Spectral estimates of least energy solutions to the Brezis-Nirenberg problem with a variable coefficient

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Abstract

Let u_ε be a least energy solution to the Brezis-Nirenberg problem:

$$-\Delta u = c_0 u^p + \varepsilon k(x) u \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N (N \geq 6)$ is a smooth bounded domain, $k \in C^2(\overline{\Omega})$ is a nonnegative function, $c_0 = N(N-2)$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small parameter.

We prove several asymptotic estimates of eigenvalues $\lambda_{i,\varepsilon}$ and corresponding eigenfunctions $v_{i,\varepsilon}$ to the eigenvalue problem:

$$\begin{cases} -\Delta v_{i,\varepsilon} = \lambda_{i,\varepsilon} \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x) \right) v_{i,\varepsilon} & \text{in } \Omega, \\ v_{i,\varepsilon} = 0 & \text{on } \partial\Omega, \\ \|v_{i,\varepsilon}\|_{L^\infty(\Omega)} = 1 \end{cases}$$

as $\varepsilon \to 0$, for $i = 1, 2, \cdots, N+1, N+2$.

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1 Introduction

Let us consider the problem

$$\begin{cases} -\Delta u = c_0 u^p + \varepsilon k(x) u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, $\varepsilon > 0$ is a small parameter and $k \in C^2(\overline{\Omega})$.

Assume $\Omega_+ := \{x \in \Omega | k(x) > 0\} \neq \phi$ and $\varepsilon > 0$ is sufficiently small such that $-\Delta - \varepsilon k(x) I$ is coercive. Then by a famous result of Brezis and Nirenberg [1], there exists a solution u_ε of (1.1) with the property that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx - \varepsilon \int_\Omega k(x) u_\varepsilon^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p+1} dx\right)^{\frac{2}{p+1}}} = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx - \varepsilon \int_\Omega k(x) u^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

u_ε is called a least energy solution to (1.1). Least energy solution u_ε is known to blow up in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon \to 0$. In the following, the symbol $\|\cdot\|$ will denote $\|\cdot\|_{L^\infty(\Omega)}$. Let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. We call any accumulation point of $\{x_\varepsilon\}$ a *blow-up point* of the sequence $\{u_\varepsilon\}$. It is also known that the set of blow-up points of $\{u_\varepsilon\}$ consists of one point $x_0 \in \overline{\Omega}$. On the location of the blow-up point x_0 of least energy solutions, the author proved [7] that $x_0 \in \Omega_+$ and x_0 is a minimum point of the function $F : \Omega_+ \to \mathbb{R}_+$, defined by

$$F(x) = \frac{R(x)^{\frac{2}{N-2}}}{k(x)}, \quad x \in \Omega_+. \tag{1.2}$$

Here,

$$R(x) = \lim_{z \to x} \left[\frac{1}{(N-2)\sigma_N} |x - z|^{2-N} - G(x, z) \right]$$

is the (positive) Robin function associated with Green's function of $-\Delta$ with the Dirichlet boundary condition $G(x, z)$, and σ_N is the volume of the unit sphere in $\mathbb{R}^N$.

In this paper, we will prove several spectral estimates for this blowing-up solution u_ε as $\varepsilon \to 0$. In the following, we assume always that

$$k \in C^2(\overline{\Omega}),\ k(x) \geq 0,\ \text{with } \Omega_+ = \{x \in \Omega | k(x) > 0\} \neq \phi.$$

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{1.3}$$

It is easy to check that (1.3) admits a countable sequence of eigenvalues $\lambda_{1,\varepsilon} < \lambda_{2,\varepsilon} \le \cdots \le \lambda_{i,\varepsilon} \le \cdots \to +\infty$ and corresponding eignefunctions $v_{1,\varepsilon}, v_{2,\varepsilon}, \cdots, v_{i,\varepsilon}, \cdots$, $\|v_{i,\varepsilon}\| = 1$ $(i \in \mathbb{N})$, such that

$$\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v_{i,\varepsilon} v_{j,\varepsilon} dx = 0, \quad (i \ne j). \tag{1.4}$$

Also, we introduce the scaled eigenfunctions

$$\tilde{v}_{i,\varepsilon}(y) = v_{i,\varepsilon}\left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{(p-1)/2}(\Omega - x_\varepsilon). \tag{1.5}$$

Now, we state our theorems.

Theorem 1.1 *Assume $N \ge 5$. As $\varepsilon \to 0$, we have*

$$\begin{aligned} &\lambda_{1,\varepsilon} \to 1/p, \\ &\tilde{v}_{1,\varepsilon}(y) \to U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} \quad \text{in } C^1_{loc}(\mathbb{R}^N), \\ &\|u_\varepsilon\|^2 v_{1,\varepsilon} \to (N-2)\sigma_N G(\cdot, x_0) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}). \end{aligned}$$

Also, $\lambda_{1,\varepsilon}$ is simple for $\varepsilon > 0$ sufficiently small.

Theorem 1.2 *Assume $N \ge 6$. Then for $i = 2, 3, \cdots, N+1$, we have*

$$\tilde{v}_{i,\varepsilon}(y) \to \sum_{j=1}^N a_{i,j} \frac{y_j}{(1+|y|^2)^{\frac{N}{2}}} \quad \text{in } C^1_{loc}(\mathbb{R}^N), \tag{1.6}$$

$$\|u_\varepsilon\|^{2+\frac{2}{N-2}} v_{i,\varepsilon}(x) \to \sigma_N \sum_{j=1}^N a_{i,j} \left(\frac{\partial G}{\partial z_j}\right)(x,z)|_{z=x_0} \quad \text{in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \tag{1.7}$$

for some $\vec{a}_i = (a_{i,1}, a_{i,2}, \cdots, a_{i,N}) \ne \vec{0}$ as $\varepsilon \to 0$. In addition,

$$\|u_\varepsilon\|^{\frac{2N}{N-2}} \left(\lambda_{i,\varepsilon} - 1\right) \to M\mu_{i-1}, \quad \varepsilon \to 0, \tag{1.8}$$

where $\mu_1 \le \mu_2 \le \cdots \le \mu_N$ are eigenvalues of the matrix

$$A(x_0) = \left(\frac{2}{N-2} \frac{R_{x_i,x_j}}{R} - \frac{k_{x_i,x_j}}{k} \right)_{1 \le i,j \le N} (x_0). \tag{1.9}$$

and

$$M = \frac{\frac{(N-2)^2}{4} \sigma_N^2 R(x_0)}{p \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy} = \frac{(N-2)\Gamma(N+2)}{2(N+2)\Gamma(N/2+1)^2} \sigma_N R(x_0) > 0.$$

Furthermore, $\vec{a}_i$ is an eigenvector of $A(x_0)$ corresponding to μ_{i-1} and $\vec{a}_i$ is perpendicular to $\vec{a}_j$ in $\mathbb{R}^N$ if $i \neq j$.

In Appendix, we prove that the matrix $A(x_0)$ is nonnegative definite, so all μ_{i-1} in (1.8) is nonnegative.

Note that the nullity of u_ε is the number of eigenvalues of (1.3) such that $\lambda_{i,\varepsilon} = 1$. Thus, as a corollary of Theorem 1.2, we have the main result of [8].

Corollary 1.3 *Under the same assumption of Theorem 1.2, if the matrix $A(x_0)$ in (1.9) is non-singular, then u_ε is nondegenerate in the sense that the linearized operator around u_ε: $L = -\Delta - c_0 p u_\varepsilon^{p-1} \cdot - \varepsilon k(x) \cdot : H_0^1(\Omega) \to H^{-1}(\Omega)$, is invertible for $\varepsilon > 0$ sufficiently small.*

Theorem 1.4 *Assume $N \ge 6$. As $\varepsilon \to 0$, we have*

$$\tilde{v}_{N+2,\varepsilon}(y) \to b_{N+2} \frac{1 - |y|^2}{(1+|y|^2)^{\frac{N}{2}}} \quad \text{in } C^1_{loc}(\mathbb{R}^N) \tag{1.10}$$

for some $b_{N+2} \neq 0$, and

$$\|u_\varepsilon\|^2 (\lambda_{N+2,\varepsilon} - 1) \to \Gamma, \tag{1.11}$$

where

$$\Gamma = \frac{(N-2)^2 (N-4) \sigma_N^2 R(x_0)}{c_0 p (\frac{N-2}{2}) \int_{\mathbb{R}^N} \frac{(1-|y|^2)^2}{(1+|y|^2)^{N+2}} dy} = 2(N-4)M > 0.$$

Let S_N be the best Sobolev constant. In [5], Grossi and Pacella considered the eigenvalue problem

$$-\Delta v = \lambda \left(c_0 (p-\varepsilon) u_\varepsilon^{p-\varepsilon-1} \right) v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad \|v\|_{L^\infty(\Omega)} = 1$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, where u_ε is a blowing-up solution of the slightly subcritical problem

$$-\Delta u = c_0 u^{p-\varepsilon},\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with the property

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p-\varepsilon+1} dx\right)^{2/(p-\varepsilon+1)}} \to S_N \quad \text{as } \varepsilon \to 0.$$

In addition to the qualitative properties of eigenfunctions, they obtained analogous results on the asymptotic behavior of eigenvalues and eigenfunctions as $\varepsilon \to 0$.

Main purpose of this paper is to study the effect of the coefficient function $k(x)$ closely on the asymptotic behavior of spectral data. We will prove above theorems along the line in [5]. However, we need more efforts to handle with the additional term involving $k(x)$ in (1.3).

Once the precise asymptotic behavior of eigenfunctions is established, the same proof in [5] with a minor modification applies well to obtain the next theorem, so we omit the details.

Theorem 1.5 *Denote $N_{i,\varepsilon} = \{x \in \Omega \mid v_{i,\varepsilon}(x) = 0\}$ for $i \in \mathbb{N}$. Then for $\varepsilon > 0$ sufficiently small, we have the followings.*

(1) The eigenfunction $v_{i,\varepsilon}$ has only two nodal regions for $i = 2, \cdots, N+1$.

(2) $\overline{N_{i,\varepsilon}} \cap \partial\Omega \neq \phi$ if Ω is convex and $i = 2, \cdots, N+1$.

(3) $\lambda_{N+2,\varepsilon}$ is simple, $v_{N+2,\varepsilon}$ has only two nodal regions and $\overline{N_{N+2,\varepsilon}} \cap \partial\Omega = \phi$.

The outline of this paper is as follows. In §2, we collect some useful lemmas and prove some preliminary result. Theorem 1.1 is proved in §3. The proof is simple, but it includes all ideas in the sequel. In §4, we will prove Theorem 1.2, while Theorem 1.4 will be proved in §5. In Appendix, we will prove that the matrix $A(x_0)$ is non-negative definite.

2 Preliminaries

In this section, we collect lemmas which are needed in the proof.

Lemma 2.1 *The following identities hold true. For any $i \in \mathbb{N}$ and for any $y \in \mathbb{R}^N$,*

$$\int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial u_\varepsilon}{\partial \nu}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial \nu}\right) ds_x = (1-\lambda_{i,\varepsilon})\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) w_\varepsilon v_{i,\varepsilon} dx$$
$$+ \varepsilon \int_\Omega u_\varepsilon v_{i,\varepsilon} \left(2k(x) + (x-y)\cdot\nabla k(x)\right) dx, \qquad (2.1)$$

where $w_\varepsilon(x) = (x-y)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon$, and

$$\int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial \nu}\right) ds_x = (1-\lambda_{i,\varepsilon})\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) v_{i,\varepsilon} dx$$
$$+ \varepsilon \int_\Omega \left(\frac{\partial k}{\partial x_j}\right) u_\varepsilon v_{i,\varepsilon} dx, \quad (j = 1, 2, \cdots, N). \quad (2.2)$$

Here $\nu = \nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

Proof. By an easy calculation, w_ε satisfies

$$-\Delta w_\varepsilon = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) w_\varepsilon + \varepsilon u_\varepsilon \left(2k(x) + (x-y)\cdot\nabla k(x)\right) \quad \text{in } \Omega. \quad (2.3)$$

Then follow the proof of Lemma 4.3 and Lemma 5.1 in [5] with (2.3). □

Lemma 2.2 *Let $G = G(x,z)$ be Green's function of $-\Delta$ under the Dirichlet boundary condition. Then we have for any $y \in \Omega$,*

$$\int_{\partial\Omega} ((x-y)\cdot\nu)\left(\frac{\partial G(x,y)}{\partial \nu}\right)^2 ds_x = (N-2)R(z)\Big|_{z=y}, \qquad (2.4)$$

$$\int_{\partial\Omega} \left(\frac{\partial G(x,y)}{\partial \nu}\right)^2 \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z)\Big|_{z=y}, \qquad (2.5)$$

$$\int_{\partial\Omega} \left(\frac{\partial G(x,y)}{\partial x_i}\right)\frac{\partial}{\partial \nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,y) ds_x = \frac{1}{2}\frac{\partial^2 R}{\partial z_i \partial z_j}(z)\Big|_{z=y}. \qquad (2.6)$$

Proof. See [6]; note that the sign of R is negative in [6]. □

Recall $v_{i,\varepsilon}$ satisfies

$$\begin{cases} -\Delta v_{i,\varepsilon} = \lambda_{i,\varepsilon}\left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v_{i,\varepsilon} & \text{in } \Omega, \\ v_{i,\varepsilon} = 0 & \text{on } \partial\Omega, \\ \|v_{i,\varepsilon}\| = 1. \end{cases} \qquad (2.7)$$

So the scaled eigenfunction $\tilde{v}_{i,\varepsilon}$ satisfies

$$\begin{cases} -\Delta \tilde{v}_{i,\varepsilon} = \lambda_{i,\varepsilon} \left(c_0 p \tilde{u}_\varepsilon^{p-1} + \frac{\varepsilon}{\|u_\varepsilon\|^{p-1}} k_\varepsilon(y) \right) \tilde{v}_{i,\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{v}_{i,\varepsilon} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_{i,\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} = 1 \quad (i \in \mathbb{N}), \end{cases} \tag{2.8}$$

where $k_\varepsilon(y) = k\left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon \right)$ and

$$\tilde{u}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon. \tag{2.9}$$

Note that $k_\varepsilon(y) \to k(x_0)$ uniformly on compact sets on $\mathbb{R}^N$. Also by a result in [6], we see

$$\tilde{u}_\varepsilon \to U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-2}{2}} \quad \text{in } C^2_{loc}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N), \tag{2.10}$$

where U is the unique solution of

$$-\Delta U = c_0 U^p \text{ in } \mathbb{R}^N, \ 0 < U \leq 1, \ U(0) = 1.$$

Lemma 2.3 *([6]. see also [8] Proposition 1.1.) Assume $N \geq 4$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Then after passing to a subsequence, we have the followings: There exists a constant $C > 0$ independent of ε such that*

$$u_\varepsilon(x) \leq C \frac{\|u_\varepsilon\|}{(1 + \|u_\varepsilon\|^{p-1}|x - x_\varepsilon|^2)^{\frac{N-2}{2}}}, \quad (\forall x \in \Omega), \tag{2.11}$$

$$\|u_\varepsilon\| u_\varepsilon \to (N-2)\sigma_N G(\cdot, x_0) \ \mathit{in} \ C^1_{loc}(\overline{\Omega} \setminus \{x_0\}), \tag{2.12}$$

as $\varepsilon \to 0$, and

$$\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^{\frac{2(N-4)}{N-2}} = \frac{(N-2)^3}{2a_N} \sigma_N \frac{R(x_0)}{k(x_0)} \qquad (N \geq 5), \tag{2.13}$$

$$\lim_{\varepsilon \to 0} \varepsilon \log \|u_\varepsilon\| = 4\sigma_4 \frac{R(x_0)}{k(x_0)} \qquad (N = 4),$$

where $a_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N-2}}$.

Theorem 2.4 *(Bianchi and Egnell [2]) The eigenvalue problem*

$$\begin{cases} -\Delta V_i = \lambda_i c_0 p U^{p-1} V_i \quad in\ \mathbb{R}^N, \\ V_i \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{2.14}$$

where $D^{1,2}(\mathbb{R}^N) = \{V \in L^{2N/(N-2)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla V|^2 dy < +\infty\}$*, has eigenvalues*

$$\lambda_1 = 1/p < \lambda_2 = \lambda_3 = \cdots = \lambda_{N+1} = \lambda_{N+2} = 1 < \lambda_{N+3} \leq \cdots$$

with eigenfunctions

$$V_1 = U = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}, \quad V_i = \frac{\partial U}{\partial y_{i-1}}, (i = 2, \cdots, N+1),$$
$$V_{N+2} = \frac{d}{d\lambda}\Big|_{\lambda=1} \lambda^{(N-2)/2} U(\lambda y) = y \cdot \nabla U + \frac{N-2}{2} U.$$

Note that the pointwise estimate (2.11) is equivalent to

$$\tilde{u}_\varepsilon(y) \leq CU(y), \quad \forall y \in \Omega_\varepsilon. \tag{2.15}$$

Also, we can obtain the following pointwise estimate and the convergence result for eigenfunctions. In the sequel, we assume always $N \geq 5$.

Lemma 2.5 *For any* $i \in \mathbb{N}$*, there exists a constant* $C > 0$ *independent of* ε *such that*

$$|\tilde{v}_{i,\varepsilon}(y)| \leq CU(y) \tag{2.16}$$

holds true for all $y \in \Omega_\varepsilon$*.*

Proof. Argue as Lemma 3.1 in [4]. □

Lemma 2.6 *Suppose* $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$*. Then*

$$\tilde{v}_{i,\varepsilon} \to V_i = \sum_{j=1}^N a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \quad in\ C^1_{loc}(\mathbb{R}^N) \tag{2.17}$$

as $\varepsilon \to 0$ *for some* $(a_{i,1}, a_{i,2}, \cdots, a_{i,N}, b_i) \neq (0, 0, \cdots, 0)$*.*

Proof. By elliptic estimates, (2.15) and (2.16), we can check that there exists some V_i such that

$$\tilde{v}_{i,\varepsilon} \to V_i \quad \text{in } C^1_{loc}(\mathbb{R}^N) \quad (i \in \mathbb{N}).$$

Also by the fact $\int_{\Omega_\varepsilon} |\nabla \tilde{v}_{i,\varepsilon}|^2 dy \le C$ (see [3]: the equation (10)), we confirm that $V_i \in D^{1,2}(\mathbb{R}^N)$. Put $\lambda_i = \lim_{\varepsilon\to 0} \lambda_{i,\varepsilon}$ ($i \in \mathbb{N}$). Then by using (2.10) in (2.8), we see

$$\begin{cases} -\Delta V_i = \lambda_i \left(c_0 p U^{p-1}\right) V_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla V_i|^2 dy < \infty. \end{cases} \tag{2.18}$$

Thus if $\lambda_i = 1$, we have some $a_{i,1}, \cdots, a_{i,N}$ and b_i such that

$$V_i = \sum_{j=1}^{N} a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$$

by Theorem 2.4. We see that $V_i \not\equiv 0$ by the estimate (2.16). Indeed, if $V_i \equiv 0$, then the maximum point y^i_ε of $\tilde{v}_{i,\varepsilon}$ would satisfy $|y^i_\varepsilon| \to +\infty$, since $\tilde{v}_{i,\varepsilon} \to V_i \equiv 0$ compact uniformly on $\mathbb{R}^N$. But this is impossible because of the estimate (2.16). □

From Lemma 2.6, we can obtain the following convergence result; see [4].

Lemma 2.7 *Suppose* $\lambda_i = \lim_{\varepsilon\to 0} \lambda_{i,\varepsilon} = 1$ *and* $b_i \ne 0$ *in (2.17). Then we have*

$$\|u_\varepsilon\|^2 v_{i,\varepsilon} \to -(N-2) b_i \sigma_N G(\cdot, x_0) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0. \tag{2.19}$$

Now, since the blow-up point x_0 is an interior point of Ω, we may assume that there exists $\rho > 0$ such that $B(x_\varepsilon, 2\rho) \subset \Omega$ for any $\varepsilon > 0$ sufficiently small. We employ a cut-off function $\phi = \phi(x)$ such that $\phi \in C^\infty_0(B(x_\varepsilon, 2\rho))$, $0 \le \phi \le 1$ and $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$. Denote

$$\psi_{j,\varepsilon}(x) = \phi(x) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right), \quad j = 1, \cdots, N, \tag{2.20}$$

$$\psi_{N+1,\varepsilon}(x) = \phi(x) \left((x - x_\varepsilon)\cdot \nabla u_\varepsilon + \frac{2}{p-1} u_\varepsilon\right). \tag{2.21}$$

Then, as Lemma 3.1 in [5], we have the following lemma.

Lemma 2.8 *$u_\varepsilon, \{\psi_{j,\varepsilon}\}_{j=1,\cdots,N}, \psi_{N+1,\varepsilon}$ are linearly independent in $H_0^1(\Omega)$.*

Proof. Assume the contrary that there exist $\alpha_{0,\varepsilon}, \alpha_{1,\varepsilon}, \cdots, \alpha_{N,\varepsilon}, \alpha_{N+1,\varepsilon}$ such that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 \neq 0$ and

$$\alpha_{0,\varepsilon} u_\varepsilon + \sum_{j=1}^{N} \alpha_{j,\varepsilon} \psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon} \psi_{N+1,\varepsilon} \equiv 0$$

in Ω. Without loss of generality, we may assume that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 = 1$.

First we claim that $\alpha_{0,\varepsilon} = 0$. Indeed, if $\alpha_{0,\varepsilon} \neq 0$, then we have

$$u_\varepsilon = \sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \tag{2.22}$$

where $\beta_{j,\varepsilon} = -\alpha_{j,\varepsilon}/\alpha_{0,\varepsilon}$. By putting $x = x_\varepsilon$ into (2.22), and noting $\phi(x_\varepsilon) = 1$ and $\nabla u_\varepsilon(x_\varepsilon) = 0$, we see $u_\varepsilon(x_\varepsilon) = \frac{2}{p-1} \beta_{N+1,\varepsilon} u_\varepsilon(x_\varepsilon)$, thus we have

$$\beta_{N+1,\varepsilon} = \frac{p-1}{2} \quad \text{if } \alpha_{0,\varepsilon} \neq 0.$$

On the other hand, by differentiating (1.1), we see for $j = 1, \cdots, N$,

$$-\Delta \psi_{j,\varepsilon} = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x) \right) \psi_{j,\varepsilon} + \varepsilon \left(\frac{\partial k}{\partial x_j} \right) u_\varepsilon \quad \text{on } B(x_\varepsilon, \rho). \tag{2.23}$$

Recall $w_\varepsilon(x) = (x - x_\varepsilon) \cdot \nabla u_\varepsilon(x) + \frac{2}{p-1} u_\varepsilon$ satisfies (2.3) (with $y = x_\varepsilon$), thus

$$-\Delta \psi_{N+1,\varepsilon} = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon \right) \psi_{N+1,\varepsilon} + \varepsilon u_\varepsilon (2k(x) + (x - x_\varepsilon) \cdot \nabla k(x)) \tag{2.24}$$

on $B(x_\varepsilon, \rho)$. From (2.23) and (2.24), we have

$$\begin{aligned} -\Delta \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) &= \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x) \right) \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) \\ &+ \varepsilon u_\varepsilon \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} \left(\frac{\partial k}{\partial x_j} \right) + \beta_{N+1,\varepsilon} (2k(x) + (x - x_\varepsilon) \cdot \nabla k(x)) \right) \end{aligned} \tag{2.25}$$

on $B(x_\varepsilon, \rho)$. Furthermore, since $u_\varepsilon = \sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon}$ is a solution to (1.1), we have

$$-\Delta \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) = \left(c_0 u_\varepsilon^{p-1} + \varepsilon k(x) \right) \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right). \tag{2.26}$$

From (2.25) and (2.26), we see

$$c_0(1-p)u_\varepsilon^{p-1} \equiv \varepsilon \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} \left(\frac{\partial k}{\partial x_j} \right) + \beta_{N+1,\varepsilon}(2k + (x - x_\varepsilon) \cdot \nabla k) \right) \quad (2.27)$$

on $B(x_\varepsilon, \rho)$. We will derive a contradiction from (2.27). For this purpose, multiplying u_ε to (2.22) and integrating, we have

$$\begin{aligned} \int_\Omega u_\varepsilon^2 dx &= \sum_{j=1}^{N} \beta_{j,\varepsilon} \int_\Omega \phi \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) u_\varepsilon dx + \beta_{N+1,\varepsilon} \int_\Omega \phi w_\varepsilon u_\varepsilon dx \\ &=: \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} I_j \right) + \beta_{N+1,\varepsilon} II. \end{aligned} \quad (2.28)$$

By (2.12) and $\phi \equiv 1$ near x_0, we have

$$\begin{aligned} I_j &= \int_\Omega \phi \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) u_\varepsilon dx = \frac{1}{2} \int_\Omega \phi \left(\frac{\partial u_\varepsilon^2}{\partial x_j} \right) dx = -\frac{1}{2} \int_\Omega u_\varepsilon^2 \left(\frac{\partial \phi}{\partial x_j} \right) dx \\ &= -\frac{1}{2} \frac{1}{\|u_\varepsilon\|^2} \int_{\Omega \setminus B(x_\varepsilon, \rho)} (\|u_\varepsilon\| u_\varepsilon)^2 \left(\frac{\partial \phi}{\partial x_j} \right) dx = O\left(\frac{1}{\|u_\varepsilon\|^2} \right), \quad (j = 1, \cdots, N). \end{aligned}$$

By the same reason,

$$\begin{aligned} &\int_\Omega \phi \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon \right) u_\varepsilon dx = \sum_{j=1}^{N} \int_\Omega \phi (x_j - (x_\varepsilon)_j) \frac{1}{2} \frac{\partial}{\partial x_j} u_\varepsilon^2 dx \\ &= -\frac{1}{2} \sum_{j=1}^{N} \int_\Omega \frac{\partial \phi}{\partial x_j} u_\varepsilon^2 dx - \frac{1}{2} \sum_{j=1}^{N} \int_\Omega \phi \frac{\partial (x_j - (x_\varepsilon)_j)}{\partial x_j} u_\varepsilon^2 dx \\ &= O\left(\frac{1}{\|u_\varepsilon\|^2} \right) - \frac{N}{2} \int_\Omega \phi u_\varepsilon^2 dx, \end{aligned}$$

thus

$$
\begin{aligned}
II &= \int_\Omega \phi \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{N-2}{2} u_\varepsilon \right) u_\varepsilon dx \\
&= O\left(\frac{1}{\|u_\varepsilon\|^2} \right) - \frac{N}{2} \int_\Omega \phi u_\varepsilon^2 dx + \left(\frac{N-2}{2} \right) \int_\Omega \phi u_\varepsilon^2 dx \\
&= O\left(\frac{1}{\|u_\varepsilon\|^2} \right) - \int_\Omega \phi u_\varepsilon^2 dx \\
&= O\left(\frac{1}{\|u_\varepsilon\|^2} \right) - \|u_\varepsilon\|^{-4/(N-2)} \int_{\Omega_\varepsilon} \phi_\varepsilon(y) \tilde{u}_\varepsilon^2 dy \\
&= O\left(\frac{1}{\|u_\varepsilon\|^2} \right) - \|u_\varepsilon\|^{-4/(N-2)} \left[\int_{\mathbb{R}^N} U^2 dy + o(1) \right],
\end{aligned}
$$

where $\phi_\varepsilon(y) = \phi(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon)$. Here we have used change of variables $x = \frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon$, $dx = \|u_\varepsilon\|^{-(\frac{p-1}{2})N} dy$, (2.15), (2.10) and the dominated convergence theorem. LHS of (2.28) is also $\|u_\varepsilon\|^{-4/(N-2)} \left[\int_{\mathbb{R}^N} U^2 dy + o(1) \right]$. Returning to (2.28), we have

$$
(1 + \beta_{N+1,\varepsilon}) \|u_\varepsilon\|^{-4/(N-2)} \left[\int_{\mathbb{R}^N} U^2 dy + o(1) \right] = \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \right) \times O\left(\frac{1}{\|u_\varepsilon\|^2} \right),
$$

which implies, with noting $\beta_{N+1,\varepsilon} = \frac{p-1}{2} = O(1)$, that

$$
\sum_{j=1}^{N+1} \beta_{j,\varepsilon} = O\left(\|u_\varepsilon\|^{2(N-4)/(N-2)} \right).
$$

By Lemma 2.3 (2.13), we know that $\varepsilon \|u_\varepsilon\|^{2(N-4)/(N-2)} = O(1)$. Going back to (2.27), inserting $x = x_\varepsilon$, and noting $\frac{\partial k}{\partial x_j}(x_\varepsilon)$ and $k(x_\varepsilon)$ are uniformly bounded, we see

$$
c_0 (1 - p) \|u_\varepsilon\|^{p-1} = \varepsilon O\left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \right) = O\left(\varepsilon \|u_\varepsilon\|^{2(N-4)/(N-2)} \right).
$$

However, this is a contradiction since the LHS tends to $-\infty$ and the RHS is $O(1)$ as $\varepsilon \to 0$. Therefore we conclude that $\alpha_{0,\varepsilon} = 0$.

Next, we claim that $\alpha_{N+1,\varepsilon} = 0$. Indeed, putting $x = x_\varepsilon$ into $\sum_{j=1}^{N} \alpha_{j,\varepsilon}\psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon}\psi_{N+1,\varepsilon} \equiv 0$ and noting $\phi(x_\varepsilon) = 1$ and $\nabla u_\varepsilon(x_\varepsilon) = 0$ as before, we see $\alpha_{N+1,\varepsilon}(\frac{2}{p-1})u_\varepsilon(x_\varepsilon) = 0$. Thus we obtain $\alpha_{N+1,\varepsilon} = 0$.

Now, we obtain $\sum_{j=1}^{N} \alpha_{j,\varepsilon}\psi_{j,\varepsilon} \equiv 0$ on Ω. By scaling, this leads to

$$\sum_{j=1}^{N} \alpha_{j,\varepsilon}\phi_\varepsilon(y)\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}(y) \equiv 0$$

for $y \in \Omega_\varepsilon$. Using (2.10), we get that

$$\sum_{j=1}^{N} \alpha_j \frac{\partial U}{\partial y_j} \equiv 0 \quad \text{on } \mathbb{R}^N,$$

where $\alpha_j = \lim_{\varepsilon\to 0} \alpha_{j,\varepsilon}$. Since $\frac{\partial U}{\partial y_j}$ are linearly independent, we have that $\alpha_j = 0$ for all $j = 1, 2, \cdots, N$. But this is impossible since $\sum_{j=1}^{N} \alpha_j^2 = \lim_{\varepsilon\to 0}(\sum_{j=1}^{N} \alpha_{j,\varepsilon}^2) = 1$. Thus we have proved Lemma 2.8. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. By the variational characterization of $\lambda_{1,\varepsilon}$, we have

$$\lambda_{1,\varepsilon} = \inf_{v\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v^2 dx}.$$

Inserting $v = u_\varepsilon$, we see

$$\lambda_{1,\varepsilon} \le \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon^2 dx} = \frac{\int_\Omega \left(c_0 u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon^2 dx}.$$

By scaling, the right hand side can be estimated as

$$\begin{aligned}
\lambda_{1,\varepsilon} &\le \frac{c_0 \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p+1} dy + \varepsilon \|u_\varepsilon\|^{-4/(N-2)} \int_{\Omega_\varepsilon} k_\varepsilon(y)\tilde{u}_\varepsilon^2 dy}{c_0 p \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p+1} dy + \varepsilon \|u_\varepsilon\|^{-4/(N-2)} \int_{\Omega_\varepsilon} k_\varepsilon(y)\tilde{u}_\varepsilon^2 dy} \\
&= \frac{c_0 \int_{\mathbb{R}^N} U^{p+1} dy + o(1)}{c_0 p \int_{\mathbb{R}^N} U^{p+1} dy + o(1)}
\end{aligned}$$

as $\varepsilon \to 0$, which implies $\limsup_{\varepsilon\to 0}\lambda_{1,\varepsilon} \le 1/p$. Hence by choosing a subsequence, we may assume that $\lambda_{1,\varepsilon} \to \lambda \in [0, 1/p]$. Now, $\tilde{v}_{1,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\tilde{v}_{1,\varepsilon} = \lambda_{1,\varepsilon}\left(c_0 p\tilde{u}_\varepsilon^{p-1} + \frac{\varepsilon k_\varepsilon(y)}{\|u_\varepsilon\|^{p-1}}\right)\tilde{v}_{1,\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{v}_{1,\varepsilon} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

As in the proof of Lemma 2.6, we see that $\tilde{v}_{1,\varepsilon}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $\tilde{v}_{1,\varepsilon} \to V_1$ for some $V_1 \in D^{1,2}(\mathbb{R}^N)$. Letting $\varepsilon \to 0$, we see V_1 satisfies

$$\begin{cases} -\Delta V_1 = \lambda\left(c_0 p U^{p-1}\right) V_1 \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla V_1|^2 dy < \infty, \quad \|V_1\|_\infty \le 1. \end{cases}$$

We confirm that $V_1 \neq 0$ by (2.16), just as in the proof of Lemma 2.6. Since there exists no eigenvalue λ less than $1/p$ by Theorem 2.4, we must have $\lambda = 1/p$ and $V_1 = U$.

Now, let us prove that $\lambda_{1,\varepsilon}$ is simple for small ε. Indeed, assume there exist two eigenfunctions $v_{1,\varepsilon}$ and $w_{1,\varepsilon}$ corresponding to $\lambda_{1,\varepsilon}$. Define $\tilde{v}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}$ as in (1.5). By the orthogonal property (1.4), we have

$$\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v_{1,\varepsilon} w_{1,\varepsilon} dx = 0,$$

which leads to

$$\int_{\Omega_\varepsilon} c_0 p \tilde{u}_\varepsilon^{p-1}\tilde{v}_{1,\varepsilon}\tilde{w}_{1,\varepsilon} dy + \varepsilon\|u_\varepsilon\|^{2-(p-1)N/2}\int_{\Omega_\varepsilon} k_\varepsilon(y)\tilde{v}_{1,\varepsilon}\tilde{w}_{1,\varepsilon} dy = 0.$$

Since $\tilde{v}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon} \to U$ as above, the dominated convergence theorem implies

$$\int_{\mathbb{R}^N} c_0 p U^{p+1} dy = 0,$$

which is a contradiction.

Finally, we see the function $\|u_\varepsilon\|^2 v_{1,\varepsilon}$ satisfies $-\Delta(\|u_\varepsilon\|^2 v_{1,\varepsilon}) = f_\varepsilon(x),\ x \in \Omega,\ \|u_\varepsilon\|^2 v_{1,\varepsilon} = 0$ on $\partial\Omega$, where $f_\varepsilon(x) = \|u_\varepsilon\|^2\lambda_{1,\varepsilon}\left(c_0 p u_\varepsilon^{p-1}(x) + \varepsilon\right) v_{1,\varepsilon}(x)$. Since $\lambda_{1,\varepsilon} \to 1/p$ and $\tilde{v}_{1,\varepsilon} \to U$ in $C^1_{loc}(\mathbb{R}^N)$ as $\varepsilon \to 0$, the same argument as in the proof of Lemma 2.7 (see [4]) implies that $\int_\Omega f_\varepsilon(x)dx \to (N-2)\sigma_N$ and $f_\varepsilon(x) \to 0$ for any $x \in \overline{\Omega}\setminus\{x_0\}$ as $\varepsilon \to 0$. Thus we obtain $\|u_\varepsilon\|^2 v_{1,\varepsilon} \to (N-2)\sigma_N G(\cdot, x_0)$ in the sense of distributions. Standard elliptic estimates assure that this convergence is valid in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$. This finish the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 along the line of [5].

Proposition 4.1 *For $i = 2, \cdots, N+1$, we have*

$$\lambda_{i,\varepsilon} \leq 1 + \frac{O(\varepsilon)}{\|u_\varepsilon\|^{\frac{6}{N-2}}} \tag{4.1}$$

as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1. \tag{4.2}$$

Proof. By the variational characterization, $\lambda_{i,\varepsilon}$ can be expressed as

$$\lambda_{i,\varepsilon} = \inf_{W \subset H_0^1(\Omega), \dim(W)=i} \max_{v \in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v^2 dx}.$$

We take

$$W = W_i = \operatorname{span}\{u_\varepsilon, \psi_{1,\varepsilon}, \cdots, \psi_{i-1,\varepsilon}\},$$

where $\psi_{j,\varepsilon}$ are defined in (2.20). For $a_0, a_1, \cdots, a_{i-1} \in \mathbb{R}$, we put

$$v = a_0 u_\varepsilon + \sum_{j=1}^{i-1} a_j \psi_{j,\varepsilon} = a_0 u_\varepsilon + \phi z_\varepsilon \in W_i,$$

where $z_\varepsilon = \sum_{j=1}^{i-1} a_j (\frac{\partial u_\varepsilon}{\partial x_j})$. Since

$$-\Delta \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) + \varepsilon u_\varepsilon \left(\frac{\partial k}{\partial x_j}\right),$$

we see z_ε satisfies

$$-\Delta z_\varepsilon = \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) z_\varepsilon + \varepsilon u_\varepsilon \left(\sum_{j=1}^{i-1} a_j \frac{\partial k}{\partial x_j}\right) \quad \text{in } \Omega,$$

and

$$\int_\Omega \nabla z_\varepsilon \cdot \nabla(\phi^2 z_\varepsilon) dx = \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) \phi^2 z_\varepsilon^2 dx$$
$$+ \varepsilon \int_\Omega \sum_{j=1}^{i-1} a_j \left(\frac{\partial k}{\partial x_j}\right) u_\varepsilon \phi^2 z_\varepsilon dx.$$

Thus,

$$
\begin{aligned}
&\int_\Omega |\nabla v|^2 dx = \int_\Omega |\nabla(a_0 u_\varepsilon + \phi z_\varepsilon)|^2 dx \\
&= a_0^2 \int_\Omega |\nabla u_\varepsilon|^2 dx + 2a_0 \int_\Omega \nabla u_\varepsilon \cdot \nabla(\phi z_\varepsilon) dx + \int_\Omega |\nabla(\phi z_\varepsilon)|^2 dx, \\
&\int_\Omega |\nabla(\phi z_\varepsilon)|^2 dx = \int_\Omega |\nabla \phi|^2 z_\varepsilon^2 dx + \int_\Omega \nabla z_\varepsilon \cdot \nabla(\phi^2 z_\varepsilon) dx \\
&= \int_\Omega |\nabla \phi|^2 z_\varepsilon^2 dx + \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) \phi^2 z_\varepsilon^2 dx + \varepsilon \int_\Omega \sum_{j=1}^{i-1} a_j \left(\frac{\partial k}{\partial x_j}\right) u_\varepsilon \phi^2 z_\varepsilon dx, \\
&\int_\Omega \nabla u_\varepsilon \cdot \nabla(\phi z_\varepsilon) dx = \int_\Omega (-\Delta u_\varepsilon) \phi z_\varepsilon dx = \int_\Omega \left(c_0 u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon \phi z_\varepsilon dx.
\end{aligned}
$$

Using these, we have

$$
\max_{v \in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v^2 dx} = \max_{a_0, a_1, \cdots, a_{i-1}} \left\{ 1 + \frac{N_\varepsilon}{D_\varepsilon} \right\}
$$

where $N_\varepsilon = N_\varepsilon^1 + N_\varepsilon^2 + N_\varepsilon^3 + N_\varepsilon^4$,

$$
\begin{aligned}
N_\varepsilon^1 &= a_0^2 c_0 (1-p) \int_\Omega u_\varepsilon^{p+1} dx, \\
N_\varepsilon^2 &= 2a_0 c_0 (1-p) \sum_{j=1}^{i-1} a_j \int_\Omega u_\varepsilon^p \phi \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx, \\
N_\varepsilon^3 &= \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx, \\
N_\varepsilon^4 &= \varepsilon \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega \phi^2 u_\varepsilon \left(\frac{\partial k}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx,
\end{aligned}
$$

and $D_\varepsilon = D_\varepsilon^1 + D_\varepsilon^2 + D_\varepsilon^3$,

$$D_\varepsilon^1 = a_0^2 \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon^2 dx,$$

$$D_\varepsilon^2 = 2a_0 \sum_{j=1}^{i-1} a_j \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon \phi \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx,$$

$$D_\varepsilon^3 = \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx.$$

N_ε^2 and N_ε^3 can be estimated as the same way in [5]:

$$\int_\Omega u_\varepsilon^p \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx = \frac{1}{p+1} \int_\Omega \phi \left(\frac{\partial u_\varepsilon^{p+1}}{\partial x_j}\right) dx = -\frac{1}{p+1} \int_\Omega u_\varepsilon^{p+1} \left(\frac{\partial \phi}{\partial x_j}\right) dx$$
$$= -\frac{1}{p+1} \frac{1}{\|u_\varepsilon\|^{p+1}} \int_{\Omega \setminus B(x_\varepsilon, \rho)} (\|u_\varepsilon\| u_\varepsilon)^{p+1} \left(\frac{\partial \phi}{\partial x_j}\right) dx = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right), \quad (4.3)$$

where we have used $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$ and (2.12). Also

$$\int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx$$
$$= \frac{1}{\|u_\varepsilon\|^2} \int_{\Omega \setminus B(x_\varepsilon, \rho)} |\nabla \phi|^2 \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_l}\right) dx = O\left(\frac{1}{\|u_\varepsilon\|^2}\right). \quad (4.4)$$

Thus we have

$$N_\varepsilon^2 = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right), \quad N_\varepsilon^3 = O\left(\frac{1}{\|u_\varepsilon\|^2}\right). \quad (4.5)$$

As for N_ε^4, by change of variables

$$x = \frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial x_j}(x) = \|u_\varepsilon\|^{\frac{p-1}{2}+1} \frac{\partial \tilde{u}_\varepsilon}{\partial y_j}(y),$$

we see

$$
\begin{aligned}
&\int_\Omega u_\varepsilon \phi^2 \left(\frac{\partial k}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx \\
&= \|u_\varepsilon\|^{1+(p+1)/2-(p-1)N/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon \phi_\varepsilon^2 \left(\frac{\partial k}{\partial x_j}\right)(x)\Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}}+x_\varepsilon} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial x_l}\right) dy \\
&= \|u_\varepsilon\|^{-2/(N-2)} \left[\left(\frac{\partial k}{\partial x_j}\right)(x_0) \int_{\mathbb{R}^N} U \left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right] \\
&= O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right) \qquad (4.6)
\end{aligned}
$$

where $\phi_\varepsilon(y)$ and $k_\varepsilon(y)$ are as before. Thus, we obtain

$$
N_\varepsilon^4 = \frac{O(\varepsilon)}{\|u_\varepsilon\|^{2/(N-2)}}. \qquad (4.7)
$$

As for D_ε^2, we write

$$
\begin{aligned}
&\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon \phi \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx \\
&= \int_\Omega \frac{c_0 p}{p+1} \phi \left(\frac{\partial u_\varepsilon^{p+1}}{\partial x_j}\right) dx + \varepsilon \int_\Omega k(x) u_\varepsilon \phi \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx.
\end{aligned}
$$

First term is $O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right)$ as above. On the other hand,

$$
\begin{aligned}
\int_\Omega k(x) u_\varepsilon \phi \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx &= \|u_\varepsilon\|^{1+(p+1)/2-(p-1)N/2} \int_{\Omega_\varepsilon} k_\varepsilon(y) \tilde{u}_\varepsilon \phi_\varepsilon \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right) dy \\
&= \|u_\varepsilon\|^{-2/(N-2)} \left[k(x_0)\phi(x_0) \int_{\mathbb{R}^N} U \left(\frac{\partial U}{\partial y_j}\right) dy + o(1)\right] \\
&= O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \qquad (4.8)
\end{aligned}
$$

Thus, we have

$$
D_\varepsilon^2 = O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}\right). \qquad (4.9)
$$

As for D_ε^3, by change of variables we see

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p-1}\phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx \\
&= \|u_\varepsilon\|^{p-1+2(\frac{p-1}{2}+1)-\frac{p-1}{2}N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1}\phi_\varepsilon^2(y) \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) dy \\
&= \|u_\varepsilon\|^{p-1} \left(\int_{\mathbb{R}^N} U^{p-1}\left(\frac{\partial U}{\partial y_j}\right)\left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right) \\
&= \|u_\varepsilon\|^{4/(N-2)} \left(\frac{\delta_{jl}}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right), \qquad (4.10)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega k(x)\phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx \\
&= \|u_\varepsilon\|^{2(\frac{p+1}{2})-(\frac{p-1}{2})N} \int_{\Omega_\varepsilon} k_\varepsilon(y)\phi_\varepsilon^2(y) \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) dy \\
&= k(x_0)\phi(x_0)\left(\frac{\delta_{jl}}{N}\int_{\mathbb{R}^N} |\nabla U|^2 dy + o(1)\right). \qquad (4.11)
\end{aligned}$$

Here, we have used the fact $\nabla \tilde{u}_\varepsilon \to \nabla U$ in $L^2(\mathbb{R}^N)$ by (2.10). Thus,

$$\begin{aligned}
D_\varepsilon^3 &= c_0 p \sum_{j=1}^{i-1} a_j^2 \|u_\varepsilon\|^{p-1}\left(\frac{1}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right) \\
&\quad + \varepsilon k(x_0)\sum_{j=1}^{i-1} a_j^2 \left(\frac{1}{N}\int_{\mathbb{R}^N} |\nabla U|^2 dy + o(1)\right) \\
&= \left(\sum_{j=1}^{i-1} a_j^2\right) O(\|u_\varepsilon\|^{4/(N-2)}). \qquad (4.12)
\end{aligned}$$

Let $(a_{0,\varepsilon}, a_{1,\varepsilon}, \cdots, a_{i-1,\varepsilon}) \in \mathbb{R}^i$ be a maximizer of $\max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i}\left\{1+\frac{N_\varepsilon}{D_\varepsilon}\right\}$. We claim that $\sum_{j=1}^{i-1} a_{j,\varepsilon}^2 = 0$ cannot hold true for any $\varepsilon > 0$ sufficiently small. Indeed, assume the contrary. Then we have some $\{\varepsilon\} \downarrow 0$ such that along the sequence,

$$\max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i}\left\{1+\frac{N_\varepsilon}{D_\varepsilon}\right\}=1+\frac{a_{0,\varepsilon}^2 c_0(1-p)\int_\Omega u_\varepsilon^{p+1}dx}{a_{0,\varepsilon}^2\int_\Omega\left(c_0pu_\varepsilon^{p-1}+\varepsilon k(x)\right)u_\varepsilon^2dx}$$
$$=1+\frac{c_0(1-p)\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p+1}dy}{c_0p\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p+1}dy+\varepsilon\|u_\varepsilon\|^{-4/(N-2)}\int_{\Omega_\varepsilon}k_\varepsilon(y)\tilde{u}_\varepsilon^2dy}$$
$$=1+\frac{c_0(1-p)\int_{\mathbb{R}^N}U^{p+1}dy+o(1)}{c_0p\int_{\mathbb{R}^N}U^{p+1}dy+o(1)}\to 1/p.$$

On the other hand, by testing $(a_0,a_1,\cdots,a_{i-1})=(0,1,\cdots,1)$, we have

$$\max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i}\left\{1+\frac{N_\varepsilon}{D_\varepsilon}\right\}\geq 1+\frac{N_\varepsilon^2+N_\varepsilon^3}{D_\varepsilon^3}\to 1,\quad \text{as } \varepsilon\to 0$$

by (4.5) and (4.12). This leads to a contradiction that $1/p\geq 1$. Thus, we have proved the claim and we may assume that $\sum_{j=1}^{i-1}a_{j,\varepsilon}^2=1$.

Then from the estimates (4.5), (4.7), (4.9) and (4.12), we have

$$\max_{v\in W_i}\frac{\int_\Omega|\nabla v|^2dx}{\int_\Omega\left(c_0pu_\varepsilon^{p-1}+\varepsilon k(x)\right)v^2dx}$$
$$=\left\{1+\frac{N_\varepsilon}{D_\varepsilon}\right\}\Big|_{(a_0,a_1,\cdots,a_{i-1})=(a_{0,\varepsilon},a_{1,\varepsilon},\cdots,a_{i-1,\varepsilon})}$$
$$=1+\frac{a_{0,\varepsilon}^2c_0(1-p)\int_\Omega u_\varepsilon^{p+1}dx+O(\frac{1}{\|u_\varepsilon\|^{p+1}})+O(\frac{1}{\|u_\varepsilon\|^2})+O(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}})}{a_{0,\varepsilon}^2\int_\Omega(c_0pu_\varepsilon^{p-1}+\varepsilon k(x))u_\varepsilon^2dx+O(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}})+O(\|u_\varepsilon\|^{4/(N-2)})}$$
$$\leq 1+\frac{O(\frac{1}{\|u_\varepsilon\|^{p+1}})+O(\frac{1}{\|u_\varepsilon\|^2})+O(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}})}{O(1)+O(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}})+O(\|u_\varepsilon\|^{4/(N-2)})}$$
$$\leq 1+O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{6/(N-2)}}\right).$$

The last inequality comes form (2.13). This proves (4.1).

To prove (4.2), first by using (4.1), we notice that $\lambda_{i,\varepsilon}\to\Lambda\in[0,1]$ for some Λ as $\varepsilon\to 0$. As in the derivation of (2.18), we have $\tilde{v}_{i,\varepsilon}\to V$ in $C^1_{loc}(\mathbb{R}^N)$ for some $V\not\equiv 0$ and V is a solution of

$$\begin{cases}-\Delta V=\Lambda c_0pU^{p-1}V \text{ in } \mathbb{R}^N,\\ V\in D^{1,2}(\mathbb{R}^N).\end{cases}$$

By Theorem 2.4, we conclude that, if $\Lambda < 1$, then we must have that $\Lambda = 1/p$ and $V = U$. However, this leads to a contradiction because $v_{i,\varepsilon}$ is orthogonal to $v_{1,\varepsilon} = u_\varepsilon/\|u_\varepsilon\|$ for $i \geq 2$. Indeed, by (1.4), we have

$$\begin{aligned}
&\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v_{1,\varepsilon} v_{i,\varepsilon} dy = 0 \\
&\Rightarrow 0 = \int_{\Omega_\varepsilon} \left(c_0 p \tilde{u}_\varepsilon^{p-1} + \varepsilon k_\varepsilon(y)/\|u_\varepsilon\|^{p-1}\right) \tilde{u}_\varepsilon \tilde{v}_{i,\varepsilon} dy \to c_0 p \int_{\mathbb{R}^N} U^p V dy,
\end{aligned}$$

thus this leads to $\int_{\mathbb{R}^N} U^{p+1} dy = 0$ if $V = U$, which is absurd. Therefore we conclude that $\Lambda = 1$ and the proof of Proposition 4.1 is finished. □

Lemma 4.2 *Assume $N \geq 6$. Let $i \in \mathbb{N}$ be such that $\lim_{\varepsilon\to 0} \lambda_{i,\varepsilon} = 1$. If b_i in (2.17) of Lemma 2.6 is not 0, then we have*

$$\lambda_{i,\varepsilon} - 1 = \frac{1}{\|u_\varepsilon\|^2}(\Gamma + o(1)) \quad \text{as } \varepsilon \to 0 \tag{4.13}$$

for some $\Gamma > 0$ independent of ε.

Proof. Assume $b_i \neq 0$. We use the integral identity (2.1) in Lemma 2.1 with $y = x_\varepsilon$. The LHS of (2.1) can be written as

$$\begin{aligned}
&\frac{1}{\|u_\varepsilon\|^3} \int_{\partial\Omega} (x - x_\varepsilon) \cdot \nu \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu}\right) \left(\frac{\partial \|u_\varepsilon\|^2 v_{i,\varepsilon}}{\partial \nu}\right) ds_x \\
&= \frac{1}{\|u_\varepsilon\|^3} \left[-(N-2)^2 \sigma_N^2 b_i \int_{\partial\Omega} (x - x_0) \cdot \nu \left(\frac{\partial G}{\partial \nu}(x, x_0)\right)^2 ds_x + o(1)\right] \\
&= \frac{1}{\|u_\varepsilon\|^3} \left[-(N-2)^3 \sigma_N^2 R(x_0) b_i + o(1)\right].
\end{aligned} \tag{4.14}$$

Here we have used (2.12), (2.19) and Lemma 2.2 (2.4).

On the other hand, the RHS of (2.1) $= I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned}
I_1 &= (1 - \lambda_{i,\varepsilon}) c_0 p \int_\Omega u_\varepsilon^{p-1} w_\varepsilon v_{i,\varepsilon} dx, \\
I_2 &= (1 - \lambda_{i,\varepsilon}) \varepsilon \int_\Omega k(x) w_\varepsilon v_{i,\varepsilon} dx, \\
I_3 &= 2\varepsilon \int_\Omega k(x) u_\varepsilon v_{i,\varepsilon} dx, \\
I_4 &= \varepsilon \int_\Omega \left((x - x_\varepsilon) \cdot \nabla k(x)\right) u_\varepsilon v_{i,\varepsilon} dx,
\end{aligned}$$

and, as before, $w_\varepsilon(x) = (x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p-1} u_\varepsilon$. Denote

$$\tilde{w}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} w_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon \right) = y \cdot \nabla_y \tilde{u}_\varepsilon(y) + \frac{2}{p-1} \tilde{u}_\varepsilon(y) \tag{4.15}$$

for $y \in \Omega_\varepsilon$. By (2.10), we see

$$\tilde{w}_\varepsilon \to y \cdot \nabla U + \frac{N-2}{2} U = \left(\frac{N-2}{2} \right) \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}}, \quad \text{in } C^1_{loc}(\mathbb{R}^N).$$

Thus,

$$\begin{aligned}
I_1 &= (1 - \lambda_{i,\varepsilon}) c_0 p \|u_\varepsilon\|^{p-(p-1)N/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1} \tilde{w}_\varepsilon \tilde{v}_{i,\varepsilon}(y) dy \\
&= (1 - \lambda_{i,\varepsilon}) c_0 p \|u_\varepsilon\|^{-1} \times \\
&\left[\int_{\mathbb{R}^N} U^{p-1} \left(y \cdot \nabla U + \frac{2}{p-1} U \right) \left(\sum_{j=1}^N a_{i,j} \frac{y_j}{(1 + |y|^2)^{N/2}} + b_i \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}} \right) dy + o(1) \right] \\
&= (1 - \lambda_{i,\varepsilon}) \|u_\varepsilon\|^{-1} b_i c_0 p \left(\frac{N-2}{2} \right) \left[\int_{\mathbb{R}^N} U^{p-1} \frac{(1 - |y|^2)^2}{(1 + |y|^2)^N} dy + o(1) \right].
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= \varepsilon (1 - \lambda_{i,\varepsilon}) \|u_\varepsilon\|^{1-(p-1)N/2} \int_\Omega k_\varepsilon(y) \tilde{w}_\varepsilon \tilde{v}_{i,\varepsilon} dy \\
&= (1 - \lambda_{i,\varepsilon}) \varepsilon k(x_0) \left(\frac{N-2}{2} \right) \|u_\varepsilon\|^{-(N+2)/(N-2)} b_i \left[\int_{\mathbb{R}^N} \frac{(1 - |y|^2)^2}{(1 + |y|^2)^N} dy + o(1) \right].
\end{aligned}$$

Finally,

$$\begin{aligned}
I_3 &= 2\varepsilon \|u_\varepsilon\|^{1-(p-1)N/2} \int_{\Omega_\varepsilon} k_\varepsilon(y) \tilde{u}_\varepsilon \tilde{v}_{i,\varepsilon} dy \\
&= 2\varepsilon k(x_0) \|u_\varepsilon\|^{-(N+2)/(N-2)} b_i \left[\int_{\mathbb{R}^N} U(y) \frac{1 - |y|^2}{(1 + |y|^2)^{N/2}} dy + o(1) \right],
\end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \varepsilon \|u_\varepsilon\|^{1-(p-1)N/2-(p-1)/2} \int_{\Omega_\varepsilon} \left(y \cdot \nabla k(x) \Big|_{x = \frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon} \right) \tilde{u}_\varepsilon \tilde{v}_{i,\varepsilon} dy \\
&= \varepsilon \|u_\varepsilon\|^{-(N+4)/(N-2)} \times \\
&\left[\int_{\mathbb{R}^N} y \cdot \nabla k(x_0) U(y) \left(\sum_{j=1}^N a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \right) dy + o(1) \right] \\
&= O(\varepsilon \|u_\varepsilon\|^{-(N+4)/(N-2)}) = o(\varepsilon \|u_\varepsilon\|^{-(N+2)/(N-2)}).
\end{aligned}
$$

Note that

$$
\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{(N-2)/2}} \frac{1-|y|^2}{(1+|y|^2)^{N/2}} O(|y|) dy = \int_{\mathbb{R}^N} (1+|y|)^{5-2N} dy < \infty
$$

if $N > 5$.

Dividing both sides of

$$
(4.14) = I_1 + I_2 + I_3 + I_4
$$

by $b_i \neq 0$, we have

$$
\frac{D + o(1)}{\|u_\varepsilon\|^3} = (1 - \lambda_{i,\varepsilon}) \left(\frac{A + o(1)}{\|u_\varepsilon\|} + \frac{\varepsilon (B + o(1))}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \right) + \frac{\varepsilon (C + o(1))}{\|u_\varepsilon\|^{(N+2)/(N-2)}}
$$

where

$$
\begin{aligned}
A &= c_0 p \left(\frac{N-2}{2} \right) \int_{\mathbb{R}^N} U^{p-1} \frac{(1-|y|^2)^2}{(1+|y|^2)^N} dy = \frac{(N-2)(N+2)\sigma_N \Gamma(N/2+1)^2}{\Gamma(N+2)}, \\
B &= k(x_0) \left(\frac{N-2}{2} \right) \int_{\mathbb{R}^N} \frac{(1-|y|^2)^2}{(1+|y|^2)^N} dy, \\
C &= 2k(x_0) \int_{\mathbb{R}^N} U(y) \frac{1-|y|^2}{(1+|y|^2)^{N/2}} dy = -k(x_0) \sigma_N \frac{2\Gamma(N/2)\Gamma(N/2-2)}{\Gamma(N-1)}, \\
D &= -(N-2)^3 \sigma_N^2 R(x_0).
\end{aligned}
$$

Recall that by Theorem 2.3 (2.13),

$$
\varepsilon = \frac{E + o(1)}{\|u_\varepsilon\|^{2(N-4)/(N-2)}} \tag{4.16}
$$

where $E = \frac{(N-2)^3}{2a_N}\sigma_N \frac{R(x_0)}{k(x_0)}$ for $N \geq 5$. Inserting this, we see

$$(\lambda_{i,\varepsilon} - 1)\left(\frac{A + o(1)}{\|u_\varepsilon\|} + \frac{EB + o(1)}{\|u_\varepsilon\|^3}\right) = \left(\frac{EC + o(1)}{\|u_\varepsilon\|^3} - \frac{D + o(1)}{\|u_\varepsilon\|^3}\right),$$

that is,

$$\lambda_{i,\varepsilon} - 1 = \frac{1}{\|u_\varepsilon\|^2}\left(\frac{CE - D}{A} + o(1)\right).$$

Now, we check that

$$\Gamma := \frac{CE - D}{A} = \frac{(N-2)(N-4)\Gamma(N+2)}{(N+2)\Gamma(N/2+1)^2}\sigma_N R(x_0) > 0, \tag{4.17}$$

since $a_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N-2}} dr = \frac{\Gamma(N/2)\Gamma(N/2-2)}{2\Gamma(N-2)}$. Thus we have proved Lemma. □

Now, by Proposition 4.1, Lemma 4.2 and (4.16), we have

$$\frac{\Gamma + o(1)}{\|u_\varepsilon\|^2} = \lambda_{i,\varepsilon} - 1 \leq \frac{O(\varepsilon)}{\|u_\varepsilon\|^{\frac{6}{N-2}}} = \frac{O(1)}{\|u_\varepsilon\|^{\frac{2N-2}{N-2}}}.$$

From this, we see

$$0 < \Gamma + o(1) \leq \frac{O(1)}{\|u_\varepsilon\|^{\frac{2}{N-2}}} \to 0,$$

a contradiction. Thus we have $b_i = 0$ in (2.17) and (1.6) in Theorem 1.2 is proved.

To prove other claims in Theorem 1.2, we need

Lemma 4.3 *Assume $N \geq 6$. For $i = 2, \cdots, N+1$, let $b_i = 0$ and $\vec{a}_i = (a_{i,1}, \cdots, a_{i,N}) \neq 0$ in (2.17). Then we have*

$$\|u_\varepsilon\|^{2+2/(N-2)} v_{i,\varepsilon} \to \sigma_N \sum_{j=1}^N a_{i,j}\left(\frac{\partial G}{\partial z_j}(x,z)\right)\Big|_{z=x_0} \tag{4.18}$$

in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof. Argue as Lemma 3.3 in [8]. Note that the restriction $N \geq 6$ is needed for this lemma. □

Now, we prove (1.8). We return to (2.2):

$$\int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial \nu}\right) ds_x = (1-\lambda_{i,\varepsilon}) \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) v_{i,\varepsilon} dx$$
$$+ \varepsilon \int_\Omega \left(\frac{\partial k}{\partial x_j}\right) u_\varepsilon v_{i,\varepsilon} dx, \quad (j = 1, 2, \cdots, N).$$

By (2.12) and Lemma 4.3, we see

$$\begin{aligned} LHS &= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}} \int_{\partial\Omega} \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial \|u_\varepsilon\|^{2+2/(N-2)} v_{i,\varepsilon}}{\partial \nu}\right) ds_x \\ &= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}} \left[(N-2)\sigma_N^2 \sum_{l=1}^N a_{i,l} \int_{\partial\Omega} \left(\frac{\partial G}{\partial x_j}\right) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_l}\right)(x, x_0) ds_x + o(1)\right] \\ &= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}} \left[\frac{N-2}{2}\sigma_N^2 \sum_{l=1}^N a_{i,l} \frac{\partial^2 R}{\partial z_j \partial z_l}(z)\Big|_{z=x_0} + o(1)\right], \end{aligned}$$

where we have used (2.6). On the other hand, write $RHS = I + II + III$ where

$$\begin{aligned} I &= (1-\lambda_{i,\varepsilon}) c_0 p \int_\Omega u_\varepsilon^{p-1} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) v_{i,\varepsilon} dx, \\ II &= (1-\lambda_{i,\varepsilon}) \varepsilon \int_\Omega k(x) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) v_{i,\varepsilon} dx, \\ III &= \varepsilon \int_\Omega \left(\frac{\partial k}{\partial x_j}\right) u_\varepsilon v_{i,\varepsilon} dx. \end{aligned}$$

As before, we have

$$\begin{aligned} I &= (1-\lambda_{i,\varepsilon}) c_0 p \|u_\varepsilon\|^{-(N-4)/(N-2)} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right) \tilde{v}_{i,\varepsilon} dy \\ &= \frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{(N-4)/(N-2)}} c_0 p \left[\int_{\mathbb{R}^N} U^{p-1} \left(\frac{\partial U}{\partial y_j}\right) \sum_{l=1}^N a_{i,l} \frac{y_l}{(1+|y|^2)^{N/2}} dy + o(1)\right] \\ &= \frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{(N-4)/(N-2)}} \frac{c_0 p}{2-N} \left[\sum_{l=1}^N a_{i,l} \int_{\mathbb{R}^N} U^{p-1} \left(\frac{\partial U}{\partial y_j}\right)\left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right] \\ &= \frac{(\lambda_{i,\varepsilon}-1)}{\|u_\varepsilon\|^{(N-4)/(N-2)}} \frac{c_0 p}{N(N-2)} a_{i,j} \left[\int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy + o(1)\right], \end{aligned}$$

$$
\begin{aligned}
II &= \frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{N/(N-2)}}\varepsilon \int_{\Omega_\varepsilon} k_\varepsilon(y)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\tilde{v}_{i,\varepsilon}dy \\
&= \frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{N/(N-2)}}\varepsilon k(x_0)\left[\int_{\mathbb{R}^N}\left(\frac{\partial U}{\partial y_j}\right)\sum_{l=1}^{N} a_{i,l}\frac{y_l}{(1+|y|^2)^{N/2}}dy + o(1)\right] \\
&= \frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{N/(N-2)}}\varepsilon k(x_0)\left[\sum_{l=1}^{N} a_{i,l}\frac{1}{2-N}\int_{\mathbb{R}^N}\left(\frac{\partial U}{\partial y_j}\right)\left(\frac{\partial U}{\partial y_l}\right)dy + o(1)\right] \\
&= \frac{(\lambda_{i,\varepsilon}-1)}{\|u_\varepsilon\|^{N/(N-2)}}\varepsilon \frac{k(x_0)}{N(N-2)}a_{i,j}\left[\int_{\mathbb{R}^N}|\nabla U|^2 dy + o(1)\right].
\end{aligned}
$$

As for III, we see by (2.10) and (2.17) that

$$
\begin{aligned}
\tilde{u}_\varepsilon \tilde{v}_{i,\varepsilon}(y) \to U(y)V(y) &= \sum_{l=1}^{N} a_{i,l}\frac{y_l}{(1+|y|^2)^{N-1}} \\
&= \sum_{l=1}^{N} a_{i,l}\frac{\partial}{\partial y_l}\left\{\frac{-1}{2(N-2)}U^2(y)\right\}
\end{aligned}
$$

in $C^1_{loc}(\mathbb{R}^N)$. Now, we exploit the solution $\psi_{i,\varepsilon}$ of the linear first order PDE

$$
\sum_{l=1}^{N} a_{i,l}\frac{\partial \psi}{\partial y_l} = \tilde{u}_\varepsilon(y)\tilde{v}_{i,\varepsilon}(y) \quad (y \in \mathbb{R}^N), \quad \psi|_{\Gamma_{\vec{a}_i}} = \frac{-1}{2(N-2)}U^2(y)
$$

where $\Gamma_{\vec{a}_i} = \{x \in \mathbb{R}^N | x \cdot \vec{a}_i = 0\}$. We can check that the unique solution satisfies $\psi_{i,\varepsilon}(y) = O(|y|^{5-2N})$ for $|y|$ large and

$$
\psi_{i,\varepsilon} \to \frac{-1}{2(N-2)}U^2
$$

uniformly on compact subsets of $\mathbb{R}^N$. Note that $\psi_{i,\varepsilon} \in L^1(\mathbb{R}^N)$ by our assumption $N > 5$. Thus,

$$
\begin{aligned}
III &= \varepsilon \int_\Omega u_\varepsilon v_{i,\varepsilon} \left(\frac{\partial k}{\partial x_j}\right) dx \\
&= \|u_\varepsilon\|^{1-(p-1)N/2} \varepsilon \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon(y) \tilde{v}_{i,\varepsilon}(y) \left(\frac{\partial k}{\partial x_j}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\Big) dy \\
&= \varepsilon \|u_\varepsilon\|^{1-2N/(N-2)} \int_{\Omega_\varepsilon} \left(\sum_{l=1}^N a_{i,l} \frac{\partial \psi_{i,\varepsilon}(y)}{\partial y_l}\right) \left(\frac{\partial k}{\partial x_j}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\Big) dy \\
&= -\varepsilon \|u_\varepsilon\|^{1-2N/(N-2)} \int_{\Omega_\varepsilon} \psi_{i,\varepsilon}(y) \sum_{l=1}^N a_{i,l} \frac{\partial}{\partial y_l} \left\{ \left(\frac{\partial k}{\partial x_j}\right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\Big) \right\} dy \\
&= -\varepsilon \|u_\varepsilon\|^{1-2N/(N-2)} \int_{\Omega_\varepsilon} \psi_{i,\varepsilon}(y) \frac{1}{\|u_\varepsilon\|^{2/(N-2)}} \sum_{l=1}^N a_{i,l} \frac{\partial}{\partial x_l} \left(\frac{\partial k}{\partial x_j}(x)\right) \Bigg|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}}+x_\varepsilon} dy \\
&= -\frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}} \frac{(N-2)^3 \sigma_N}{2a_N} \frac{R(x_0)}{k(x_0)} \times \\
&\qquad \sum_{l=1}^N a_{i,l} \frac{\partial^2 k}{\partial x_l \partial x_j}(x_0) \left[\frac{-1}{2(N-2)} \int_{\mathbb{R}^N} U^2(y) dy + o(1)\right] \\
&= \frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}} \left[\frac{(N-2)^2 \sigma_N^2}{4} \frac{R(x_0)}{k(x_0)} \sum_{l=1}^N a_{i,l} \frac{\partial^2 k}{\partial x_l \partial x_j}(x_0) + o(1)\right].
\end{aligned}
$$

Here again we have used Proposition 2.3 (4.16) and the dominated convergence theorem. Note that $\sigma_N a_N = \int_{\mathbb{R}^N} U^2 dy$.

Returning to $LHS = I + II + III$, multiplying $\|u_\varepsilon\|^{3+2/(N-2)}$ to both sides, we have

$$
\begin{aligned}
&\frac{N-2}{2} \sigma_N^2 \left[\sum_{l=1}^N a_{i,l} \frac{\partial^2 R}{\partial x_l \partial x_j}(x_0) - \frac{(N-2)}{2} \frac{R(x_0)}{k(x_0)} \sum_{l=1}^N a_{i,l} \frac{\partial^2 k}{\partial x_l \partial x_j}(x_0) + o(1)\right] \\
&= (\lambda_{i,\varepsilon} - 1) \|u_\varepsilon\|^{2N/(N-2)} \left[p a_{i,j} \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy + o(1)\right].
\end{aligned}
$$

Thus we obtain

$$
\|u_\varepsilon\|^{2N/(N-2)} (\lambda_{i,\varepsilon} - 1) \to M\eta_i, \quad (\varepsilon \to 0)
$$

where

$$M = \frac{(\frac{N-2}{2})^2 \sigma_N^2 R(x_0)}{p \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy}$$

and

$$\eta_i = \frac{\sum_{l=1}^N a_{i,l} \left[\frac{2}{N-2} \frac{1}{R(x_0)} \frac{\partial^2 R}{\partial x_l \partial x_j}(x_0) - \frac{1}{k(x_0)} \frac{\partial^2 k}{\partial x_l \partial x_j}(x_0) \right]}{a_{i,j}}.$$

By the definition of η_i, we see

$$\sum_{l=1}^N a_{i,l} A_{l,j}(x_0) = \eta_i a_{i,j}$$

where $A(x_0) = (A_{l,j}(x_0))_{1 \le l,j \le N}$ is defined as (1.9). This means η_i is an eigenvalue of the matrix $A(x_0)$ and $\vec{a}_i$ is a corresponding eigenvector. If $i \neq j$, we see that $\vec{a}_i$ and $\vec{a}_j$ is perpendicular to each other in $\mathbb{R}^N$. Indeed, for fixed ε, $v_{i,\varepsilon}$ and $v_{j,\varepsilon}$ is orthogonal in the sense of (1.4):

$$\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x) \right) v_{i,\varepsilon} v_{j,\varepsilon} dx = 0.$$

From this, we have

$$\|u_\varepsilon\|^{-2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1} \tilde{v}_{i,\varepsilon} \tilde{v}_{j,\varepsilon} dy + O(\|u_\varepsilon\|^{-4}) \int_{\Omega_\varepsilon} k_\varepsilon(y) \tilde{v}_{i,\varepsilon} \tilde{v}_{j,\varepsilon} dy = 0.$$

Multiplying $\|u_\varepsilon\|^2$ to the both sides and letting $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^N} U^{p-1} \left(\sum_{h=1}^N a_{i,h} \frac{y_h}{(1+|y|^2)^{N/2}} \right) \left(\sum_{l=1}^N a_{j,l} \frac{y_l}{(1+|y|^2)^{N/2}} \right) dy = 0,$$

where we have used (2.10) and (1.6). From this, we have

$$\begin{aligned} 0 &= \sum_{h,l=1}^N \int_{\mathbb{R}^N} U^{p-1} a_{i,h} a_{j,l} \frac{y_h y_l}{(1+|y|^2)^N} dy \\ &= \sum_{h=1}^N a_{i,h} a_{j,h} \left(\frac{1}{N} \int_{\mathbb{R}^N} U^{p-1} \frac{|y|^2}{(1+|y|^2)^N} dy \right), \end{aligned}$$

which implies $\vec{a}_i \cdot \vec{a}_j = 0$. Thus, all η_i is one of N eigenvalues of $A(x_0)$ and we have $\eta_i = \mu_{i-1}$ for $i = 2, \cdots, N+1$. This ends the proof of Theorem 1.2. □

5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. First, we prove

Lemma 5.1

$$\lambda_{N+2,\varepsilon} \to 1 \quad \textit{as } \varepsilon \to 0. \tag{5.1}$$

Proof. Since we know $\liminf_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \geq 1$ by Proposition 4.1, we have to check that $\limsup_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \leq 1$. For this purpose, we use a variational characterization of $\lambda_{N+2,\varepsilon}$ to obtain

$$\lambda_{N+2,\varepsilon} \leq \max_{v\in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) v^2 dx}, \tag{5.2}$$

where $W = \text{span}\{u_\varepsilon, \phi(\frac{\partial u_\varepsilon}{\partial x_1}), \cdots, \phi(\frac{\partial u_\varepsilon}{\partial x_N}), \phi w_\varepsilon\}$, ϕ is a cut-off function as in Lemma 2.8, and, as before, $w_\varepsilon(x) = (x - x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon$. For $a_0, a_1, \cdots, a_N, d \in \mathbb{R}$, we set

$$\hat{z}_\varepsilon(x) = \sum_{j=1}^N a_j \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) + d w_\varepsilon(x).$$

Direct calculation shows that $\hat{z}_\varepsilon$ satisfies the equation

$$-\Delta \hat{z}_\varepsilon = (c_0 p u_\varepsilon^{p-1} + \varepsilon k(x))\hat{z}_\varepsilon + 2\varepsilon d k(x) u_\varepsilon + \varepsilon u_\varepsilon h_\varepsilon(x),$$

where

$$h_\varepsilon(x) = \sum_{j=1}^N a_j \left(\frac{\partial k}{\partial x_j}\right) + d(x - x_\varepsilon)\cdot\nabla k(x).$$

Note that

$$\tilde{h}_\varepsilon(y) := h_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p-1}{2}}} + x_\varepsilon\right) \to \sum_{j=1}^N a_j \left(\frac{\partial k}{\partial x_j}\right)(x_0) \tag{5.3}$$

uniformly on compact sets of $\mathbb{R}^N$.

We test (5.2) by $v = a_0 u_\varepsilon + \phi\hat{z}_\varepsilon \in W$. As in the proof of Proposition 4.1, we have

$$\max_{v\in W} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega (c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)) v^2 dx} = \max_{a_0, a_1, \cdots, a_N, d} \left\{1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon}\right\},$$

where $\hat{N}_\varepsilon = \hat{N}_\varepsilon^1 + \hat{N}_\varepsilon^2 + \hat{N}_\varepsilon^3 + \hat{N}_\varepsilon^4 + \hat{N}_\varepsilon^5$,

$$\hat{N}_\varepsilon^1 = a_0^2 c_0 (1-p) \int_\Omega u_\varepsilon^{p+1} dx,$$

$$\begin{aligned}\hat{N}_\varepsilon^2 &= 2a_0 c_0 (1-p) \int_\Omega u_\varepsilon^p \phi \hat{z}_\varepsilon dx \\ &= 2a_0 c_0 (1-p) \left\{ \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^p \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) \phi dx + d \int_\Omega u_\varepsilon^p \phi w_\varepsilon(x) dx \right\},\end{aligned}$$

$$\begin{aligned}\hat{N}_\varepsilon^3 &= \int_\Omega |\nabla \phi|^2 \hat{z}_\varepsilon^2 dx = \sum_{j,l=1}^N a_j a_l \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) \left(\frac{\partial u_\varepsilon}{\partial x_l} \right) dx \\ &+ 2d \sum_{j=1}^N a_j \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) w_\varepsilon dx + d^2 \int_\Omega |\nabla \phi|^2 w_\varepsilon^2 dx,\end{aligned}$$

$$\begin{aligned}\hat{N}_\varepsilon^4 &= 2d\varepsilon \int_\Omega k(x) \phi^2 \hat{z}_\varepsilon u_\varepsilon dx \\ &= 2d\varepsilon \sum_{j=1}^N a_j \int_\Omega k(x) \phi^2 u_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) dx + 2d^2 \varepsilon \int_\Omega k(x) \phi^2 u_\varepsilon w_\varepsilon dx,\end{aligned}$$

$$\begin{aligned}\hat{N}_\varepsilon^5 &= \varepsilon \int_\Omega u_\varepsilon \phi^2 \hat{z}_\varepsilon h_\varepsilon dx \\ &= \varepsilon \sum_{j=1}^N a_j \int_\Omega u_\varepsilon \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) h_\varepsilon(x) dx + \varepsilon d \int_\Omega u_\varepsilon \phi^2 w_\varepsilon h_\varepsilon(x) dx,\end{aligned}$$

and $\hat{D}_\varepsilon = \hat{D}^1_\varepsilon + \hat{D}^2_\varepsilon + \hat{D}^3_\varepsilon$,

$$\hat{D}^1_\varepsilon = a_0^2 \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon^2 dx,$$

$$\begin{aligned}
\hat{D}^2_\varepsilon &= 2a_0 \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) u_\varepsilon \phi \hat{z}_\varepsilon dx \\
&= 2a_0 \sum_{j=1}^N a_j \left\{ c_0 p \int_\Omega u_\varepsilon^p \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx + \varepsilon \int_\Omega k(x) u_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx \right\} \\
&+ 2a_0 d \left\{ c_0 p \int_\Omega u_\varepsilon^p \phi w_\varepsilon dx + \varepsilon \int_\Omega k(x) u_\varepsilon \phi w_\varepsilon dx \right\},
\end{aligned}$$

$$\begin{aligned}
\hat{D}^3_\varepsilon &= \int_\Omega \left(c_0 p u_\varepsilon^{p-1} + \varepsilon k(x)\right) \phi^2 \hat{z}_\varepsilon^2 dx \\
&= \sum_{j,l=1}^N a_j a_l \left\{ c_0 p \int_\Omega u_\varepsilon^{p-1} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) \phi^2 dx + \varepsilon \int_\Omega k(x) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) \phi^2 dx \right\} \\
&+ 2d \sum_{j=1}^N a_j \left\{ c_0 p \int_\Omega u_\varepsilon^{p-1} \phi^2 w_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx + \varepsilon \int_\Omega k(x) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) w_\varepsilon \phi^2 dx \right\} \\
&+ d^2 \left\{ c_0 p \int_\Omega u_\varepsilon^{p-1} \phi^2 w_\varepsilon^2 dx + \varepsilon \int_\Omega k(x) \phi^2 w_\varepsilon^2 dx \right\}.
\end{aligned}$$

Let $(a_0, a_1, \cdots, a_N, d)$ denote a maximizer of $\max_{a_0, a_1, \cdots, a_N, d} \left\{ 1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon} \right\}$ which is normalized as $a_0^2 + \sum_{j=1}^N a_j^2 + d^2 = 1$. Since the case $a_0 = 1$ is obvious, we consider only the case $\sum_{j=1}^N a_j^2 + d^2 \neq 0$.

We calculate, as the derivation of (7.8), (7.9), (7.10) in [5],

$$\begin{aligned}
&\int_\Omega u_\varepsilon^p \phi w_\varepsilon dx = \int_\Omega u_\varepsilon^p \phi \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p-1} u_\varepsilon \right) dx \\
&= \int_\Omega \frac{\phi}{p+1} \sum_{j=1}^N \frac{\partial}{\partial x_j} \left\{ (x_j - (x_\varepsilon)_j) u_\varepsilon^{p+1} \right\} - \left(\frac{N}{p+1} - \frac{2}{p-1} \right) u_\varepsilon^{p+1} \phi dx \\
&= -\frac{1}{p+1} \int_\Omega \sum_{j=1}^N \frac{\partial \phi}{\partial x_j} (x_j - (x_\varepsilon)_j) u_\varepsilon^{p+1} dx = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}} \right), \qquad (5.4)
\end{aligned}$$

and

$$\int_\Omega |\nabla\phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) w_\varepsilon dx = O\left(\frac{1}{\|u_\varepsilon\|^2}\right), \tag{5.5}$$

$$\int_\Omega |\nabla\phi|^2 w_\varepsilon^2 dx = O\left(\frac{1}{\|u_\varepsilon\|^2}\right) \tag{5.6}$$

since (2.12) and $\nabla\phi \equiv 0$ near x_0. Thus by (4.3), (4.4), (5.4), (5.5), (5.6), we have

$$\hat{N}_\varepsilon^2 = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right), \quad \hat{N}_\varepsilon^3 = O\left(\frac{1}{\|u_\varepsilon\|^2}\right). \tag{5.7}$$

Also, as (7.11), (7.12) in [5], we have

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p-1}\phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) w_\varepsilon dx = \int_\Omega u_\varepsilon^{p-1}\phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left((x-x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p-1}u_\varepsilon\right) dx \\
&= \sum_{l=1}^N \int_\Omega u_\varepsilon^{p-1}\phi^2 (x_l - (x_\varepsilon)_l)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx + \frac{2}{p-1}\int_\Omega u_\varepsilon^p \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx \\
&= \|u_\varepsilon\|^{2/(N-2)} \sum_{l=1}^N \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1}\phi_\varepsilon^2(y) y_l \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right) dy + O(\frac{1}{\|u_\varepsilon\|^{p+1}}) \\
&= \|u_\varepsilon\|^{2/(N-2)} \left(\sum_{l=1}^N \int_{\mathbb{R}^N} U^{p-1} y_l \left(\frac{\partial U}{\partial y_l}\right)\left(\frac{\partial U}{\partial y_j}\right) dy + o(1)\right) \\
&= \|u_\varepsilon\|^{2/(N-2)} o(1),
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p-1}\phi^2 w_\varepsilon^2 dx = \|u_\varepsilon\|^{p+1-(p-1)N/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p-1}\phi_\varepsilon^2(y)\left(y\cdot\nabla\tilde{u}_\varepsilon + \frac{2}{p-1}\tilde{u}_\varepsilon\right)^2 dy \\
&= \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} U^{p-1}(y)\left(\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\right)^2 dy + o(1).
\end{aligned} \tag{5.9}$$

Moreover,

$$\begin{aligned}
&\int_\Omega k(x)\phi^2 u_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) dx \\
&= \|u_\varepsilon\|^{2+(p-1)/2-(p-1)N/2} \int_{\Omega_\varepsilon} k_\varepsilon(y)\phi_\varepsilon^2 \tilde{u}_\varepsilon \frac{\partial \tilde{u}_\varepsilon}{\partial y_j} dy \\
&= \|u_\varepsilon\|^{-2/(N-2)}\left[k(x_0)\int_{\mathbb{R}^N} U\frac{\partial U}{\partial y_j} dy + o(1)\right] = O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right)
\end{aligned} \tag{5.10}$$

and

$$\int_\Omega k(x)\phi^2 u_\varepsilon w_\varepsilon dx = \|u_\varepsilon\|^{2-(p-1)N/2}\int_{\Omega_\varepsilon} k_\varepsilon(y)\phi_\varepsilon^2\tilde{u}_\varepsilon\tilde{w}_\varepsilon dy$$
$$= \|u_\varepsilon\|^{-4/(N-2)}\left[k(x_0)\int_{\mathbb{R}^N} U\left(y\cdot\nabla U + \frac{2}{p-1}U\right)dy + o(1)\right]$$
$$= O\left(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}}\right). \tag{5.11}$$

Thus $\hat{N}_\varepsilon^4$ can be estimated as

$$\hat{N}_\varepsilon^4 = O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}\right) + O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}\right) = O\left(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}}\right) \tag{5.12}$$

by (5.10), (5.11) and (4.16).

Similarly by (5.3), we have

$$\int_\Omega \phi^2 u_\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)h_\varepsilon(x)dx$$
$$= \|u_\varepsilon\|^{1+(p+1)/2-(p-1)N/2}\int_{\Omega_\varepsilon}\tilde{h}_\varepsilon(y)\phi_\varepsilon^2\tilde{u}_\varepsilon\left(\frac{\partial\tilde{u}_\varepsilon}{\partial y_j}\right)dy$$
$$= \|u_\varepsilon\|^{-2/(N-2)}\left[\sum_{j=1}^N a_j\left(\frac{\partial k}{\partial x_j}\right)(x_0)\int_{\mathbb{R}^N} U\frac{\partial U}{\partial y_j}dy + o(1)\right]$$
$$= O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right) \tag{5.13}$$

and

$$\int_\Omega \phi^2 u_\varepsilon w_\varepsilon h_\varepsilon(x)dx = \|u_\varepsilon\|^{2-(p-1)N/2}\int_{\Omega_\varepsilon}\phi_\varepsilon^2\tilde{u}_\varepsilon\tilde{w}_\varepsilon\tilde{h}_\varepsilon(y)dy$$
$$= \|u_\varepsilon\|^{-4/(N-2)}\left[\sum_{j=1}^N a_j\left(\frac{\partial k}{\partial x_j}\right)(x_0)\int_{\mathbb{R}^N} U\left(y\cdot\nabla U + \frac{2}{p-1}U\right)dy + o(1)\right]$$
$$= O\left(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}}\right). \tag{5.14}$$

Thus $\hat{N}_\varepsilon^5$ can be estimated as

$$\hat{N}_\varepsilon^5 = O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}\right) + O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}\right) = O\left(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}}\right) \tag{5.15}$$

by (5.13), (5.14) and (4.16).

Therefore by (5.7), (5.12) and (5.15), we have

$$\hat{N}_\varepsilon = \hat{N}_\varepsilon^1 + \hat{N}_\varepsilon^2 + \hat{N}_\varepsilon^3 + \hat{N}_\varepsilon^4 + \hat{N}_\varepsilon^5$$
$$= a_0^2 c_0 (1-p) \int_\Omega u_\varepsilon^{p+1} dx + O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right) + O\left(\frac{1}{\|u_\varepsilon\|^2}\right) + O\left(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}}\right)$$
$$\leq O\left(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}}\right). \tag{5.16}$$

Next, we estimate $\hat{D}_\varepsilon$ from the below. Calculation shows

$$\int_\Omega k(x) \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) w_\varepsilon \phi^2 dx$$
$$= \|u_\varepsilon\|^{1+(p+1)/2-(p-1)N/2} \int_{\Omega_\varepsilon} k_\varepsilon(y) \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right) \tilde{w}_\varepsilon \phi_\varepsilon^2 dy$$
$$= \|u_\varepsilon\|^{-2/(N-2)} \left[k(x_0) \int_{\mathbb{R}^N} \left(\frac{\partial U}{\partial y_j}\right) \left(y \cdot \nabla U + \frac{N-2}{2} U\right) dy + o(1)\right]$$
$$= \left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right), \tag{5.17}$$

$$\int_\Omega k(x) w_\varepsilon^2 \phi^2 dx$$
$$= \|u_\varepsilon\|^{2-(p-1)N/2} \int_{\Omega_\varepsilon} k_\varepsilon(y) \tilde{w}_\varepsilon^2 \phi_\varepsilon^2 dy$$
$$= \|u_\varepsilon\|^{-4/(N-2)} \left[k(x_0) \int_{\mathbb{R}^N} \left(y \cdot \nabla U + \frac{N-2}{2} U\right)^2 dy + o(1)\right]$$
$$= \left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \tag{5.18}$$

Thus by (4.9), (5.4), (5.10), (5.11) and (4.16), we have

$$\hat{D}_\varepsilon^2 = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right) + O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}\right) + O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right) + O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}\right)$$
$$= O\left(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}}\right), \tag{5.19}$$

and by (4.10), (4.11), (5.8), (5.9), (5.17) and (5.18),

$$\hat{D}_\varepsilon^3 = c_0 p \left(\sum_{j=1}^N a_j^2\right) \|u_\varepsilon\|^{4/(N-2)} \left(\frac{1}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right)$$
$$+ O(\varepsilon) + d\left(\sum_{j=1}^N a_j\right) o(\|u_\varepsilon\|^{2/(N-2)}) + O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{2/(N-2)}}\right)$$
$$+ d^2 \left(c_0 p \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} U^{p-1}(y)\left(\frac{(1-|y|^2)}{(1+|y|^2)^{N/2}}\right)^2 dy + o(1)\right)$$
$$+ O\left(\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}\right). \tag{5.20}$$

From these, we can estimate $\hat{D}_\varepsilon$ from below, just as in Grossi and Pacella [5]:

$$\hat{D}_\varepsilon \geq \hat{D}_\varepsilon^2 + \hat{D}_\varepsilon^3$$
$$\geq \gamma_1 \left(\sum_{j=1}^N a_j^2\right) \|u_\varepsilon\|^{4/(N-2)} + d\left(\sum_{j=1}^N a_j\right) o(\|u_\varepsilon\|^{2/(N-2)}) + \gamma_2 d^2$$
$$\geq (\gamma_1/2)\left(\sum_{j=1}^N a_j^2\right) \|u_\varepsilon\|^{4/(N-2)} + (\gamma_2/2) d^2 \geq \delta \tag{5.21}$$

for some $\gamma_1, \gamma_2 > 0$ and $\delta > 0$, since $\sum_{j=1}^N a_j^2$ and d^2 can not vanish simultaneously. Therefore, using (5.16) and (5.21), we have

$$\limsup_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \leq \limsup_{\varepsilon\to 0}\left\{1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon}\right\} \leq 1 + \lim_{\varepsilon\to 0}\frac{O(\frac{1}{\|u_\varepsilon\|^{2-2/(N-2)}})}{\delta} = 1.$$

Thus we have proved Lemma 5.1 □

Since we have checked (5.1), we know by Lemma 2.6 that

$$\tilde{v}_{N+2,\varepsilon} \to \sum_{j=1}^N a_{N+2,j}\frac{y_j}{(1+|y|^2)^{N/2}} + b_{N+2}\frac{1-|y|^2}{(1+|y|^2)^{N/2}}$$

in $C^1_{loc}(\mathbb{R}^N)$. Now, for fixed ε, $v_{N+2,\varepsilon}$ and $v_{i,\varepsilon}$ is orthogonal in the sense of (1.4) for $i = 2, \cdots, N+1$. By the same argument as in the last part of the proof of Theorem 1.2, we have that $\vec{a}_{N+2} \cdot \vec{a}_i = 0$ for any $i = 2, \cdots N+1$. Since $\vec{a}_i$ are linearly independent in $\mathbb{R}^N$, we have that $\vec{a}_{N+2} = \vec{0}$. Thus we obtain (1.10).

Also by $b_{N+2} \neq 0$, Lemma 2.7 assures that

$$\|u_\varepsilon\|^2 v_{N+2,\varepsilon} \to -(N-2)\sigma_N b_{N+2} G(\cdot, x_0), \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0.$$

Then, we can repeat the same proof of Lemma 4.2 (with $i = N+2$) to obtain

$$\|u_\varepsilon\|^2 (\lambda_{N+2,\varepsilon} - 1) \to \Gamma,$$

where Γ is defined in (4.17). Calculation shows $\Gamma = 2(N-4)M$. This proves Theorem 1.4. □

6 Appendix

In this appendix, we prove that the matrix $A(x_0)$ in (1.9) is nonnegative definite, where $x_0 \in \Omega_+$ is a blow-up point of least energy solutions $\{u_\varepsilon\}$.

Since x_0 minimizes $\log F(x)$ for F in (1.2), the Hessian matrix $(\operatorname{Hess} \log F)(x_0)$ is nonnegative definite. $(\operatorname{Hess} \log F)(x_0)$ is

$$\left[\left(\frac{2}{N-2} \frac{R_{x_i,x_j}}{R} - \frac{k_{x_i,x_j}}{k} \right) - \left(\frac{2}{N-2} \frac{R_{x_i} R_{x_j}}{R^2} - \frac{k_{x_i} k_{x_j}}{k^2} \right) \right]_{1 \le i,j \le N} (x_0).$$

Since

$$\frac{\partial}{\partial x_i} \log F = \frac{2}{N-2} \frac{R_{x_i}}{R} - \frac{k_{x_i}}{k} = 0$$

at x_0, we have

$$\left(\frac{2}{N-2} \right)^2 \frac{R_{x_i} R_{x_j}}{R^2}(x_0) = \frac{k_{x_i} k_{x_j}}{k^2}(x_0).$$

Thus

$$\begin{aligned} B(x_0) &:= \left(\frac{2}{N-2} \frac{R_{x_i} R_{x_j}}{R^2} - \frac{k_{x_i} k_{x_j}}{k^2} \right)_{1 \le i,j \le N} (x_0) \\ &= \left(\frac{2}{N-2} - \left(\frac{2}{N-2} \right)^2 \right) \left(\frac{R_{x_i} R_{x_j}}{R^2} \right)_{1 \le i,j \le N} (x_0) \\ &= \frac{2(N-4)}{(N-2)^2} \left(\frac{R_{x_i} R_{x_j}}{R^2} \right)_{1 \le i,j \le N} (x_0), \end{aligned}$$

which can be written as

$$B(x_0) = C_N(\nabla \log R)(x_0) \otimes (\nabla \log R)(x_0)$$

for $C_N = \frac{2(N-4)}{(N-2)^2}$.

In general, for any vector $\vec{a} = (a_1, \cdots, a_N) \in \mathbb{R}^N$, the matrix $\vec{a} \otimes \vec{a} = (a_i a_j)_{1 \le i,j \le N}$ is nonnegative definite, because determinants of all $l \times l$ principal minor of $\vec{a} \otimes \vec{a}$ satisfy

$$\begin{vmatrix} a_{i_1}a_{i_1} & a_{i_1}a_{i_2} & \cdots & a_{i_1}a_{i_l} \\ \vdots & & & \vdots \\ a_{i_l}a_{i_1} & a_{i_l}a_{i_2} & \cdots & a_{i_l}a_{i_l} \end{vmatrix} \ge 0, \quad (1 \le i_1 < i_2 < \cdots < i_l \le N).$$

Therefore $B(x_0)$ is nonnegative definite and

$$A(x_0) = (\text{Hess} \log F)(x_0) + B(x_0)$$

is also nonnegative definite. □

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