DIFFERENTIAL GEOMETRY OF LAGRANGIAN SUBMANIFOLDS AND HAMILTONIAN VARIATIONAL PROBLEMS

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Abstract. In this article we shall provide a survey on our recent works ([25],[26]) and their environs on differential geometry of Lagrangian submanifolds in specific symplectic Kähler manifolds, such as complex projective spaces, complex space forms, Hermitian symmetric spaces and Kähler C-spaces. We shall discuss (1) Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds in Hamiltonian volume minimizing problem, (2) classification problem of homogeneous Lagrangian submanifolds from the viewpoint of Lagrangian orbits and moment maps, (3) tightness problem of Lagrangian submanifolds. Moreover we shall give attention to Lagrangian submanifolds in complex hyperquadrics, which are compact Hermitian symmetric spaces of rank 2. The relationship between certain minimal Lagrangian submanifold in complex hyperquadrics and isoparametric hypersurfaces in spheres will be emphasized. Recently we gave a complete classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics and we determined the Hamiltonian stability of ALL compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as the Gauss images of homogeneous isoparametric hypersurfaces in spheres.

Introduction

A Lagrangian submanifold L immersed in a symplectic manifold (M^{2n}, ω) is an n-dimensional submanifold on which the pull-back of the symplectic form ω vanishes. The study of Lagrangian submanifolds L in Kähler manifolds (M^{2n}, ω, J, g) is a fruitful area in differential geometry of submanifolds. From various viewpoints of Riemannian geometry and symplectic geometry, there appear many interesting works on Lagrangian submanifolds in specific Kähler manifolds such as complex space forms, Hermitian symmetric spaces, generalized flag manifolds and so on. Throughout this article, we treat compact immersed or embedded Lagrangian submanifolds without boundary.

In this article we shall explain our recent works on Lagrangian submanifolds in complex hyperquadrics $M^{2n} = Q_n(\mathbb{C})$ and their environs. The complex hyperquadric $M^{2n} = Q_n(\mathbb{C})$ is a compact Hermitian symmetric space of rank

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2. There is a relationship between Lagrangian geometry in the complex hyperquadrics $Q_n(\mathbb{C})$ and hypersurface geometry in the unit standard sphere $S^{n+1}(1)$. Via the "Gauss maps "oriented hypersurfaces in $S^{n+1}(1)$ give Lagrangian submanifolds immersed in $Q_n(\mathbb{C})$. Especially the Gauss images of oriented hypersurfaces with constant principal curvatures, so called "isoparametric hypersurfaces", in $S^{n+1}(1)$ provide a nice class of compact minimal Lagrangian submanifolds embedded in $Q_n(\mathbb{C})$. By using the results of isoparametric hypersurface theory, we shall discuss the properties of such Lagrangian submanifolds, a classification of compact homogeneous Lagrangian submanifolds and the Hamiltonian stability/instability of the Gauss images of homogeneous isoparametric hypersurfaces, in the complex hyperquadrics.

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1. Lagrangian submanifolds in symplectic manifolds and Hamiltonian deformations

Let (M, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω. A *Lagrangian immersion* $\varphi : L \longrightarrow M$ is a smooth immersion of an n-dimensional (maximal dimensional) smooth manifold L into M satisfying the condition $\varphi^*\omega = 0$. For a Lagrangian immersion $\varphi : L \longrightarrow M$, by the non-degeneracy of ω the natural linear bundle map $\varphi^{-1}TM/\varphi_*TL \ni v \mapsto \alpha_v := \omega(v, \varphi_*(\cdot)) \in T^*L$ becomes a linear bundle isomorphism and thus we have a linear isomorphism $C^\infty(\varphi^{-1}TM/\varphi_*TL) \to \Omega^1(L)$. Here $\Omega^i(L)$ denotes the vector space of smooth i-forms on L.

A *Lagrangian deformation* is defined as a one-parameter smooth family of Lagrangian immersions $\varphi_t : L \longrightarrow M$ with $\varphi = \varphi_0$. Let α_{V_t} be the 1-form on L corresponding to its variational vector field $V_t := \partial\varphi_t/\partial t \in C^\infty(\varphi_t^{-1}TM)$. The Lagrangian deformation is characterized by the condition that for each t, α_{V_t} is closed, i.e., $\alpha_{V_t} \in Z^1(L)$. Furthermore, if α_{V_t} is exact, i.e., $\alpha_{V_t} \in B^1(L)$, for each t, then $\{\varphi_t\}$ is called a *Hamiltonian deformation* of $\varphi = \varphi_0$. Here $Z^i(L)$ denotes the vector space of smooth closed i-forms on L and $B^i(L)$ denotes the vector space of smooth exact i-forms on L.

There is a characterization of a Hamiltonian deformation in terms of "isomonodromy deformation"as follows (cf. [25], [44]). Suppose that $[(1/2\pi)\omega] \in H^2(M, \mathbb{R})$ is an integral cohomology class. Then there is a complex line bundle $\mathcal{L}$ over M with a $U(1)$-connection ∇ whose curvature coincides with $\sqrt{-1}\omega$. Let $\varphi_t : L \longrightarrow M$ be a Lagrangian deformation. For each t, we take the pull-back complex line bundle $\varphi_t^{-1}\mathcal{L}$ over L with the pull-back connection $\varphi_t^{-1}\nabla$ through φ_t and thus we have a family of flat connections $\{\varphi_t^{-1}\nabla\}$. Then a Lagrangian deformation $\{\varphi_t\}$ is a Hamiltonian deformation if and only if a family of flat connections $\{\varphi_t^{-1}\nabla\}$ have the same holonomy homomorphism $\pi_1(L) \longrightarrow U(1)$.

A fundamental fact on the relationship of Lagrangian orbits and moment maps of the Hamiltonian group action on a symplectic manifold is as follows : any Lagrangian orbit of Hamiltonian group action G on a symplectic manifold (M, ω) with moment map μ appears as components of the level set $\mu^{-1}(\alpha)$ for some $\alpha \in \mathfrak{z}(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual space of Lie algebra $\mathfrak{g}$ of G and

$$\mathfrak{z}(\mathfrak{g}^*) := \{\alpha \in \mathfrak{g}^* \mid \mathrm{Ad}^*(a)\alpha = \alpha \text{ for all } a \in G\}.$$

If M and G are compact and connected, then each Lagrangian orbit coincides with the level set $\mu^{-1}(\alpha)$ for some $\alpha \in \mathfrak{z}(\mathfrak{g}^*) \cong \mathfrak{c}(\mathfrak{g})$, the center of $\mathfrak{g}$.

Here we mention about elementary examples of the moment maps related to Hermitian symmetric spaces (cf. [42]). Suppose that (U, K) is an Hermitian symmetric pair of compact type or noncompact type. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of the Hermitian symmetric Lie algebra of (U, K) and there is $Z \in \mathfrak{c}(\mathfrak{k})$ such that $\mathrm{ad}(Z)|_{\mathfrak{p}}$ corresponds with the standard complex structure J on $\mathfrak{p}$ invariant under the linear isotropy action of K. The moment map $\tilde{\mu} : \mathfrak{p} \to \mathfrak{k}^* \cong \mathfrak{k}$ of the isotropy representation of K on $\mathfrak{p}$ is given by the formula

$$\tilde{\mu}(p) - \tilde{\mu}(0) = (\mathrm{ad}(p))^2(Z) \tag{1.1}$$

for each $p \in \mathfrak{p}$, where $\tilde{\mu}(0) \in \mathfrak{c}(\mathfrak{k})$. The moment map $\mu : U/K \to \mathfrak{k}^* \cong \mathfrak{k}$ of the isotropy action of K on U/K is given by

$$\langle \mu(aK), \xi \rangle - \langle \mu(eK), \xi \rangle = \langle (\mathrm{Ad}(a) - 1)Z, \xi \rangle \tag{1.2}$$

for each $\xi \in \mathfrak{k}$ and each $aK \in U/K$, where $\mu(eK) \in \mathfrak{c}(\mathfrak{k})$. If we define μ as $\mu(eK) := Z$, then

$$\langle \mu(aK), \xi \rangle = \langle \mathrm{Ad}(a)Z, \xi \rangle$$

for each $\xi \in \mathfrak{k}$. Using the projection $\pi_{\mathfrak{k}} : \mathfrak{u} \to \mathfrak{k}$ with respect to $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, we can express μ as

$$\mu(aK) - \mu(eK) = \pi_{\mathfrak{k}} \circ (\mathrm{Ad}(a) - 1)(Z) = \pi_{\mathfrak{k}}(\mathrm{Ad}(a)(Z)) - Z$$

for each $aK \in U/K$.

2. Lagrangian submanifolds in Kähler manifolds

2.1. **Hamiltonian minimality and Hamiltonian stability.** Let (M, ω, J, g) be a Kähler manifold with complex structure J and Kähler metric g and $\varphi : L \longrightarrow M$ be a Lagrangian immersion. Let B and A denote the second fundamental form and the shape operator of submanifold L in (M, g). Let H denote the mean curvature vector field of φ and the corresponding 1-form $\alpha_{\mathrm{H}} \in \Omega(L)$ is called the *mean curvature form* of φ. Submanifolds with vanishing mean curvature vector field $\mathrm{H} = 0$ are called *minimal submanifolds* in Riemannian geometry. It is known ([13]) that the mean curvature form of a Lagrangian immersion always satisfies the identity

$$d\alpha_{\mathrm{H}} = \varphi^* \rho_M, \tag{2.1}$$

where ρ_M denotes the Ricci form of M defined by $\rho_M(X, Y) = \mathrm{Ric}^M(JX, Y)$ and Ric^M denotes the Ricci tensor field of (M, ω, J, g). It follows from the Codazzi equation of Riemannian submanifolds. Thus if (M, ω, J, g) is an Einstein-Kähler manifold, then α_{H} is closed, i. e. $\alpha_{\mathrm{H}} \in Z^1(L)$.

The notions of Hamiltonian minimality and Hamiltonian stability were introduced and investigated first by Y. G. Oh ([35], [37]).

Definition 2.1. A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian minimal*, shortly *H-minimal*, or *Hamiltonian stationary* if it has extremal volume under every Hamiltonian deformation of φ.

The Euler-Lagrange equation of this variational problem is

$$\delta\alpha_{\mathrm{H}} = 0, \tag{2.2}$$

where δ denotes the co-differential operator with respect to the induced metric $\varphi^* g$ on L.

Definition 2.2. A Hamiltonian minimal Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian stable*, shortly *H-stable* if the second variation of the volume is nonnegative under every Hamiltonian deformation of φ.

A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian volume minimizing*, or *globally Hamiltonian stable*, shortly *globally H-stable*, if φ has minimum volume under every Hamiltonian deformation of φ.

The Hamiltonian version of the second variational formula is given as follows ([37]) : Suppose that φ is an H-minimal Lagrangian immersion. For each Hamiltonian deformation $\varphi_t : L \to M$ with $\varphi_0 = \varphi$ and $V_t = \partial\varphi_t/\partial t \in C^\infty(\varphi_t^{-1}TM)$,

$$\begin{aligned}&\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0}\\ &= \int_L \left(\langle \Delta_L^1 \alpha, \alpha\rangle - \langle \overline{R}(\alpha), \alpha\rangle - 2\langle \alpha\otimes\alpha\otimes\alpha_{\mathrm{H}}, S\rangle + \langle \alpha_{\mathrm{H}}, \alpha\rangle^2\right) dv,\end{aligned} \tag{2.3}$$

where dv denotes the Riemannian measure of $\varphi^* g$ and Δ_L^1 denotes the Hodge-de Rham Laplace operator of $(L, \varphi^* g)$ acting on the vector space $\Omega^1(L)$ of smooth 1-forms on L and $\alpha := \alpha_{V_0} \in B^1(L)$. Here,

$$\langle \overline{R}(\alpha), \alpha\rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j),$$

where $\{e_i\}$ is a local orthonormal frame on L and

$$S(X, Y, Z) := \omega(B(X, Y), Z) = g(JB(X, Y), Z) \tag{2.4}$$

for each $X, Y, Z \in C^\infty(TL)$, which is a symmetric 3-tensor field on L defined by the second fundamental form B of submanifold L in M. The *Jacobi differential operator* or *second variational operator* of a Hamiltonian minimal Lagrangian immersion φ is the self-adjoint linear differential operator $\mathcal{J}_\varphi : B^1(L) \to B^1(L)$ defined by

$$\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} = \int_L \langle \mathcal{J}_\varphi(\alpha), \alpha\rangle dv, \tag{2.5}$$

where

$$\mathcal{J}_\varphi(\alpha) = \Delta_L^1\alpha - \overline{R}(\alpha) - 2A^*_{\mathrm{H}}(\alpha) + \langle \alpha_H, \alpha\rangle\alpha_H. \tag{2.6}$$

Note that $A^*_H(\alpha) = \alpha \circ A_H$. The *linearized Hamiltonian minimal Lagrangian equation* is the linear differential equation

$$\mathcal{J}_\varphi(\alpha) = \Delta^1_L \alpha - \overline{R}(\alpha) - 2A^*_{\mathrm{H}}(\alpha) + \langle \alpha_H, \alpha \rangle \alpha_H = 0 \tag{2.7}$$

for $\alpha \in B^1(L)$.

For an H-minimal Lagrangian immersion $\varphi : L \to M$, denote by $E_0(\varphi) := \mathrm{Ker}(\mathcal{J}_\varphi)$ the null space of the second variation on $B^1(L)$, or equivalently, the vector space of all solutions to the linearized Hamiltonian minimal Lagrangian equation (2.7), and then the dimension $n(\varphi) := \dim(E_0(\varphi))$ of $E_0(\varphi)$ is called the *nullity* of φ.

Suppose that X is a holomorphic Killing vector field defined on M. Then the corresponding 1-form $\alpha_X := \omega(X, \cdot)$ on M is closed. If $H^1(M, \mathbb{R}) = \{0\}$, then $\alpha_X = \omega(X, \cdot)$ is exact, i.e. X is a Hamiltonian vector field on M.

Suppose that M is simply connected, more generally $H^1(M, \mathbb{R}) = \{0\}$. Then each holomorphic Killing vector field X of M generates a volume-preserving Hamiltonian deformation of φ and thus

$$\{\varphi^* \alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\} \subset E_0(\varphi) \subset B^1(L).$$

Set $n_{hk}(\varphi) := \dim\{\varphi^* \alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\}$, which is called the *holomorphic Killing nullity* of φ. Such a Hamiltonian deformation of φ is called *trivial*.

Definition 2.3. Assume that φ is an H-minimal Lagrangian immersion. Then we call φ *strictly Hamiltonian stable* if the following two conditions are satisfied :

(i) φ is Hamiltonian stable.
(ii) The null space of the second variation on Hamiltonian deformations coincides with the vector subspace consisting of infinitesimal deformations induced by trivial Hamiltonian deformations of φ. Namely, $n_{hk}(\varphi) = n(\varphi)$.

Note that if L is strictly Hamiltonian stable, then L has an isolated local minimum volume, up to the congruence, under every Hamiltonian deformation.

An H-minimal Lagrangian immersion φ is called *Hamiltonian rigid* if $n_{hk}(\varphi) = n(\varphi)$ (cf. Yng-Ing Lee [23]).

Definition 2.4. Assume that (M, ω, J, g) is a Kähler manifold and G is an analytic subgroup of its automorphism group $\mathrm{Aut}(M, \omega, J, g)$. We call a Lagrangian orbit $L = G \cdot x \subset M$ of G a *homogeneous Lagrangian submanifold* of M.

Then the following is an easy but useful observation.

Proposition 2.1. *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is always Hamiltonian minimal.*

Proof. Since α_{H} is an invariant 1-form on L, $\delta\alpha_{\mathrm{H}}$ is a constant function on L. Hence by the divergence theorem we obtain $\delta\alpha_{\mathrm{H}} = 0$. □

Set

$$\tilde{G} := \{a \in \mathrm{Aut}(M, \omega, J, g) \mid a(L) = L\}.$$

Then $G \subset \tilde{G}$ and $\tilde{G}$ is the maximal subgroup of $\mathrm{Aut}(M, \omega, J, g)$ preserving L. Moreover we have $n_{hk}(\varphi) = \dim(\mathrm{Aut}(M, \omega, J, g)) - \dim(\tilde{G})$.

2.2. **First eigenvalue of minimal Lagrangian submanifolds in Einstein-Kähler manifolds.** In the case of minimal Lagrangian submanifolds in Einstein-Kähler manifolds, by the second variational formula (2.3), the Hamiltonian stability condition is simplified as follows :

Corollary 2.1 (B. Y. Chen - P. F. Leung - T. Nagano [11], Y. G. Oh [35])**.** *Assume M is an Einstein-Kähler manifold with Einstein constant κ and $\varphi : L \to M$ is a minimal Lagrangian immersion of a compact smooth manifold L into M (i.e. $\alpha_{\mathrm{H}} \equiv 0$). Then L is Hamiltonian stable if and only if*

$$\lambda_1 \geq \kappa \ ,$$

where λ_1 denotes the first (positive) eigenvalue of the Laplacian of L acting on $C^\infty(L)$.

On the other hand, there is an upper bound of the first eigenvalue λ_1 in the following homogeneous Einstein-Kähler manifold case:

Theorem 2.1 ([45], [46], [5])**.** *Assume that M is a compact homogeneous Einstein-Kähler manifold with Einstein constant $\kappa > 0$. Let $L \hookrightarrow M$ be a compact minimal Lagrangian submanifold immersed in M. Then*

$$\lambda_1 \leq \kappa \ .$$

It is a natural question what compact minimal Lagrangian submanifolds attain its equality. Combining it with Corollary 2.1, in this case we get a variational characterization of such minimal Lagrangian submanifolds as follows : L is Hamiltonian stable if and only if $\lambda_1 = \kappa$.

2.3. **Examples of Hamiltonian stable Lagrangian submanifolds.**

Question. What compact Lagrangian submanifolds in a Kähler manifold is a Hamiltonian stable H-minimal Lagrangian submanifold ?

Example 2.1.
(1) Circles on a plane $S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$, great circles and small circles on a sphere $S^1 \subset S^2 \cong \mathbb{C}P^1$, closed circles on a hyperbolic plane $S^1 \subset H^2 \cong \mathbb{C}H^1$ are elementary examples of compact strictly Hamiltonian stable H-minimal Lagrangian submanifolds.
(2) The real projective space embedded in the complex projective space $\mathbb{R}P^n \subset \mathbb{C}P^n$ as a totally geodesic Lagrangian submanifold is strictly Hamiltonian stable ([35]). In fact, it is Hamiltonian volume minimizing (Kleiner and Oh, cf. [35], [36]). See also [2].
(3) A product of $n+1$ circles $S^1(r_0) \times \cdots \times S^1(r_n) \subset \mathbb{C}^{n+1}$ and the quotient space $T^n \subset \mathbb{C}P^n$ by the standard S^1-action are strictly Hamiltonian stable ([37], [48]). Note that $T^n \subset \mathbb{C}P^n$ is minimal if and only if $r_0 = \cdots = r_n$. F. Urbano ([60]) showed that any Hamiltonian stable minimal Lagrangian torus in $\mathbb{C}P^2$ is congruent to $T^2 \subset \mathbb{C}P^2$ with $r_0 = r_1 = r_2$ (see also [10]).

(4) Compact irreducible minimal Lagrangian submanifolds

$$SU(p)/SO(p)\cdot\mathbb{Z}_p,\ SU(p)/\mathbb{Z}_p,\ SU(2p)/Sp(p)\cdot\mathbb{Z}_{2p} \text{ and } E_6/F_4\cdot\mathbb{Z}_3$$

standardly embedded in $\mathbb{C}P^n$ are strictly Hamiltonian stable ([5], cf. [2]). Remark that they are not totally geodesic but they all satisfy $\nabla S = 0$.

(5) Let (V_3, ρ_3) be the irreducible unitary representation of $SU(2)$ of degree 3, where the representation space V_3 consists of all complex homogeneous polynomials with respect to z_0, z_1 of degree 3. The minimal Lagrangian orbit $\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3$ is a 3-dimensional compact embedded strictly Hamiltonian stable Lagrangian submanifold with $\nabla S \neq 0$ (L. Bedulli and A. Gori [7], [43]). It gives a negative answer to Problem 4.2 in [5, p.606].

(6) M. Takeuchi ([56]) classified all compact totally geodesic Lagrangian submanifolds in compact irreducible Hermitian symmetric spaces. He proved that they all are *real forms* of Hermitian symmetric spaces, i.e. the fixed point subset of anti-holomorphic isometries, and are given as symmetric R-spaces L canonically embedded in compact Hermitian symmetric spaces M. If a symmetric R-space L is of Hermitian type, then L is canonically embedded in a compact Hermitian symmetric space $M = L \times L$ as a diagonal subset and thus L is homologically volume-minimizing in $M = L \times L$, in particular L is Hamiltonian stable and moreover L is strictly Hamiltonian stable, because of theorems of Lichnerowicz and Matsushima on the first eigenvalue of compact Einstein-Kähler manifolds with positive Einstein constant. In the case when a symmetric R-space L is not of Hermitian type, the Hamiltonian stability of L are given as in Table 1 ([56], [35], [5], [25, p. 775]). Then a compact totally geodesic Lagrangian submanifold L embedded in a compact irreducible Hermitian symmetric space M is NOT Hamiltonian stable if and only if

$$(L, M) = \begin{cases} (Q_{p,q}(\mathbb{R}) = (S^{p-1}\times S^{q-1})/\mathbb{Z}_2, Q_{p+q-2}(\mathbb{C}))\ (q-p\geq 3), \\ (U(2p)/Sp(p), SO(4p)/U(2p))\ (p\geq 3), \\ (T\cdot E_6/F_4, E_7/T\cdot E_6). \end{cases}$$

Here $G_{p,q}(\mathbb{F})$ denotes the Grassmann manifold of all p-dimensional subspaces of $\mathbb{F}^{p+q}$ for each $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $P_2(\mathbb{K})$ the Cayley projective plane. $Q_{p,q}(\mathbb{R})$ and $Q_n(\mathbb{C})$ denote the real and complex hyperquadric of dimension n. Here each M is equipped with the standard Kähler metric of Einstein constant $1/2$ and λ_1 denotes the first eigenvalue of the Laplacian of L on smooth functions.

The totally geodesic Lagrangian torus $S^1\times S^1 \cong Q_{2,2}(\mathbb{R}) \subset Q_2(\mathbb{C}) \cong S^2\times S^2$ in $S^2\times S^2$ is Hamiltonian stable, and more strongly it is Hamiltonian volume minimizing (H. Iriyeh, H. Ono and T. Sakai [21]).

At present non-trivial known examples of Hamiltonian volume minimizing Lagrangian submanifolds in Kähler manifolds are only $\mathbb{R}P^n \subset \mathbb{C}P^n$ and totally geodesic Lagrangian torus $S^1\times S^1 \subset S^2\times S^2$. It is a natural problem to investigate whether a given Hamiltonian stable Lagrangian submanifold in a Kähler manifold is Hamiltonian volume minimizing or not.

Table 1. Symmetric R-space L of non-Hermitian type canonically embedded in compact Hermitian symmetric spaces

M	L	Einstein	λ_1	H-stable	stable
$G_{p,q}(\mathbb{C}), p \leq q$	$G_{p,q}(\mathbb{R})$	Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbb{C}), p \leq q$	$G_{p,q}(\mathbb{H})$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbb{C})$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$SO(2m)/U(m)$	$SO(m), m \geq 5$	Yes	$\frac{1}{2}$	Yes	No
$SO(4m)/U(2m), m \geq 3$	$U(2m)/Sp(m)$	No	$\frac{m}{4m-2}$	No	No
$Sp(2m)/U(2m)$	$Sp(m), m \geq 2$	Yes	$\frac{1}{2}$	Yes	Yes
$Sp(m)/U(m)$	$U(m)/O(m)$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbb{C}), q-p \geq 3$	$Q_{p,q}(\mathbb{R}), p \geq 2$	No	$\frac{p}{p+q-2}$	No	No
$Q_{p+q-2}(\mathbb{C}), 0 \leq q-p < 3$	$Q_{p,q}(\mathbb{R}), p \geq 2$	No	$\frac{1}{2}$	Yes	No
$Q_{q-1}(\mathbb{C}), q \geq 3$	$Q_{1,q}(\mathbb{R})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$P_2(\mathbb{K})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$G_{2,2}(\mathbb{H})/\mathbb{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$SU(8)/Sp(4)\mathbb{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$\frac{1}{6}$	No	No

Let $\widetilde{M}(c)$ be a simply connected complete complex space form with constant holomorphic sectional curvature c, that is, $\widetilde{M}(c)$ is a complex Euclidean space $\mathbb{C}^n$, a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space form $\mathbb{C}H^n$. Lagrangian submanifolds with $\nabla S = 0$ in complex space forms were completely classified by Hiroo Naitoh and Masaru Takeuchi [31], [32], [33], [34].

Theorem 2.2 ([4],[3],[6])**.** *Let $L^n \hookrightarrow \widetilde{M}(c)$ be a compact Lagrangian submanifold with $\nabla S = 0$ embedded in $\mathbb{C}^n$ or $\mathbb{C}P^n$. Then L is strictly Hamiltonian stable.*

It gives a positive answer to Problem 4.1 in [5, p.606].

Problem. Let $L \hookrightarrow \mathbb{C}P^n$ be a compact minimal Lagrangian submanifold *embedded* in a complex projective space. Is it true that $\lambda_1 = \kappa$, or equivalently, L is Hamiltonian stable? (At present we do not know any counter example yet.)

2.4. **Classification of homogeneous Lagrangian submanifolds in complex projective spaces.** By using the classification theory of prehomogeneous vector spaces due to M. Sato and T. Kimura [52], L. Bedulli and A. Gori [8] provided a classification of compact homogeneous Lagrangian submanifolds in $\mathbb{C}P^n$ which are obtained as Lagrangian orbits of compact connected simple Lie subgroups of $SU(n+1)$. Such Lagrangian submanifolds are classified into 16 types of examples, which consist of 5 types of examples with $\nabla S = 0$ ((2), (4) in Example 2.1) and 11 types of examples with $\nabla S \neq 0$ (including (5) in Example 2.1 as the simplest non-trivial one).

The classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbb{C})$ are described in Subsection 3.5.

2.5. **Tightness of Lagrangian submanifolds.** Suppose that $M^{2n} = G/K$ is an Hermitian symmetric space of compact type and L^n is a compact Lagrangian submanifold embedded in $M = G/K$. Let $\mathrm{Symp}(M)$ and $\mathrm{Ham}(M)$ denote the group of all symplectic diffeomorphisms of M and the group of all Hamiltonian diffeomorphisms of M.

Let L be a real form of an Hermitian symmetric space of compact type. Assume that the minimal Maslov numbger $\Sigma_L \geq 2$. Then for each $\phi \in \mathrm{Hamil}(M)$ with transversal intersection $L \cap aL$ the Arnold-Givental inequality

$$\sharp(L \cap \phi L) \geq SB(L, \mathbb{Z}_2)$$

holds (Y.-G. Oh [39], [40], [41]).

Definition 2.5. We call L *globally tight* (resp. *locally tight*) if for each $a \in G$ (resp. a in a neighborhood of the identity of G) the intersection $L \cap aL$ is not empty and the equality

$$\sharp(L \cap aL) = SB(L, \mathbb{Z}_2)$$

holds provided that the intersection $L \cap aL$ is transverse. Here

$$SB(L, \mathbb{Z}_2) := \sum_{i=0}^{n} \mathrm{rank}(H_i(L, \mathbb{Z}_2))$$

is the sum of Betti numbers of L over $\mathbb{Z}_2$.

Problem. Classify compact globally tight or locally tight Lagrangian submanifolds in compact Hermitian symmetric spaces, more generally simply connected compact homogeneous Kähler manifolds, that is, Kähler C-spaces or generalized flag manifolds with invariant Kähler metrics.

Theorem 2.3 ([36])**.** *Let L be a compact locally tight Lagrangian submanifold embedded in $\mathbb{C}P^n$. Then L is a totally geodesic Lagrangian submanifold $\mathbb{R}P^n$ $(n \geq 2)$ of $\mathbb{C}P^n$, or L is a great or small circle of $S^2 \cong \mathbb{C}P^1$ $(n = 1)$.*

Theorem 2.4 ([22])**.** *Let L be a compact locally tight Lagrangian submanifold embedded in $S^2 \times S^2 \cong Q_2(\mathbb{C})$. Then L is a totally geodesic Lagrangian submanifold S^2 embedded in $S^2 \times S^2$ as a diagonal subset, or $L = S^1(a) \times S^1(b) \subset S^2 \times S^2$.*

Due to the results of Masaru Takeuchi and Shoshichi Kobayashi on the standard embeddings of R-spaces, we know

Theorem 2.5 ([57])**.** *Any real form of an Hermitian symmetric space of compact type is locally tight.*

In his theory on integral geometry of homogeneous spaces, R. Howard gives

Theorem 2.6 ([18])**.** *A totally geodesic Lagrangian submanifold $\mathbb{R}P^n$ of $\mathbb{C}P^n$ is globally tight.*

The global tightness was applied to prove the Hamiltonian volume minimizing property for $\mathbb{R}P^n \subset \mathbb{C}P^n$ (Kleiner-Oh, cf. [36]).

More generally, recently Makiko Sumi Tanaka and H. Tasaki proved

Theorem 2.7 ([58], [59]). *Any real form of an Hermitian symmetric space of compact type is globally tight.*

Concerned with the tightness for real forms of Kähler C-spaces, very recently H. Iriyeh, T. Sakai, H. Tasaki showed

Theorem 2.8 (H. Iriyeh, T. Sakai and H. Tasaki, 2010). *A real form $F_{k_1,\cdots,k_r}(\mathbb{R})$ of $F_{k_1,\cdots,k_r}(\mathbb{C})$ is globally tight.*

3. Lagrangian Submanifolds in Complex Hyperquadrics

3.1. **Complex hyperquadrics and real Grassmann manifolds of oriented 2-planes.** The *complex hyperquadric*

$$Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong SO(n+2)/(SO(2)\times SO(n))$$

is a compact Hermitian symmetric space of rank 2, where

$$Q_n(\mathbb{C}) := \{[\mathbf{z}] \in \mathbb{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\},$$

$$\widetilde{Gr}_2(\mathbf{R}^{n+2}) := \{W \mid \text{oriented 2-dimensional vector subspace of } \mathbb{R}^{n+2}\}.$$

The identification between $Q_n(\mathbb{C})$ and $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ is given by

$$\mathbb{C}P^{n+1} \supset Q_n(\mathbb{C}) \ni\ [\mathbf{a}+\sqrt{-1}\mathbf{b}] \longleftrightarrow W = \mathbf{a}\wedge\mathbf{b}\ \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2}.$$

Here $\{\mathbf{a},\mathbf{b}\}$ is an orthonormal basis of W compatible with its orientation. In case $n=2$, then $Q_2(\mathbb{C}) \cong S^2\times S^2$. If $n \geq 3$, then $Q_n(\mathbb{C})$ is irreducible.

Note that the Einstein constant κ of the standard Kähler metric on $Q_n(\mathbb{C}) \cong \widetilde{\mathrm{Gr}}_2(\mathbb{R}^{n+2})$ induced from the standard inner product of $\mathbb{R}^{n+2}$ is equal to n.

3.2. **Lagrangian submanifolds in complex hyperquadrics and hypersurfaces in spheres.** Let $N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an oriented hypersurface immersed or embedded in the $(n+1)$-dimensional unit standard sphere. Let $\mathbf{x}$ and $\mathbf{n}$ denote the position vector of points of N^n and the unit normal vector field of N^n in $S^{n+1}(1)$, respectively. It is a fundamental fact in symplectic geometry that the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p)+\sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p)\wedge\mathbf{n}(p) \in Q_n(\mathbb{C})$$

is always a Lagrangian immersion.

Proposition 3.1 ([25]). *Any deformation of an oriented hypersurface N^n in $S^{n+1}(1)$ gives a Hamiltonian deformation of $\mathcal{G}$ in $Q_n(\mathbb{C})$. Conversely, any small Hamiltonian deformation of $\mathcal{G}$ in $Q_n(\mathbb{C})$ is obtained from a deformation of an oriented hypersurface N^n in $S^{n+1}(1)$.*

The $(2n+1)$-dimensional real Stiefel manifold

$$V_2(\{\mathbb{R}^{n+2}) := \{(\mathbf{a},\mathbf{b}) \mid \mathbf{a},\mathbf{b}\in\mathbb{R}^{n+2} \text{ orthonormal}\} \cong SO(n+2)/SO(n)$$

of oriented 2-frames in $\mathbb{R}^{n+2}$ has the standard η-Einstein Sasakian manifold structure over $Q_n(\mathbb{C})$. Define the natural projections p_1, p_2 as

$$p_1 : V_2(\mathbb{R}^{n+2}) \ni (\mathbf{a},\mathbf{b}) \longmapsto \mathbf{a} \in S^{n+1}(1),$$

$$p_2 : V_2(\mathbb{R}^{n+2}) \ni (\mathbf{a},\mathbf{b}) \longmapsto \mathbf{a}\wedge\mathbf{b} \in Q_n(\mathbb{C}).$$

$$\begin{tikzcd}
\tilde{N}^n \arrow[r, "\psi", "\text{Legend.}"'] \arrow[d, "\cong"'] & V_2(\mathbb{R}^{n+2}) \arrow[d, "p_1"', "S^n"] \arrow[r, phantom, "="] & V_2(\mathbb{R}^{n+2}) \arrow[d, "p_2"', "S^1"] \\
N^n \arrow[r, "\text{ori.hypsurf.}"'] & S^{n+1}(1) & Q_n(\mathbb{C}) \supset p_2(\psi(N^n)) = \mathcal{G}(N^n) \\
 & & \text{Lagr.}
\end{tikzcd}$$

Here the Legendrian lift $\tilde{N}^n$ of $N^n \hookrightarrow S^{n+1}(1)$ to $V_2(\mathbb{R}^{n+2})$ is defined by $N^n \ni p \longmapsto (\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbb{R}^{n+2})$.

More generally, assume that N^m is an oriented m-dimensional submanifold immersed in $S^{n+1}(1)$. It is a classical fact that the conormal bundle ν_N^* is a Lagrangian submanifold in the cotangent vector bundle $T^*S^{n+1}(1)$ of the unit standard sphere $S^{n+1}(1)$. Notice that the unit cotangent bundle $U(T^*S^{n+1}(1))$ is diffeomorphic to $V_2(\mathbb{R}^{n+2})$. Furthermore, the unit cotangent bundle $U(T^*S^{n+1}(1))$ is a circle bundle over the space $\mathrm{Geod}^+(S^{n+1}(1))$ of oriented geodesics of S^{n+1}, which is isomorphic to the real Grassmann manifold $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ of oriented 2-planes in $\mathbb{R}^{n+2}$ and thus the complex hyperquadric $Q_n(\mathbb{C})$. Hence $U(T^*S^{n+1}(1))$ carries the canonical contact structure and then the unit conormal bundle $U(\nu_N^*)$ of N is a Legendrian submanifold of $U(T^*S^{n+1}(1))$. Then the projection of $U(\nu_N^*)$ gives a Lagrangian immersion in $Q_n(\mathbb{C})$. We have the following diagram :

$$\begin{tikzcd}
\nu_N^* \arrow[r, "\text{Lag.}"'] \arrow[d] & T^*S^{n+1}(1) \arrow[d] & \\
\mathrm{U}(\nu_N^*) \arrow[r, "\text{Leg.}"'] \arrow[d] & U(T^*S^{n+1}(1)) \arrow[d, "p_2"', "S^1"] \arrow[r, phantom, "\cong"] & V_2(\mathbb{R}^{n+2}) \arrow[d, "p_1"', "S^n"] \\
p_2(\mathrm{U}(\nu_N^*)) \arrow[r, "\text{Lag.}"'] & Q_n(\mathbb{C}) & S^{n+1}(1) \supset N^m \\
 & & \text{imm. submfd.}
\end{tikzcd}$$

3.3. **The mean curvature form formula.** Let $g^{std}_{Q_n(\mathbb{C})}$ be the standard Kähler metric of $Q_n(\mathbb{C})$ induced from the standard inner product of $\mathbb{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbb{C})}$ is equal to n. Let κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$. Choose an orthonormal frame $\{e_i\}$ on $N^n \subset S^{n+1}(1)$ such that the second fundamental form h of N^n in $S^{n+1}(1)$ with respect to $\mathbf{n}$ is diagonalized as $h(e_i, e_j) = \kappa_i \delta_{ij}$ and let $\{\theta^i\}$ be its dual coframe. Then the induced metric $\mathcal{G}^* g^{std}_{Q_n(\mathbb{C})}$ on N^n by the Gauss map $\mathcal{G}$ is given as

$$\mathcal{G}^* g^{std}_{Q_n(\mathbb{C})} = \sum_{i=1}^{n} (1 + \kappa_i^2) \theta^i \otimes \theta^i.$$

Let H denote the mean curvature vector field of $\mathcal{G}$. Then the mean curvature form of the Gauss map $\mathcal{G}$ is expressed in terms of the principal curvatures as follows:

Lemma 3.1 (Palmer [51])**.**

$$\alpha_{\mathrm{H}} = d\left(\mathrm{Im}\left(\log\prod_{i=1}^{n}(1+\sqrt{-1}\kappa_i)\right)\right).$$

In case $n = 2$, if $N^2 \subset S^3(1)$ is a minimal surface, then the Gauss map $\mathcal{G} : N^2 \longrightarrow \widetilde{Gr}_2(\mathbb{R}^4) \cong Q_2(\mathbb{C}) \cong S^2 \times S^2$ is a minimal Lagrangian immersion. See also Castro-Urbano [9]. More generally, if $N^n \subset S^{n+1}(1)$ is an oriented *austere* hypersurface in $S^{n+1}(1)$, introduced by Harvey-Lawson ([17]), then the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

And if $N^n \subset S^{n+1}(1)$ is an oriented hypersurface in $S^{n+1}(1)$ with constant principal curvatures, then the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

Note that more minimal Lagrangian submanifolds of complex hyperquadrics can be obtained from Gauss maps of certain oriented hypersurfaces in spheres through Palmer's formula and there are compact oriented rotational non-minimal hypersurfaces embedded in spheres with $n \geq 3$ whose Gauss maps are minimal Lagrangian immersions. ([24]).

3.4. **The Gauss maps of isoparametric hypersurfaces in $S^{n+1}(1)$.** Assume that $N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ is a compact oriented hypersurface embedded in the standard sphere with constant principal curvatures, so called "*isoparametric hypersurface*". Let g denote the number of distinct principal curvatures of N^n in $S^{n+1}(1)$ and $m_1, m_2, \cdots, m_g$ denote the multiplicities of the principal curvatures $k_1 < \cdots < k_g$. Then the image of the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a compact minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$ and the Gauss map gives a covering map

$$N^n \underset{\mathbb{Z}_g}{\longrightarrow} L^n = \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C})$$

with Deck transformation group $\mathbb{Z}_g$.

By the famous theorems of Münzner ([29], [30]), we know that m_i ($i = 1, \cdots, g$) satisfy $m_i = m_{i+2}$ for each i, i.e., $m_1 = m_3 = \cdots$, $m_2 = m_4 = \cdots$, and g must be 1, 2, 3, 4 or 6. By using the results of [38], [47], we can show

Proposition 3.2 ([26])**.** *$L = \mathcal{G}(N^n)$ is a compact monotone and cyclic Lagrangian submanifold embedded in $Q_n(\mathbb{C})$ and its minimal Maslov number Σ_L is given by*

$$\Sigma_L = \frac{2n}{g} = m_1 + m_2\,.$$

All isoparametric hypersurfaces in spheres are classified into homogeneous one and non-homogeneous ones. A hypersurface N^n in $S^{n+1}(1)$ is *homogeneous* if it is obtained as an orbit of a compact connected subgroup G of $SO(n+2)$. Obviously a homogeneous hypersurface in $S^{n+1}(1)$ is an isoparametric hypersurface.

Due to W. Y. Hsiang and H. B. Lawson, Jr. ([19]), see also Ryoichi Takagi and Tsunero Takahashi ([53]), any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a Riemannian symmetric pair (U, K) of rank 2 (see Table 2).

TABLE 2. Homogeneous isoparametric hypersurfaces in spheres

g	*Type*	(U,K)	dimN	m_1, m_2	K/K_0
1	$S^1\times$ BDII	$(S^1\times SO(n+2), SO(n+1))$ $n\geq 1$, $[\mathbb{R}\oplus A_1]$	n	n	S^n
2	BDII$\times$ BDII	$(SO(p+2)\times SO(n+2-p),$ $SO(p+1)\times SO(n+1-p))$ $1\leq p\leq n-1$, $[A_1\oplus A_1]$	n	$p, n-p$	$S^p\times S^{n-p}$
3	AI_2	$(SU(3), SO(3))\,[A_2]$	3	$1,1$	$\frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3)\times SU(3), SU(3))\,[A_2]$	6	$2,2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))\,[A_2]$	12	$4,4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	$(E_6, F_4)\,[A_2]$	24	$8,8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5)\times SO(5), SO(5))\,[B_2]$	8	$2,2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(2)\times U(m)))$ $m\geq 2$, $[BC_2](m\geq 3), [B_2](m=2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2), SO(2)\times SO(m))$ $m\geq 3$, $[B_2]$	$2m-2$	$1,$ $m-2$	$\frac{SO(2)\times SO(m)}{\mathbb{Z}_2\times SO(m-2)}$
4	CII_2	$(Sp(m+2), Sp(2)\times Sp(m))$ $m\geq 2$, $[BC_2](m\geq 3), [B_2](m=2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$
4	DIII_2	$(SO(10), U(5))\,[BC_2]$	18	$4,5$	$\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$
4	EIII	$(E_6, U(1)\cdot Spin(10))\,[BC_2]$	30	$6,9$	$\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$
6	$\mathfrak{g}_2$	$(G_2\times G_2, G_2)\,[G_2]$	12	$2,2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))\,[G_2]$	6	$1,1$	$\frac{SO(4)}{\mathbb{Z}_2+\mathbb{Z}_2}$

Cartan showed that any isoparametric hypersurface in a sphere with $g\leq 3$ distinct principal curvatures must be homogeneous. The classification of isoparametric hypersurfaces in spheres with four or six distinct principal curvatures are still open. In the case $g=6$, all of the principal curvatures must have the same multiplicity $m_1=m_2$ ([30]), and $m_1=m_2$ must be 1 or 2 ([1]). In either case, the known examples belong to the family of homogeneous hypersurfaces. In the case $m=1$, the isoparametric hypersurface must be homogeneous due to Dormeister-Neher [14] (See also Miyaoka [27]), but for $m_1=m_2=2$ the classification is still open so far. In the case $g=4$, the Clifford algebra construction of non-homogeneous isoparametric hypersurfaces in the sphere were discovered first by Hideki Ozeki and Masaru Takeuchi ([49], [50]) and generalized by D. Ferus, H. Karcher and H. F. Münzner [15] (so called "isoparametric hypersurfaces of OT-FKM type"(cf. [28]). Recently T. Cecil, Q.-S. Chi and G. Jensen ([12]) and S. Immervoll ([20]) showed that isoparametric hypersurfaces in the sphere with $g=4$ except for the cases of $(m_1,m_2)=(3,4),(4,5),(6,9),(7,8)$ are either homogeneous or of OT-FKM type.

Proposition 3.3 ([25]). *N^n is homogeneous if and only if $\mathcal{G}(N^n)$ is homogeneous.*

In [25] we classified all compact homogeneous Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbb{C})$ by using the theory of homogeneous isoparametric hypersurfaces. We shall mention it in the next subsection.

Consider

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2}.$$

Here $\bigwedge^2 \mathbb{R}^{n+2} \cong \mathfrak{o}(n+2)$ can be identified with the Lie algebra of all (holomorphic) Killing vector fields on $S^{n+1}(1)$ or $\widetilde{Gr}_2(\mathbb{R}^{n+2})$. Let $\tilde{\mathfrak{k}}$ be the Lie subalgebra of $\mathfrak{o}(n+2)$ consisting of all Killing vector fields tangent to N^n or $\mathcal{G}(N^n)$ and $\tilde{K}$ be an analytic subgroup of $SO(n+2)$ generated by $\tilde{\mathfrak{k}}$. Take the orthogonal direct sum

$$\bigwedge^2 \mathbb{R}^{n+2} = \tilde{\mathfrak{k}} + \mathcal{V},$$

where $\mathcal{V}$ is a vector subspace of $\mathfrak{o}(n+2)$. The linear map

$$\mathcal{V} \ni X \longmapsto \alpha_X|_{\mathcal{G}}(N^n) \in E_0(\mathcal{G}) \subset B^1(\mathcal{G}(N^n))$$

is injective and $n_{hk}(\mathcal{G}) = \dim \mathcal{V}$. Then $\mathcal{G}(N^n) \subset \mathcal{V}$ and thus $\mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V}$. Indeed, for each $X \in \tilde{\mathfrak{k}}$ and each $p \in N^n$,

$$\begin{aligned}\langle X, \mathbf{x}(p) \wedge \mathbf{n}(p)\rangle &= \langle X\mathbf{x}(p), \mathbf{n}(p)\rangle - \langle \mathbf{x}(p), X\mathbf{n}(p)\rangle \\ &= 2\langle X\mathbf{x}(p), \mathbf{n}(p)\rangle = 0.\end{aligned}$$

Note that $\mathcal{G}(N^n)$ is a compact minimal submanifold embedded in the unit hypersphere of $\mathcal{V}$ and by the theorem of Tsunero Takahashi [54] each coordinate function of $\mathcal{V}$ restricted to $\mathcal{G}(N^n)$ is an eigenfunction of the Laplace operator with eigenvalue n. If n is just the first (positive) eigenvalue of $\mathcal{G}(N^n)$, then $\mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is Hamiltonian stable. Moreover if the dimension of the vector space $\mathcal{V}$ is equal to the multiplicity of the first eigenvalue n, then $\mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is strictly Hamiltonian stable.

Proposition 3.4. *If N^n is homogeneous, then*

$$\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V}.$$

Proof. Assume that $\mathbb{R}^{n+2} = \mathfrak{p}$ and $N^n = \mathrm{Ad}_{\mathfrak{p}}(K)\xi$ ($\xi \in \mathfrak{a}$ regular) for a compact Riemannian symmetric pair (U, K) of rank 2. If $X \wedge Y \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V}$, then for any $T \in \mathfrak{k}$

$$\begin{aligned}0 =&\langle \mathrm{ad}_{\mathfrak{p}}(T), X \wedge Y\rangle \\ =&\langle [T, X], Y\rangle - \langle [T, Y], X\rangle \\ =&2\langle [T, X], Y\rangle \\ =&2\langle T, [X, Y]\rangle\end{aligned}$$

and thus we have $[X, Y] = 0$. Hence we obtain $X \wedge Y \in \mathrm{Ad}_{\mathfrak{p}}(K)[\mathfrak{a}] = \mathcal{G}(N^n)$. $\square$

Define $\mu : \widetilde{Gr}_2(\mathbb{R}^{n+2}) \to \bigwedge^2 \mathbb{R}^{n+2}$ in the following way:

$$\mu : \widetilde{Gr}_2(\mathbb{R}^{n+2}) \ni [W] \longmapsto \mathbf{a} \wedge \mathbf{b} \in \bigwedge^2 \mathbb{R}^{n+2} \cong \mathfrak{o}(n+2) = \tilde{\mathfrak{k}} + \mathcal{V}.$$

The moment map of the action $\tilde{K}$ on $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ is given by $\mu_{\tilde{\mathfrak{k}}} := \pi_{\tilde{\mathfrak{k}}} \circ \mu$: $\widetilde{Gr}_2(\mathbb{R}^{n+2}) \to \tilde{\mathfrak{k}}$. For any $p \in N^n$,

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subset \mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

It is obvious that N^n is homogeneous if and only if $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$.

Proposition 3.5. *Assume that $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V}$. Then $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$, that is, N^n is homogeneous.*

Proof. Suppose that $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \neq \mathcal{G}(N^n)$. $\tilde{K}\mathbf{x}(p) \neq N^n$. Then there is a unit vector $\mathbf{c} \in T_pN^n$ such that $\mathbf{c} \perp T_p\tilde{K}\mathbf{x}(p)$. Since $\mathbf{x}(p) \wedge \mathbf{c} \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V} = \mathcal{G}(N^n)$, there is $q \in N^n$ such that $\mathbf{x}(p) \wedge \mathbf{c} = \mathbf{x}(q) \wedge \mathbf{n}(q)$. By the isoparametric property of N^n we have $\mathbf{c} = \pm\mathbf{n}(p)$, a contradiction. $\square$

Corollary 3.1. *N^n is not homogeneous if and only if*

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subsetneqq \mathcal{G}(N^n) \subsetneqq \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

3.5. **Classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics.** Suppose that $G \subset SO(n+2)$ is a compact connected Lie subgroup and $L = G \cdot [W] \subset Q_n(\mathbb{C})$ is a Lagrangian orbit of G through a point $[W] \in Q_n(\mathbb{C})$, where W is an oriented 2-dimensional vector subspace of $\mathbb{R}^{n+2}$. Denote a unit circle of W by

$$S^1(W) := \{v \in W \mid \|v\| = 1\}.$$

Then we can show that there is a finite subset $w_1, \cdots, w_d$ of $S^1(W)$ such that for each $w \in S^1(W) \setminus \{w_1, \cdots, w_d\}$ the orbit $G \cdot w$ of G through w on $S^{n+1}(1) \subset \mathbb{R}^{n+2}$ is a compact homogeneous hypersurface in $S^{n+1}(1)$ ([25]). Set $N^n := G \cdot w$.

By the theorem of W. H. Hsiang-H. B. Lawson, Jr., there is a compact Riemannian symmetric pair (U, K) of rank 2 such that

$$N^n = \mathrm{Ad}_{\mathfrak{p}}(K)v \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} = \mathfrak{p} \quad \text{for some } v \in S^{n+1}(1),$$

where $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ is the canonical decomposition of the symmetric pair (U, K). Here we may assume that $\mathrm{Ad}_{\mathfrak{p}}(K) \subset SO(n+2)$ is a maximal compact subgroup of $SO(n+2)$ containing G which is orbit-equivalent to the action of G on $S^{n+1}(1)$. Then we obtain

Theorem 3.1 (Ma-Ohnita [25])**.** *There exists a compact homogeneous isoparametric hypersurface $N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}$ such that*

(i) *$L = \mathcal{G}(N)$ and L is a compact minimal Lagrangian submanifold, or*
(ii) *L is contained in a Lagrangian deformation of $\mathcal{G}(N)$ consisting of compact homogeneous Lagrangian submanifolds.*

We shall explain about the second case (ii) a little more in detail. The second case (ii) happens only when (U, K) is one of

(1) $(S^1 \times SO(3), SO(2))$,
(2) $(SO(3) \times SO(3), SO(2) \times SO(2))$,
(3) $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 3)$,
(4) $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2, m \geq 3)$.

In the first two cases (1) and (2), it is elementary and well-known to describe all Lagrangian orbits of the natural actions of $K = SO(2)$ on $Q_1(\{\mathbb{C}) \cong S^2$ and $K = SO(2) \times SO(2)$ on $Q_2(\{\mathbb{C}) \cong S^2 \times S^2$. Also in the last two cases (3) and (4), there exist one-parameter families of Lagrangian K-orbits in $Q_n(\mathbb{C})$ and each family contains Lagrangian submanifolds which can NOT be obtained as the Gauss image of any homogeneous isoparametric hypersurface in a sphere. The fourth one is a new family of Lagrangian orbits :

(1) If (U, K) is $(S^1 \times SO(3), SO(2))$, then L is a small or great circle in $Q_1(\mathbb{C}) \cong S^2$.
(2) If (U, K) is $(SO(3) \times SO(3), SO(2) \times SO(2))$, then L is a product of small or great circles of S^2 in $Q_2(\mathbb{C}) \cong S^2 \times S^2$.
(3) If (U, K) is $(SO(3) \times SO(n + 1), SO(2) \times SO(n))$ $(n \geq 3)$, then
$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$
where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic K-orbits satisfying
 (a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold in $Q_n(\mathbb{C})$.
 (b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,
$$K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbb{Z}_2 \cong Q_{2,n}(\mathbb{R})$$
 is an H-minimal Lagrangian submanifold in $Q_n(\mathbb{C})$ with $\nabla S = 0$ and thus $\nabla \alpha_H = 0$.
 (c) $K \cdot [W_{\pm\sqrt{-1}}]$ are isotropic submanifolds in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = 0$ (points !).
(4) If (U, K) is $(SO(m + 2), SO(2) \times SO(m))$ $(n = 2m - 2)$, then
$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$
where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic orbits satisfying
 (a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a minimal (NOT totally geodesic) Lagrangian submanifold in $Q_n(\mathbb{C})$.
 (b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,
$$K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times SO(m - 2))$$
 is an H-minimal Lagrangian submanifold in $Q_n(\mathbb{C})$ with $\nabla S \neq 0$ and $\nabla \alpha_H = 0$.
 (c) $K \cdot [W_{\pm\sqrt{-1}}] \cong SO(m)/S(O(1) \times O(m-1)) \cong \mathbb{R}P^{m-1}$ are isotropic submanifolds in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = m - 1$.

3.6. **Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$.** Suppose that N^n is a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. Palmer ([51]) showed that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$ $(g = 1)$.

Problem. Investigate the Hamiltonian stability of its Gauss image $\mathcal{G}(N^n) = N^n/\mathbb{Z}_g$ embedded in $Q_n(\mathbb{C})$ as a compact minimal Lagrangian submanifold.

$g = 1$: $N^n = S^n$ is a great or small sphere and $\mathcal{G}(N^n) \cong S^n$ is strictly Hamiltonian stable. More strongly, it is stable as a minimal submanifold and homologically volume-minimizing because it is a calibrated submanifold.

$g = 2$: $N^n = S^{m_1} \times S^{m_2}$ $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$ are the so called Clifford hypersurfaces and $\mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbb{R}) \subset Q_n(\mathbb{C})$. Then $m_2 - m_1 \geq 3$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is NOT Hamiltonian stable. In case $m_2 - m_1 \geq 3$, the spherical harmonics of degree 2 on the sphere $S^{m_1} \subset \mathbb{R}^{m_1+1}$ of smaller dimension give volume-decreasing Hamiltonian deformations of $\mathcal{G}(N^n)$. If $m_2 - m_1 = 2$, then it is Hamiltonian stable but not strictly Hamiltonian stable. If $m_2 - m_1 = 0$ or 1, then it is strictly Hamiltonian stable.

$g = 3$: All isoparametric hypersurfaces in the sphere with $g = 3$ were classified by E. Cartan and they all are homogeneous, so called "Cartan hypersurfaces".

Theorem 3.2 (Ma-Ohnita [25]). *If $g = 3$, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is strictly Hamiltonian stable.*

Remark. In case $g = 3$, each induced metric from $Q_n(\mathbb{C})$ is a normal homogeneous metric. It never holds in cases $g = 4, 6$.

$g = 6$: Only homogeneous examples are known now(Dorfmeister-Neher [14], Reiko Miyaoka [27]).

Theorem 3.3 (Ma-Ohnita [26]). *If $g = 6$ and N^n is homogeneous, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is strictly Hamiltonian stable.*

$g = 4$: In the case $g = 4$ and N^n is homogeneous, we obtain the Hamiltonian stability of Gauss images for ALL homogeneous isoparametric hypersurfaces as follows.

Theorem 3.4 (Ma-Ohnita [26]). *If $g = 4$ and N^n is homogeneous, then*

(1)
$$L = \mathcal{G}(N^n) = SO(5)/T^2 \cdot \mathbb{Z}_4$$
is strictly Hamiltonian stable.

(2)
$$L = \mathcal{G}(N^n) = U(5)/(SU(2) \times SU(2) \times U(1)) \cdot \mathbb{Z}_4$$
is strictly Hamiltonian stable.

(3)
$$L = \mathcal{G}(N^n) = (SO(2) \times SO(m))/(\mathbb{Z}_2 \times SO(m-2)) \cdot \mathbb{Z}_4 \ (m \geq 3)$$
is NOT Hamiltonian stable if and only if $m \geq 6$, i.e. $m_2 - m_1 = (m-2) - 1 \geq 3$. If $m_2 - m_1 = (m-2) - 1 = 2$, i.e. $m = 5$, then it is Hamiltonian stable but not strictly Hamiltonian stable. If $m_2 - m_1 = (m-2) - 1 = 0$ or 1, i.e. $m = 3$ or 4, then it is strictly Hamiltonian stable.

(4)
$$L = \mathcal{G}(N^n) = S(U(2) \times U(m))/S(U(1) \times U(1) \times U(m-2)) \cdot \mathbb{Z}_4 \ (m \geq 2)$$

is NOT Hamiltonian stable if and only if $m \geq 4$, *i.e.* $m_2 - m_1 = (2m-3) - 2 \geq 3$. *If* $m_2 - m_1 = -1$ *or* 1, *i.e.* $m = 2$ *or* 3, *then it is strictly Hamiltonian stable.*

(5)

$$L = \mathcal{G}(N^n) = (Sp(2) \times Sp(m))/(Sp(1) \times Sp(1) \times Sp(m-2)) \cdot \mathbb{Z}_4 \ (m \geq 2)$$

is NOT Hamiltonian stable if and only if $m \geq 3$, *i.e.* $m_2 - m_1 = (4m-5) - 4 \geq 3$. *If* $m_2 - m_1 = 1$, *i.e.* $m = 2$, *then it is strictly Hamiltonian stable.*

(6)

$$L = \mathcal{G}(N^n) = (U(1) \cdot Spin(10))/(S^1 \cdot Spin(6)) \cdot \mathbb{Z}_4$$

is strictly Hamiltonian stable.

The last case is obtained as the Gauss image of a principal isotropy orbit of a Riemannian symmetric pair $(U,K) = (E_6, U(1) \cdot Spin(10))$ of exceptional type EIII. For all homogeneous isoparametric hypersurfaces in spheres except for this case, we can observe that $\mathcal{G}(N^n)$ is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$. However in the last case $(m_1, m_2) = (6, 9)$ and thus $m_2 - m_1 = 3$ but it is Hamiltonian stable.

In a summary, we obtain the Hamiltonian stability of the Gauss images of ALL homogeneous isoparametric hypersurfaces in spheres as follows :

Theorem 3.5 ([26]). *Suppose that* (U,K) *is not of type* EIII, *that is,* $(U,K) \neq (E_6, U(1) \cdot Spin(10))$. *Then* $L = \mathcal{G}(N)$ *is NOT Hamiltonian stable if and only if* $m_2 - m_1 \geq 3$. *Moreover if* (U,K) *is of type* EIII, *that is,* $(U,K) = (E_6, U(1) \cdot Spin(10))$, *then* $(m_1, m_2) = (6, 9)$ *but* $L = \mathcal{G}(N)$ *is strictly Hamiltonian stable.*

Problem. Investigate the Hamiltonian stability and other properties of the Gauss images of compact non-homogeneous isoparametric hypersurfaces in the sphere with $g = 4$ as compact minimal Lagrangian submanifolds embedded in complex hyperquadrics.

References

[1] U. Abresch, *Isoparametric hypersurfaces with four and six principal curvatures*, Math. Ann. **264** (1983), 283–302.

[2] A. Amarzaya and Y. Ohnita, *On Hamiltonian stability of certain H-minimal Lagrangian submanifolds in Hermitian symmetric spaces*, RIMS Kokyuroku 1236, *Geometry of Submanifolds and Related Topics*, Nov 2001, RIMS, Kyoto University, Kyoto Japan, 31–48. (revised version: http://math01.sci.osaka-cu.ac.jp/ ohnita/paper/AmarRIMS01c.pdf)

[3] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain symmetric R-spaces embedded in complex Euclidean spaces*, preprint, Tokyo Metropolitan University, 2002.

[4] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain H-minimal Lagrangian submanifolds and related problems*, Surikaisekikenkyusho Kokyuroku 1292, *General study on Riemannian submanifolds*, 72–93. TMU Math. Preprint Ser. no.23, 2002. (revised version: http://math01.sci.osaka-cu.ac.jp/ ohnita/paper/RIMS02c(Kokyuroku).pdf)

[5] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces*, Tohoku Math. J. **55** (2003), 583–610. (a list of misprints : http://math01.sci.osaka-cu.ac.jp/ ohnita/paper/ListOfCorrect.pdf)

[6] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of parallel Lagrangian submanifolds embedded in complex space forms*, a preprint.

[7] L. Bedulli and A. Gori, A Hamiltonian stable minimal Lagrangian submanifolds of projective spaces with nonparallel second fundamental form. Transformation Groups **12** (2007) no.4, 611–617. math.DG/0603528

[8] L. Bedulli and A. Gori, *Homogeneous Lagrangian submanifolds*, Comm. Anal. Geom. **16** (2008), no. 3, 591–615. ArXiv:math.DG/0604169.

[9] I. Castro and F. Urbano, *Minimal Lagrangian surfaces in* $S^2 \times S^2$, Comm. Anal. Geom. **15** (2007), 217–248.

[10] S. Chang, *On Hamiltonian stable minimal Lagrangian surfaces in* $\mathbb{C}P^2$, The Journal of Geometric Analysis **10** (2000), 243-255.

[11] B. Y. Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, 1981.

[12] T. Cecil, Q.-S. Chi and G. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. of Math. **166** (2007), 1–76.

[13] P. Dazord, *Sur la geometrie des sous-fibres et des feuilletages lagrangiens,* (French) [*On the geometry of subbundles and Lagrange foliations*] Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 4, 465–480 (1982).

[14] J. Dorfmeister and E. Neher, *Isoparametric hypersurfaces, case* $g = 6$, $m = 1$, Comm. Algebra **13** (1985), 2299–2368.

[15] D. Ferus, H. Karcher and H. F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*. Math. Z. **177** (1981), 479–502.

[16] E. Goldstein, *Some estimates related to Oh́s conjecture for the Clifford tori in* $\mathbb{C}P^n$, arXiv:math/0311460v1.

[17] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries* Acta Mathematica **148** (1982), 47–157.

[18] R. Howard, *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc. No. 509, **106**, (1993).

[19] W.-Y. Hsiang and H. B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom. **5** (1971), 1–38.

[20] S. Immervoll, *On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres*, Ann. of Math. **168** (2008), 1011–1024.

[21] H. Iriyeh, H. Ono and T. Sakai, *Integral geometry and Hamiltonian volume minimizing property of a totally geodesic Lagrangian torus in* $S^2 \times S^2$, Proc. Japan Acad. **79**, Ser.A, No.10, (2003), 167–170.

[22] H. Iriyeh and T. Sakai, *Tight Lagrangian surfaces in* $S^2 \times S^2$, Geom. Dedicata (2009) doi:10.1007/s10711-009-9398-6

[23] D. Joyce, Y.-I. Lee and R. Schoen, *On the existence of Hamiltonian stationary Lagrangian submanifolds in symplectic manifolds*, arXiv:0902.3338v1 [math.DG].

[24] H. Li, H. Ma and G-X. Wei, *A class of rotational hypersurfaces in the sphere*, in preparation, (2009).

[25] H. Ma and Y. Ohnita, *On Lagrangian submanifolds in complex hyperquadrics and isoparametric hypersurfaces in spheres*, Math. Z. **261** (2009), 749–785.

[26] H. Ma and Y. Ohnita, *Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces*, in preparation (2009).

[27] R. Miyaoka, *The Dorfmeister-Neher theorem on isoparametric hypersurfaces*, to appear in Osaka J. Math. arXiv:0602519 [math.DG].

[28] R. Miyaoka, *Classification of isoparametric hypersurfaces*, In Report of the Fukuoka University Geometry meeting "Geometry and Something"2008, Fukuoka University Seminar House, 2008.11.21-11.24.

[29] H. F. Münzner, *Isoparametrische Hyperfläche in Sphären*, Math. Ann. **251** (1980), 57–71.

[30] H. F. Münzner, *Isoparametrische Hyperfläche in Sphären,* II, Math. Ann. **256** (1981), 215–232.

[31] H. Naitoh, *Totally real parallel submanifolds in $P^n(c)$*, Tokyo J. Math. **4** (1981), 279-306.

[32] H. Naitoh, *Parallel submanifolds of complex space forms. I*, Nagoya Math. J. **90** (1983), 85-117.

[33] H. Naitoh, *Parallel submanifolds of complex space forms. II*, Nagoya Math. J. **91** (1983), 119-149.

[34] H. Naitoh and M. Takeuchi, *Totally real submanifolds and symmetric bounded domains*, Osaka Math. J. **19** (1982), 717-731.

[35] Y. G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. math. **101** (1990), 501–519.

[36] Y. G. Oh, *Tight Lagrangian submanifolds in $\mathbb{C}P^n$*, Math. Z. **207** (1991), 409-416.

[37] Y. G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. **212** (1993), 175–192.

[38] Y. G. Oh, *Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kähler manifolds*, Math. Z. **216** (1994), 471-482.

[39] Y. G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I*, Comm. Pure. Appl. Math. **46** (1993), 949-993. *Appendum to "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I"*, Comm. Pure. Appl. Math. **48** (1995), 1299-1302.

[40] Y. G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, II : $(\mathbb{C}P^n, \mathbb{R}P^n)$*, Comm. Pure. Appl. Math. **46** (1993), 949-994.

[41] Y. G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, III : Arnold-Givental Conjecture*, The Floer Memorial Volume, H. Hofer, C. H. Taubes, A. Weinsitein, E. Zehnder ed., Progress in Mathematics **33** (1995), 555-573, Birkhäuser Verlag, Basel, Boston, Berlin.

[42] Y. Ohnita, *Moment maps and symmetric Lagrangian submanifolds*, in Report of the meeting "Yuzawa Submanifold 2004".
(http://math01.sci.osaka-cu.ac.jp/ ohnita/paper/Yuzawa04.pdf)

[43] Y. Ohnita, *Stability and rigidity of special Lagrangian cones over certain minimal Legendrian orbits*, Osaka J. Math. **44** no.2 (2007), 305-334.

[44] Y. Ohnita, *Differential geometry of Lagrangian submanifolds and related variational problems*, Proceedings of The Twelfth International Workshop on Differential Geometry and Related Fields, **12** (2008), 91–114, ed. by Y.-J. Suh, J. D. Pérez, Y.-S. Choi, Korean Math. Soc. and Research Group in Real and Complex Grassmann Manifolds.

[45] H. Ono, *Minimal Lagrangian submanifolds in adjoint orbits and upper bounds on the first eigenvalue of the Laplacian*, J. Math. Soc. Japan **55** (2003), 243–254.

[46] H. Ono, *Minimality and Hamiltonian stability of Lagrangian submanifolds in adjoint orbits*, Tokyo J. Math. **26** (2003), 83–106.

[47] H. Ono, *Integral formula of Maslov index and its applications*, Japan J. Math. (N.S.) **30** (2004), 413–421.

[48] H. Ono, *Hamiltonian stability of Lagrangian tori in toric Kähler manifolds*, Ann. Glob. Anal. Geom. **37** (2007), 329–343.

[49] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres* I, Tohoku Math. J.(2) **27** (1975), 515–559.

[50] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres* II, Tohoku Math. J.(2) **28** (1976), 7–55.

[51] B. Palmer, *Hamiltonian minimality of Hamiltonian stability of Gauss maps*, Diff. Geom. and its Appl. **7** (1997), 51–58.

[52] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1-155.

[53] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a unit sphere*, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, (1972), 469–481.

[54] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

[55] M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo **12** (1965), 81-192.

[56] M. Takeuchi, *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tohoku Math. J. (2) **36** (1984), 293–314.

[57] M. Takeuchi and S. Kobayashi, *Minimal imbeddings of R-spaces*, J. Differential Geom. **2** (1968), 203-215.

[58] H. Tasaki, *The intersection of two real forms in the complex hyperquadrics*, a preprint 2009.

[59] M. Sumi Tanaka and H. Tasaki, *The intersection of two real forms in Hermitian symmetric spaces of compact type*, a preprint 2010.

[60] F. Urbano, *Index of Lagrangian submanifolds of* $\mathbb{C}P^n$ *and the Laplacian of* 1*-forms*, Geometria Dedicata **48** (1993), 309-318.

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