.

Remark 9.4. According to our classification results in Theorem 1, for torus manifolds with some assumptions, we can easily show that there is a similar principal T^a-bundle like the above moment-angle manifold:

$$\mathcal{T}(\mathfrak{a}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} S^{2l_i+1} \times S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \longrightarrow M(\mathfrak{a});$$

$$\mathcal{T}(\mathfrak{b}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a-1} S^{2l_i+1} \times S(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}) \longrightarrow M(\mathfrak{b});$$

$$\mathcal{T}(\mathfrak{c}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} S^{2l_i+1} \times S(\mathbb{R}^{2k} \oplus \mathbb{R}) \longrightarrow M(\mathfrak{c}),$$

where $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \cong S(\mathbb{R}^{2k} \oplus \mathbb{R}) \cong S^{2k}$. Moreover, all codimension 1 extended G-actions on $M(\mathfrak{a})$, $M(\mathfrak{b})$ and $M(\mathfrak{c})$ can be lifted to G-actions on $\mathcal{T}(\mathfrak{a})$, $\mathcal{T}(\mathfrak{b})$ and $\mathcal{T}(\mathfrak{c})$ (they are denoted by $\mathcal{T}$) with codimension 1 principal orbits respectively. In other wards, all of codimension 1 extended G-actions on M in Theorem 1 can be induced from G-actions on $\mathcal{T}$ with codimension 1 principal orbits.

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