On the optimal singularity of the critical Sobolev space and a related Sobolev type inequality with a logarithmic weight

Sei Nagayasu[1] and Hidemitsu Wadade[2]

[1] *Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan*

[2] *Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan*

Abstract

In this paper, we investigate the optimal singularity for the critical Sobolev space $H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $n \in \mathbb{N}$ and $1 < p < \infty$. The same authors of this paper have already proved that the function behaving as $[\log(\frac{1}{|x|})]^\tau$ near $x = 0$ belongs to $H^{\frac{n}{p},p}(\mathbb{R}^n)$ if $\tau < \frac{1}{p'} = 1 - \frac{1}{p}$. The purpose of this article is to give more precise characterization of $H^{\frac{n}{p},p}(\mathbb{R}^n)$ by using multiple logarithmic functions.

On the other hand, the authors of this paper also have proved the following Sobolev type embedding with a logarithmic weight: for $1 < p < r < \infty$,

$$H^{\frac{n}{p},p}(\mathbb{R}^n) \hookrightarrow L^{(r-1)p'}(\mathbb{R}^n\,; w_r(x)dx), \text{ where } w_r(x) = \frac{1}{[\log(e + \frac{1}{|x|})]^r |x|^n}. \tag{1}$$

We observe that the embedding (1) is closely related to the optimal singularity for $H^{\frac{n}{p},p}(\mathbb{R}^n)$. In the end, we shall prove that the embedding (1) is strongly sharp in the sense that the weight w_r cannot be replaced by $w_r\varphi$ with any function φ satisfying $\varphi(x) \to \infty$ as $x \to 0$.

2000 Mathematics Subject Classification. Primary 46E35 ; Secondary 26D10.

Key words : critical Sobolev space, weighted Sobolev inequality, optimal singularity of logarithmic type

1 Introduction and main results

In this paper, we investigate the optimal singularity of the critical Sobolev space $H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $n \in \mathbb{N}$ and $1 < p < \infty$. The Sobolev embedding theorem states that $H^{\frac{n}{p},p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ holds for all $p \le q < \infty$, but $H^{\frac{n}{p},p}(\mathbb{R}^n) \not\subset L^\infty(\mathbb{R}^n)$ which implies $H^{\frac{n}{p},p}(\mathbb{R}^n)$ possibly can have a local singularity. Indeed, at least in the case of $n \ge 2$ and $\frac{n}{n-1} \le p < \infty$, we observe that the function behaving as $\left[\log(\frac{1}{|x|})\right]^\tau$ near $x = 0$ belongs to $H^{\frac{n}{p},p}(\mathbb{R}^n)$ if $0 < \tau < \frac{1}{p'}$, which was proved by the same authors of this paper, see [2, Lemma 2.6]. Here, $p' := \frac{p}{p-1}$ denotes the Hölder conjugate exponent of p. The purpose of this paper is to obtain the optimal singularity so that the functions having the logarithmic type growth order near $x = 0$ belong to the critical Sobolev space $H^{\frac{n}{p},p}(\mathbb{R}^n)$.

To state our main theorem, we define functions involving the multiple logarithm as follows. Let $\tau > 0$, and let $\eta \in C^\infty(\mathbb{R}^n)$ be any fixed non-negative function on $\mathbb{R}^n$ satisfying

$$\operatorname{supp}\eta \subset \{x \in \mathbb{R}^n \,:\, |x| < \delta\} \text{ and } \eta \equiv 1 \text{ for } |x| < \frac{\delta}{2}$$

for some small $\delta > 0$. For simplicity of notation, we define the j-ple logarithm $\log^j(t)$ by

$$\log^j(t) := \underbrace{\log \circ \cdots \circ \log}_{j}(t) \text{ for } j \in \mathbb{N} \text{ and large } t > 0.$$

Furthermore, define the functions $v_{j,\tau}(x)$ by

$$\begin{cases} v_{1,\tau}(x) := \left[\log\left(\frac{1}{|x|}\right)\right]^{\tau} \eta(x), \\ v_{j,\tau}(x) := \left[\log\left(\frac{1}{|x|}\right)\right]^{\frac{1}{p'}} \left(\prod_{k=2}^{j-1}\left[\log^k\left(\frac{1}{|x|}\right)\right]^{-\frac{1}{p}}\right)\left[\log^j\left(\frac{1}{|x|}\right)\right]^{-\tau} \eta(x) \text{ for } j \geq 2, \end{cases}$$

where we put $\prod_{k=2}^{1}\left[\log^k(|x|^{-1})\right]^{-1/p} := 1$ for the convenience. Then we shall show the following theorem:

Theorem 1.1. (i) *Let $n \geq 2$ and $\frac{n}{n-1} \leq p < \infty$. Then it holds*

$$v_{j,\tau} \in H^{\frac{n}{p},p}(\mathbb{R}^n) \text{ if } \begin{cases} 0 < \tau < \frac{1}{p'} & \text{when } j = 1, \\ \tau > \frac{1}{p} & \text{when } j \geq 2. \end{cases}$$

(ii) *Let $n \in \mathbb{N}$ and $1 < p < \infty$. Then it holds*

$$v_{j,\tau} \notin H^{\frac{n}{p},p}(\mathbb{R}^n) \text{ if } \begin{cases} \tau \geq \frac{1}{p'} & \text{when } j = 1, \\ 0 < \tau < \frac{1}{p} & \text{when } j = 2. \end{cases}$$

Remark 1.2. Theorem 1.1 implies that the critical exponents τ so that the function $v_{j,\tau}$ belongs to $H^{\frac{n}{p},p}(\mathbb{R}^n)$ become $\tau = \frac{1}{p'}$ when $j = 1$, and $\tau = \frac{1}{p}$ when $j = 2$ provided that $n \geq 2$ and $\frac{n}{n-1} \leq p < \infty$. Unfortunately, we do not know whether $\tau = \frac{1}{p}$ is optimal or not when $j \geq 3$ for the technical reason. The case $j = 1$ in the assertion (i) was proved in [2, Lemma 2.6].

We now introduce weighted Lebesgue space $L^p(\mathbb{R}^n\,;\,w(x)dx)$ for $1 \leq p < \infty$ and the non-negative measurable function w, and we adopt the norm of $L^p(\mathbb{R}^n\,;\,w(x)\,dx)$ as

$$\|u\|_{L^p(\mathbb{R}^n\,;\,w(x)dx)} := \left(\int_{\mathbb{R}^n} |u(x)|^p w(x)\,dx\right)^{\frac{1}{p}}.$$

Theorem 1.1 is closely related to the following weighted Sobolev type embedding proved by [2, Theorem 1.5]:

Theorem A ([2, Theorem 1.5]). *Let $n \in \mathbb{N}$, $1 < p < r < \infty$ and $p \le q \le (r-1)p'$. Then the continuous embedding*

$$H^{\frac{n}{p},p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n\,;\, w_r(x)dx) \tag{1.1}$$

holds, where the weight function w_r is defined by

$$w_r(x) := \frac{1}{\left[\log(e + \frac{1}{|x|})\right]^r |x|^n} \quad \textit{for } x \in \mathbb{R}^n \setminus \{0\}.$$

We shall show the assertion (ii) in Theorem 1.1 as a direct consequence of Theorem A in Section 2. In [2], it was also proved that the upper bound of the exponent $q = (r-1)p'$ is optimal in the sense that if $q > (r-1)p'$, the embedding (1.1) cannot hold. Indeed, by the direct computation we can see that

$$v_{1,\tau} \in L^q(\mathbb{R}^n\,;\, w_r(x)\,dx) \text{ if and only if } \tau < \frac{r-1}{q},$$

which gives

$$\begin{cases} v_{1,\tau} \in L^q(\mathbb{R}^n\,;\, w_r(x)dx) & \text{for all } 0 < \tau < \dfrac{1}{p'} \quad \text{when} \quad p \le q \le (r-1)p', \\ v_{1,\tau} \notin L^q(\mathbb{R}^n\,;\, w_r(x)dx) & \text{for some } 0 < \tau < \dfrac{1}{p'} \quad \text{when} \quad q > (r-1)p'. \end{cases}$$

Hence, by the above fact and Theorem 1.1 (i) with $j = 1$, we can see the optimality of the upper bound $q = (r-1)p'$ provided that $n \ge 2$ and $\frac{n}{n-1} \le p < \infty$.

Another purpose of this paper is to investigate sharper optimality of Theorem A with the critical case $q = (r-1)p'$. As stated above, the case $q = (r-1)p'$ is optimal in the sense that the case $q > (r-1)p'$ makes the embedding (1.1) fail to hold. However, for the critical case $q = (r-1)p'$, we shall explore the possibility whether the weight $w_r(x)$ can be replaced by another weight having a slightly stronger singularity at the origin. This exploration is motivated by Theorem 1.1 which says that the characterization of $H^{\frac{n}{p},p}(\mathbb{R}^n)$ can be given by not only the single logarithm but also the multiple logarithms. However, against our expectation, we can solve this question negatively as follows:

Theorem 1.3. *Let $n \in \mathbb{N}$ and $1 < p < r < \infty$, and let $\varphi \in C(\mathbb{R}^n \setminus \{0\})$ be a positive function such that φ is radially symmetric and non-increasing with respect to the radial direction $r = |x| \in (0, \infty)$. In addition, assume $\lim_{|x| \downarrow 0} \varphi(x) = \infty$. Then it holds either* (i) *or* (ii) *as follows:*

(i) *It holds $H^{\frac{n}{p},p}(\mathbb{R}^n) \not\subset L^{(r-1)p'}(\mathbb{R}^n\,;\, (w_r\varphi)(x)dx)$.*

(ii) *It holds $H^{\frac{n}{p},p}(\mathbb{R}^n) \subset L^{(r-1)p'}(\mathbb{R}^n\,;\, (w_r\varphi)(x)dx)$, and*

$$\sup_{u \in H^{\frac{n}{p},p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|u\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|u\|_{H^{\frac{n}{p},p}(\mathbb{R}^n)}} = \infty. \tag{1.2}$$

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We first show the affirmative part (i).

Proof of Theorem 1.1 (i). The assertion with the case $j = 1$ was already shown in [2, Lemma 2.6]. Hence, we consider the case $j \geq 2$ below. However, we basically follow the strategy used in [2].

Define for $l \in \mathbb{N}$,

$$\tilde{v}_{j,l,\tau}(t) := t^{-l} \left(\prod_{k=1}^{j-1} \left[\log^k \left(\frac{1}{t} \right) \right]^{-\frac{1}{p}} \right) \left[\log^j \left(\frac{1}{t} \right) \right]^{-\tau} \chi_{[0,\delta]}(t) \quad \text{for } t > 0.$$

The function $\tilde{v}_{j,l,\tau}$ can be non-increasing on $(0, \infty)$ by choosing $\delta > 0$ small enough. Then the direct computation gives

$$\left| \partial_x^\beta v_{j,\tau}(x) \right| \leq C\, \tilde{v}_{j,l,\tau}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \tag{2.1}$$

where $1 \leq |\beta| \leq l$, and C depends only on j, l, τ and δ.

In this proof, C denotes a positive constant depending only on n, p, j, τ and δ, which may vary from line to line. It is easy to show $v_{j,\tau} \in L^p(\mathbb{R}^n)$. Now let $\frac{n}{p} = m + \alpha$, where m is a non-negative integer and $\alpha \in [0, 1)$. If $\alpha = 0$, we can prove $\partial_x^\beta v_{j,\tau} \in L^p(\mathbb{R}^n)$ for all $1 \leq |\beta| \leq m$ directly by applying the estimate (2.1). Thus hereafter we may assume $\alpha \in (0, 1)$. Note that $0 \leq m \leq n-2$ by the assumptions $p \geq \frac{n}{n-1}$ and $\alpha \neq 0$. We shall make use of the characterization of $H^{\frac{n}{p},p}(\mathbb{R}^n)$ in [4, §1.7, §2.1]. Thus it is enough to show that

$$J(\cdot) := \int_{\mathbb{R}^n} \frac{|(\partial_x^\beta v_{j,\tau})(\cdot + y) - (\partial_x^\beta v_{j,\tau})(\cdot)|}{|y|^{n+\alpha}}\, dy \in L^p(\mathbb{R}^n) \quad \text{for } |\beta| \leq m \tag{2.2}$$

since we already know $v_{j,\tau} \in L^p(\mathbb{R}^n)$. In order to prove (2.2), we first divide the integral into three parts as follows:

$$\begin{aligned}
J(x) &\leq \int_{\mathbb{R}^n} \int_0^1 \left| (\nabla \partial_x^\beta v_{j,\tau})(x + ty) \right| dt\, |y|^{-n-\alpha+1}\, dy \\
&\leq C \int_{\mathbb{R}^n} \int_0^1 \tilde{v}_{j,m+1,\tau}\big(|x + ty|\big)\, dt\, |y|^{-n-\alpha+1}\, dy \\
&\leq C \bigg(\int_{\{|y| < \frac{|x|}{2}\}} \int_0^1 \tilde{v}_{j,m+1,\tau}\big(|x + ty|\big)\, dt\, |y|^{-n-\alpha+1}\, dy \\
&\qquad + \int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \int_0^1 \tilde{v}_{j,m+1,\tau}\big(|x + ty|\big)\, dt\, |y|^{-n-\alpha+1}\, dy \\
&\qquad + \int_{\{|y| > 2|x|\}} \int_0^1 \tilde{v}_{j,m+1,\tau}\big(|x + ty|\big)\, dt\, |y|^{-n-\alpha+1}\, dy \bigg)
\end{aligned}$$

$$=: C\big(J_1(x)+J_2(x)+J_3(x)\big).$$

Since it holds $|x+ty|>\frac{|x|}{2}$ for any $|y|<\frac{|x|}{2}$ and $0\le t\le 1$, we can estimate J_1 as follows:

$$J_1(x)\le \int_{\{|y|<\frac{|x|}{2}\}}\int_0^1 \tilde v_{j,m+1,\tau}\left(\frac{|x|}{2}\right)dt\,|y|^{-n-\alpha+1}\,dy\le C\,|x|^{1-\alpha}\,\tilde v_{j,m+1,\tau}\left(\frac{|x|}{2}\right).$$

Next, we estimate J_2. By changing a variable $x+ty=z$, we have

$$\begin{aligned}
J_2(x)&\le C|x|^{-n-\alpha+1}\int_0^1\int_{\{\frac{|x|}{2}\le|y|\le 2|x|\}}\tilde v_{j,m+1,\tau}\big(|x+ty|\big)\,dy\,dt\\
&=C|x|^{-n-\alpha+1}\int_0^1\int_{\{\frac{t|x|}{2}\le|z-x|\le 2t|x|\}}\tilde v_{j,m+1,\tau}\big(|z|\big)\,dz\,t^{-n}\,dt\\
&=C|x|^{-n-\alpha+1}\Bigg[\int_{\{\frac{|x|}{2}\le|z-x|\le 2|x|\}}\int_{\frac{|z-x|}{2|x|}}^1 t^{-n}\,dt\,\tilde v_{j,m+1,\tau}\big(|z|\big)\,dz\\
&\qquad+\int_{\{|z-x|<\frac{|x|}{2}\}}\int_{\frac{|z-x|}{2|x|}}^{\frac{2|z-x|}{|x|}} t^{-n}\,dt\,\tilde v_{j,m+1,\tau}\big(|z|\big)\,dz\Bigg]\\
&=: C|x|^{-n-\alpha+1}\big(J_{21}(x)+J_{22}(x)\big).
\end{aligned}$$

Note that $\frac{|x|}{2}\le|z-x|\le 2|x|$ implies $\frac{|z-x|}{2|x|}\ge\frac14$ and $|z|\le 3|x|$. We now define $\exp^j(t)$ by

$$\exp^j(t):=\underbrace{\exp\circ\cdots\circ\exp}_{j}(t)\ \text{ for }\ j\in\mathbb{N}\ \text{ and }\ t>0.$$

Then by changing variables $z=\rho\,\omega$ ($\rho>0$ and $|\omega|=1$), $\sigma=\log^j(1/\rho)$ and using the condition $m\le n-2$, we can estimate J_{21} as

$$\begin{aligned}
J_{21}(x)&\le C\int_{\{\frac{|x|}{2}\le|z-x|\le 2|x|\}}\tilde v_{j,m+1,\tau}\big(|z|\big)\,dz\le C\int_{\{|z|\le 3|x|\}}\tilde v_{j,m+1,\tau}\big(|z|\big)\,dz\\
&\le C\int_{\log^j\left(\frac{1}{\min\{\delta,3|x|\}}\right)}^{\infty}[\exp^j(\sigma)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{1-\frac1p}\right)\sigma^{-\tau}\,d\sigma\\
&\le\begin{cases} C & \text{if } |x|\ge\dfrac{\delta}{3},\\[2mm] C|x|^{n-m-1}\left(\displaystyle\prod_{k=1}^{j-1}\left[\log^k\left(\frac{1}{3|x|}\right)\right]^{-\frac1p}\right)\left[\log^j\left(\frac{1}{3|x|}\right)\right]^{-\tau} & \text{if } |x|<\dfrac{\delta}{3},\end{cases}
\end{aligned}$$

where we used the following claim:

Claim. The estimate

$$\int_T^\infty[\exp^j(\sigma)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{1-\frac1p}\right)\sigma^{-\tau}d\sigma$$

$$\leq \frac{1}{n-m-1}[\exp^j(T)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(T)]^{-\frac{1}{p}}\right)T^{-\tau}$$

holds for any $T>0$.

Indeed, this claim is shown as

$$\begin{aligned}
&\int_T^\infty [\exp^j(\sigma)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{1-\frac{1}{p}}\right)\sigma^{-\tau}d\sigma\\
&= -\frac{1}{n-m-1}\int_T^\infty \frac{d}{d\sigma}\left\{[\exp^j(\sigma)]^{-(n-m-1)}\right\}\left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{-\frac{1}{p}}\right)\sigma^{-\tau}d\sigma\\
&= \frac{1}{n-m-1}[\exp^j(T)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(T)]^{-\frac{1}{p}}\right)T^{-\tau}\\
&\quad + \frac{1}{n-m-1}\int_T^\infty [\exp^j(\sigma)]^{-(n-m-1)}\frac{d}{d\sigma}\left\{\left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{-\frac{1}{p}}\right)\sigma^{-\tau}\right\}d\sigma\\
&\leq \frac{1}{n-m-1}[\exp^j(T)]^{-(n-m-1)}\left(\prod_{k=1}^{j-1}[\exp^k(T)]^{-\frac{1}{p}}\right)T^{-\tau}
\end{aligned}$$

since the function $\sigma \mapsto \left(\prod_{k=1}^{j-1}[\exp^k(\sigma)]^{-\frac{1}{p}}\right)\sigma^{-\tau}$ is non-increasing.

On the other hand, we can estimate J_{22} as

$$\begin{aligned}
J_{22}(x) &= C\,|x|^{n-1}\int_{\{|z-x|<\frac{|x|}{2}\}}\frac{1}{|z-x|^{n-1}}\,\tilde{v}_{j,m+1,\tau}\big(|z|\big)\,dz\\
&\leq C\,|x|^{n-1}\,\tilde{v}_{j,m+1,\tau}\left(\frac{|x|}{2}\right)\int_{\{|z-x|<\frac{|x|}{2}\}}\frac{1}{|z-x|^{n-1}}dz = C\,|x|^n\,\tilde{v}_{j,m+1,\tau}\left(\frac{|x|}{2}\right)
\end{aligned}$$

since $|z|>\frac{|x|}{2}$ holds for $|z-x|<\frac{|x|}{2}$. Lastly, we estimate J_3. By changing a variable $ty=z$, we divide the integral into two parts as follows:

$$\begin{aligned}
J_3(x) &= \int_0^1\int_{\{|z|>2t|x|\}}\tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n-\alpha+1}t^{\alpha-1}\,dt\,dz\\
&= \int_{\{|z|>2|x|\}}\int_0^1 t^{\alpha-1}\,dt\,\tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n-\alpha+1}\,dz\\
&\quad + \int_{\{|z|\leq 2|x|\}}\int_0^{\frac{|z|}{2|x|}} t^{\alpha-1}\,dt\,\tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n-\alpha+1}\,dz =: J_{31}(x)+J_{32}(x).
\end{aligned}$$

We now estimate J_{31}. We first remark that we have $J_{31}(x)=0$ for $|x|\geq\delta$ since $|x+z|>\delta$ holds for $|z|>2|x|$ and $|x|\geq\delta$. Then we consider $|x|<\delta$. Since we have $|x+z|>\frac{|z|}{2}$ for any $|z|>2|x|$, by changing variables $z=\rho\,\omega$ ($\rho>0$ and $|\omega|=1$) and $\sigma=\log^j(2/\rho)$, we have

$$J_{31}(x) = \frac{1}{\alpha}\int_{\{|x|>2|x|\}}\tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n-\alpha+1}\,dz \leq \frac{1}{\alpha}\int_{\{|x|>2|x|\}}\tilde{v}_{j,m+1,\tau}\left(\frac{|z|}{2}\right)|z|^{-n-\alpha+1}\,dz$$

$$
\begin{aligned}
&= C \int_{\log^j\left(\frac{1}{\delta}\right)}^{\log^j\left(\frac{1}{|x|}\right)} \left[\exp^j(\sigma)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(\sigma)\right]^{1-\frac{1}{p}} \right) \sigma^{-\tau}\, d\sigma \\
&\le C \left[\exp^j(\sigma)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(\sigma)\right]^{-\frac{1}{p}} \right) \sigma^{-\tau} \Bigg|_{\sigma=\log^j\left(\frac{1}{|x|}\right)} \\
&= C|x|^{-\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\log^k\left(\frac{1}{|x|}\right)\right]^{-\frac{1}{p}} \right) \left[\log^j\left(\frac{1}{|x|}\right)\right]^{-\tau},
\end{aligned}
$$

where we used the following claim:

Claim. Fix $a > 0$. Then there exists a positive constant C_a depending only on n, p, j, τ and a such that

$$
\int_a^t \left[\exp^j(\sigma)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(\sigma)\right]^{1-\frac{1}{p}} \right) \sigma^{-\tau}\, d\sigma \le C_a \left[\exp^j(t)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(t)\right]^{-\frac{1}{p}} \right) t^{-\tau} \tag{2.3}
$$

for any $t > a$.

Indeed, this claim is shown as follows. We first remark that

$$
\begin{aligned}
&\frac{d}{dt} h_0(t\,; t_0) \\
&:= \frac{d}{dt} \left[\frac{2p}{n} \left[\exp^j(t)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(t)\right]^{-\frac{1}{p}} \right) t^{-\tau} - \int_{t_0}^t \left[\exp^j(\sigma)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(\sigma)\right]^{1-\frac{1}{p}} \right) \sigma^{-\tau}\, d\sigma \right] \\
&= \left[\exp^j(t)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(t)\right]^{1-\frac{1}{p}} \right) t^{-\tau} \left(1 - \frac{2p}{n} R_0(t) \right),
\end{aligned}
$$

where

$$
R_0(t) := \frac{1}{p} \sum_{k=1}^{j-1} \frac{1}{\prod_{\lambda=1}^{k} \exp^{j-\lambda}(t)} + \frac{\tau}{t \prod_{\lambda=1}^{j-1} \exp^{\lambda}(t)} \to 0 \text{ as } t \to \infty.
$$

Then there exists $t_0 > 0$ depending only on n, p, j, τ such that $R_0(t) < \frac{n}{4p}$ holds for any $t \ge t_0$. Thus for any $t \ge t_0$, we have $\frac{d}{dt} h_0(t\,; t_0) > 0$ which implies $h_0(t\,; t_0) \ge h_0(t_0\,; t_0) > 0$. Hence, the inequality (2.3) with $a = t_0$ and $C_a = \frac{2p}{n}$ holds. Therefore, it is enough to consider $a < t_0$. In this case, we see

$$
\begin{cases}
\displaystyle \int_a^{t_0} \left[\exp^j(\sigma)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(\sigma)\right]^{1-\frac{1}{p}} \right) \sigma^{-\tau}\, d\sigma \le C'_a, \\
\displaystyle \min_{t \in [a, t_0]} \left[\exp^j(t)\right]^{\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\exp^k(t)\right]^{-\frac{1}{p}} \right) t^{-\tau} > 0.
\end{cases}
$$

Hence, the inequality (2.3) holds for $a < t_0$. Thus the claim is proved.

We proceed to the estimate of J_{32}. We divide it into two parts as follows:

$$\begin{aligned}
J_{32}(x) &= C|x|^{-\alpha} \int_{\{|z|\le 2|x|\}} \tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n+1}\,dz \\
&= C|x|^{-\alpha} \left(\int_{\{|z|<\frac{|x|}{2}\}} \tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n+1}\,dz + \int_{\{\frac{|x|}{2}\le |z|\le 2|x|\}} \tilde{v}_{j,m+1,\tau}\big(|x+z|\big)|z|^{-n+1}\,dz \right) \\
&=: C|x|^{-\alpha}\big(J_{321}(x) + J_{322}(x)\big).
\end{aligned}$$

Since $|z| < \frac{|x|}{2}$ yields $|x+z| > \frac{|x|}{2}$, we have

$$J_{321}(x) \le \int_{\{|z|<\frac{|x|}{2}\}} \tilde{v}_{j,m+1,\tau}\left(\frac{|x|}{2}\right)|z|^{-n+1}\,dz = C\,|x|\,\tilde{v}_{j,m+1,\tau}\left(\frac{|x|}{2}\right).$$

On the other hand, note that $|x+z| \le 3|x|$ holds for $|z| \le 2|x|$. Hence, recalling $m \le n-2$ and in the same way as the estimate of J_{21}, we see

$$\begin{aligned}
J_{322}(x) &\le C\,|x|^{-n+1} \int_{\{\frac{|x|}{2}\le |z|\le 2|x|\}} \tilde{v}_{j,m+1,\tau}\big(|x+z|\big)\,dz \\
&\le C\,|x|^{-n+1} \int_{\{|x+z|\le 3|x|\}} \tilde{v}_{j,m+1,\tau}\big(|x+z|\big)\,dz \\
&\le \begin{cases} C\,|x|^{-n+1} & \text{if } |x| \ge \dfrac{\delta}{3}, \\ C\,|x|^{-m} \left(\displaystyle\prod_{k=1}^{j-1} \left[\log^k\left(\frac{1}{3|x|}\right)\right]^{-\frac{1}{p}} \right) \left[\log^j\left(\frac{1}{3|x|}\right)\right]^{-\tau} & \text{if } |x| < \dfrac{\delta}{3}. \end{cases}
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
&J(x) \\
&\le C\left[\sum_{\mu=\frac{1}{2},1,3} |x|^{-\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\log^k\left(\frac{1}{\mu|x|}\right)\right]^{-\frac{1}{p}} \right) \left[\log^j\left(\frac{1}{\mu|x|}\right)\right]^{-\tau} \chi_{[0,\frac{\delta}{\mu}]}\big(|x|\big) + |x|^{-n-\alpha+1}\chi_{[\frac{\delta}{3},\infty]}\big(|x|\big) \right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|J\|_{L^p(\mathbb{R}^n)} &\le C \sum_{\mu=\frac{1}{2},1,3} \left\| |\cdot|^{-\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\log^k\left(\frac{1}{\mu|\cdot|}\right)\right]^{-\frac{1}{p}} \right) \left[\log^j\left(\frac{1}{\mu|\cdot|}\right)\right]^{-\tau} \chi_{[0,\frac{\delta}{\mu}]}\big(|\cdot|\big) \right\|_{L^p(\mathbb{R}^n)} \\
&\quad + C\left\| |\cdot|^{-n-\alpha+1}\chi_{[\frac{\delta}{3},\infty]}\big(|\cdot|\big) \right\|_{L^p(\mathbb{R}^n)} \\
&\le C \left\| |\cdot|^{-\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\log^k\left(\frac{1}{|\cdot|}\right)\right]^{-\frac{1}{p}} \right) \left[\log^j\left(\frac{1}{|\cdot|}\right)\right]^{-\tau} \right\|_{L^p(\{|x|<\delta\})} + C\int_{\frac{\delta}{3}}^{\infty} \rho^{-p\,(n+\alpha-1)+n-1}\,d\rho.
\end{aligned}$$

Furthermore, the direct computation gives

$$\left\| |\cdot|^{-\frac{n}{p}} \left(\prod_{k=1}^{j-1} \left[\log^k \left(\frac{1}{|\cdot|} \right) \right]^{-\frac{1}{p}} \right) \left[\log^j \left(\frac{1}{|\cdot|} \right) \right]^{-\tau} \right\|_{L^p(\{|x|<\delta\})}^p = \frac{1}{\tau p - 1} \left[\log^j \left(\frac{1}{\delta} \right) \right]^{-\tau p + 1}$$

and

$$\int_{\frac{\delta}{3}}^{\infty} \rho^{-p\,(n+\alpha-1)+n-1}\, d\rho = \frac{1}{p(n-m-1)} \left(\frac{\delta}{3} \right)^{-p(n-m-1)}$$

since $m \le n-2$ and $\tau > \frac{1}{p}$. Thus we obtain the desired estimate. □

Next, we shall prove Theorem 1.1 (ii) as a corollary of Theorem A.

Proof of Theorem 1.1 (ii). We first consider the case $j = 1$ by a contradiction argument. Suppose $v_{1,\tau} \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ for some $\tau \ge \frac{1}{p'}$. Then Theorem A guarantees $v_{1,\tau} \in L^{(r-1)p'}(\mathbb{R}^n\,; w_r(x)dx)$ for all r with $p < r < \infty$. However, we see

$$\infty > \|v_{1,\tau}\|_{L^{(r-1)p'}(\mathbb{R}^n\,; w_r(x)dx)}^{(r-1)p'} \ge C \int_{B_{\frac{\delta}{2}}(0)} \left[\log \left(\frac{1}{|x|} \right) \right]^{\tau(r-1)p'-r} \frac{dx}{|x|^n} = C \int_{\log(\frac{2}{\delta})}^{\infty} \sigma^{\tau(r-1)p'-r} d\sigma,$$

where C is some positive constant. Thus we must have $\tau(r-1)p' - (r-1) < 0$, i.e., $\tau < \frac{1}{p'}$, which is a contradiction to the assumption $\tau \ge \frac{1}{p'}$.

Next, we consider the case $j = 2$. In the same way as the case $j = 1$, suppose $v_{2,\tau} \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ for some $0 < \tau < \frac{1}{p}$. Then $v_{2,\tau}$ belongs to $L^{(r-1)p'}(\mathbb{R}^n\,; w_r(x)dx)$ for all r with $p < r < \infty$ by Theorem A. On the other hand, the direct computation shows

$$\begin{aligned} \infty > \|v_{2,\tau}\|_{L^{(r-1)p'}(\mathbb{R}^n\,; w_r(x)dx)}^{(r-1)p'} &\ge \int_{B_{\frac{\delta}{2}}(0)} \left(\left[\log \left(\frac{1}{|x|} \right) \right]^{\frac{1}{p'}} \left[\log^2 \left(\frac{1}{|x|} \right) \right]^{-\tau} \right)^{(r-1)p'} w_r(x)dx \\ &\ge C \int_{B_{\frac{\delta}{2}}(0)} \left[\log \left(\frac{1}{|x|} \right) \right]^{-1} \left[\log^2 \left(\frac{1}{|x|} \right) \right]^{-\tau(r-1)p'} \frac{dx}{|x|^n} = C \int_{\log^2(\frac{2}{\delta})}^{\infty} \sigma^{-\tau(r-1)p'} d\sigma, \end{aligned}$$

where C is some positive constant. Thus we must have $-\tau(r-1)p' + 1 < 0$, i.e., $\tau > \frac{1}{(r-1)p'}$ for all r with $p < r < \infty$, which is a contradiction to the assumption $\tau < \frac{1}{p}$. □

Remark 2.1. If Theorem A held with the case $p = r$, the remaining cases of Theorem 1.1 (ii) could be solved in the quite same manner as the cases $j = 1$ and $j = 2$ as follows:

$$v_{j,\tau} \notin H^{\frac{n}{p},p}(\mathbb{R}^n) \ \text{ if } \ \begin{cases} \tau \ge \dfrac{1}{p'} \ \text{ when } \ j = 1, \\ 0 < \tau \le \dfrac{1}{p} \ \text{ when } \ j \ge 2. \end{cases}$$

However, the proof of Theorem A in [2] cannot work when $p = r$ since they made use of the generalized Young's inequality and the case $p = r$ corresponds to its marginal case where the inequality fails.

3 Proof of Theorem 1.3

In this section, we shall give the proof of Theorem 1.3. First, we recall the Riesz kernel $I_\alpha(x)$ and the Bessel kernel $G_\alpha(x)$ defined as follows. For $n \in \mathbb{N}$ and $0 < \alpha < n$,

$$\begin{cases} I_\alpha(x) := \dfrac{1}{\gamma(\alpha)}|x|^{-(n-\alpha)}, \\ G_\alpha(x) := \dfrac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})} \displaystyle\int_0^\infty e^{-\frac{\pi|x|^2}{\sigma}} e^{-\frac{\sigma}{4\pi}} \sigma^{-\frac{n-\alpha}{2}} \frac{d\sigma}{\sigma} \end{cases}$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\gamma(\alpha) := \pi^{n/2}\, 2^\alpha\, \Gamma(\frac{\alpha}{2}) / \Gamma(\frac{n-\alpha}{2})$ and Γ denotes the Gamma function. For the relation between I_α and G_α, it is well-known that

$$\begin{cases} G_\alpha(x) \le I_\alpha(x) \quad \text{for all } \ x \in \mathbb{R}^n \setminus \{0\}, \\ G_\alpha(x) = I_\alpha(x) + o(|x|^{-(n-\alpha)}) \quad \text{as } \ |x| \to 0. \end{cases} \tag{3.1}$$

Among others, we refer to [3] for more detailed properties of I_α and G_α.

Then Theorem 1.3 can be reformulated in terms of G_α as the following equivalent form :

Theorem 3.1. *Let $n \in \mathbb{N}$ and $1 < p < r < \infty$, and let $\varphi \in C(\mathbb{R}^n \setminus \{0\})$ as in Theorem 1.3. Then it holds either* (i) *or* (ii) *as follows :*

(i) *There exists $f_0 \in L^p(\mathbb{R}^n)$ such that $G_{\frac{n}{p}} * f_0 \notin L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)$.*

(ii) *It holds $G_{\frac{n}{p}} * f \in L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)$ for all $f \in L^p(\mathbb{R}^n)$, and*

$$\sup_{f \in L^p(\mathbb{R}^n)\setminus\{0\}} \frac{\|G_{\frac{n}{p}} * f\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|f\|_{L^p(\mathbb{R}^n)}} = \infty.$$

It is easy to see the equivalence between Theorem 1.3 and Theorem 3.1. However, we will show it for the completeness of the paper.

Proof of the equivalence. First, we will check that Theorem 3.1 yields Theorem 1.3. Assume that the condition (i) in Theorem 3.1. Set $u_0 := G_{\frac{n}{p}} * f_0$. Then it holds

$$\|u_0\|_{H^{\frac{n}{p},p}(\mathbb{R}^n)} = \|(I-\Delta)^{\frac{n}{2p}} u_0\|_{L^p(\mathbb{R}^n)} = \|f_0\|_{L^p(\mathbb{R}^n)} < \infty.$$

Thus $u_0 \in H^{\frac{n}{p},p}(\mathbb{R}^n)$, but $u_0 \notin L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)$. Hence, the condition (i) in Theorem 1.3 holds.

Next, assume the condition (ii) in Theorem 3.1. Take any element $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$, and set $f := (I-\Delta)^{\frac{n}{2p}} u$. Then $f \in L^p(\mathbb{R}^n)$, and then by the assumption, we have

$$u = G_{\frac{n}{p}} * (I-\Delta)^{\frac{n}{2p}} u = G_{\frac{n}{p}} * f \in L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx).$$

Thus $H^{\frac{n}{p},p}(\mathbb{R}^n) \subset L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)$. Moreover, by the assumption, take a sequence $\{f_j\}_{j\in\mathbb{N}} \in L^p(\mathbb{R}^n) \setminus \{0\}$ such that

$$\lim_{j\to\infty} \frac{\|G_{\frac{n}{p}} * f_j\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|f_j\|_{L^p(\mathbb{R}^n)}} = \infty.$$

Set $u_j := G_{\frac{n}{p}} * f_j$. Then $u_j \in H^{\frac{n}{p},p}(\mathbb{R}^n) \setminus \{0\}$ and

$$\frac{\|u_j\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|u_j\|_{H^{\frac{n}{p},p}(\mathbb{R}^n)}} = \frac{\|G_{\frac{n}{p}} * f_j\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|f_j\|_{L^p(\mathbb{R}^n)}} \to \infty \quad \text{as } j \to \infty,$$

which implies (1.2).

The direction from Theorem 1.3 to Theorem 3.1 will be seen in a quite same way as above, and we omit the details. □

Hence we will concentrate on the proof of Theorem 3.1 below. In order to prove Theorem 3.1, we will apply the following theorem in [1]:

Theorem B ([1, Theorem 2.1]). *Let $n \in \mathbb{N}$ and $1 < p \le q < \infty$, and let U and V be positive weight functions in $\mathbb{R}^n$. Assume that*

$$\sup_{R>0} \left(\int_{\{|x|<R\}} U(x)\,dx \right)^{\frac{1}{q}} \left(\int_{\{|x|>R\}} V(x)^{-(p'-1)}\,dx \right)^{\frac{1}{p'}} = \infty.$$

Then it holds either (i) *or* (ii):

(i) *There exists a non-negative function $f_0 \in L^p(\mathbb{R}^n\,;\,V(x)dx)$ such that $\int_{\{|y|>|\cdot|\}} f_0(y)dy \notin L^q(\mathbb{R}^n\,;\,U(x)dx)$.*

(ii) *$\int_{\{|y|>|\cdot|\}} f(y)dy \in L^q(\mathbb{R}^n\,;\,U(x)dx)$ for all non-negative functions $f \in L^p(\mathbb{R}^n\,;\,V(x)dx)$, and*

$$\sup_{\substack{f\in L^p(\mathbb{R}^n\,;\,V(x)dx)\setminus\{0\},\\ f\,:\,\text{non-negative}}} \frac{\left\|\int_{\{|y|>|\cdot|\}} f(y)\,dy\right\|_{L^q(\mathbb{R}^n\,;\,U(x)dx)}}{\|f\|_{L^p(\mathbb{R}^n\,;\,V(x)dx)}} = \infty.$$

Remark 3.2. In [1], the following Hardy type inequality in the n-dimensional case was proved: if

$$\sup_{R>0} \left(\int_{\{|x|<R\}} U(x)\,dx \right)^{\frac{1}{q}} \left(\int_{\{|x|>R\}} V(x)^{-(p'-1)}\,dx \right)^{\frac{1}{p'}} < \infty \tag{3.2}$$

then $\int_{\{|y|>|\cdot|\}} f(y)\,dy \in L^q(\mathbb{R}^n\,;\,U(x)dx)$ holds for all non-negative functions $f \in L^p(\mathbb{R}^n\,;\,V(x)dx)$, and

$$\sup_{\substack{f\in L^p(\mathbb{R}^n\,;\,V(x)dx)\setminus\{0\},\\ f\,:\,\text{non-negative}}} \frac{\left\|\int_{\{|y|>|\cdot|\}} f(y)\,dy\right\|_{L^q(\mathbb{R}^n\,;\,U(x)dx)}}{\|f\|_{L^p(\mathbb{R}^n\,;\,V(x)dx)}} < \infty.$$

Thus, the condition (3.2) is necessary and sufficient for the n-dimensional Hardy inequality to hold.

We are now in the position to prove Theorem 3.1 :

Proof of Theorem 3.1. Let f be a non-negative function in $L^p(\mathbb{R}^n)$. Then since $|y| > |x|$ implies $|x-y| < 2|y|$, we see

$$\begin{aligned}\|G_{\frac{n}{p}} * f\|^{(r-1)p'}_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)} &\geq \int_{\mathbb{R}^n} \left(\int_{\{|y|>|x|\}} G_{\frac{n}{p}}(x-y) f(y)\, dy \right)^{(r-1)p'} (w_r\varphi)(x)\, dx \\ &\geq \int_{\mathbb{R}^n} \left(\int_{\{|y|>|x|\}} G_{\frac{n}{p}}(2y) f(y)\, dy \right)^{(r-1)p'} (w_r\varphi)(x)\, dx.\end{aligned}$$

Thus, it is enough to show that either (i) or (ii) holds as follows :

(i) There exists a non-negative function $f_0 \in L^p(\mathbb{R}^n)$ such that

$$\int_{\{|y|>|\cdot|\}} G_{\frac{n}{p}}(2y) f_0(y) dy \notin L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx).$$

(ii) For all non-negative functions $f \in L^p(\mathbb{R}^n)$ it holds

$$\int_{\{|y|>|\cdot|\}} G_{\frac{n}{p}}(2y) f(y) dy \in L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx),$$

and

$$\sup_{\substack{f\in L^p(\mathbb{R}^n)\setminus\{0\},\\ f\,:\,\text{non-negative}}} \frac{\left\| \int_{\{|y|>|\cdot|\}} G_{\frac{n}{p}}(2y) f(y)\, dy \right\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)\,dx)}}{\|f\|_{L^p(\mathbb{R}^n)}} = \infty.$$

Furthermore, by Theorem B, it suffices to show that

$$\sup_{R>0} \left(\int_{\{|x|<R\}} U_0(x) dx \right)^{\frac{1}{(r-1)p'}} \left(\int_{\{|x|>R\}} V_0(x)^{-(p'-1)} dx \right)^{\frac{1}{p'}} = \infty, \tag{3.3}$$

where

$$(U_0, V_0) := \left(w_r\varphi, G_{\frac{n}{p}}(2\cdot)^{-p} \right).$$

Indeed, once (3.3) has been established, then by applying Theorem B, we obtain either (i) or (ii) as follows :

(i) There exists a non-negative function $f_0 \in L^p(\mathbb{R}^n\,;\,G_{\frac{n}{p}}(2x)^{-p}dx)$ such that $\int_{\{|y|>|\cdot|\}} f_0(y)dy \notin L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)$.

(ii) For all non-negative functions $f \in L^p(\mathbb{R}^n\,;\,G_{\frac{n}{p}}(2x)^{-p}dx)$ it holds

$$\int_{\{|y|>|\cdot|\}} f(y)\, dy \in L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx),$$

and

$$\sup_{\substack{f\in L^p(\mathbb{R}^n\,;\,G_{\frac{n}{p}}(2x)^{-p}dx)\setminus\{0\},\\ f\,:\,\text{non-negative}}} \frac{\left\|\int_{\{|y|>|\cdot|\}} f(y)\,dy\right\|_{L^{(r-1)p'}(\mathbb{R}^n\,;\,(w_r\varphi)(x)dx)}}{\|f\|_{L^p(\mathbb{R}^n\,;\,G_{\frac{n}{p}}(2x)^{-p}dx)}} = \infty.$$

By setting $\tilde{f}_0(x) := f_0(x)G_{\frac{n}{p}}(2x)^{-1}$ or $\tilde{f}(x) := f(x)G_{\frac{n}{p}}(2x)^{-1}$, we have the desired facts. Hence, it remains to prove (3.3).

Obviously, we may assume $\int_{\{|x|<R\}}(w_r\varphi)(x)\,dx < \infty$ for any $R>0$. Note that

$$\max_{t\geq 2}\frac{\log(e+t)}{\log t} = \frac{\log(e+2)}{\log 2}.$$

Then we see for any $0<R<\frac{1}{2}$,

$$\begin{aligned}
\int_{\{|x|<R\}}(w_r\varphi)(x)\,dx &= \int_{\{|x|<R\}}\left[\log\left(e+\frac{1}{|x|}\right)\right]^{-r}|x|^{-n}\varphi(x)\,dx\\
&\geq \left[\frac{\log(e+2)}{\log 2}\right]^{-r}\int_{\{|x|<R\}}\left[\log\left(\frac{1}{|x|}\right)\right]^{-r}|x|^{-n}\varphi(x)\,dx\\
&= C\int_0^R\left[\log\left(\frac{1}{t}\right)\right]^{-r}\tilde{\varphi}(t)\,\frac{dt}{t},
\end{aligned}$$

where we set $\tilde{\varphi}(t) := \varphi(x)$ with $|x| = t\in(0,\infty)$. Define $g(R)$ by

$$g(R) := \frac{\int_0^R\left[\log(\frac{1}{t})\right]^{-r}\tilde{\varphi}(t)\,\frac{dt}{t}}{\left[\log(\frac{1}{R})\right]^{-(r-1)}} \quad \text{for } 0<R<\frac{1}{2}.$$

Then by using L'Hopital's rule, we obtain

$$\lim_{R\downarrow 0} g(R) = \lim_{R\downarrow 0}\frac{\left[\log(\frac{1}{R})\right]^{-r}\tilde{\varphi}(R)\frac{1}{R}}{(r-1)\left[\log(\frac{1}{R})\right]^{-r}\frac{1}{R}} = \lim_{R\downarrow 0}\frac{\tilde{\varphi}(R)}{r-1} = \infty.$$

Next, we consider the integral $\int_{\{|x|>R\}} G_{\frac{n}{p}}(2x)^{p'}dx$ for $R>0$. The latter estimate in (3.1) implies that there exists a positive constant $0<\delta_0<1$ such that $I_{\frac{n}{p}}(x)\leq 2G_{\frac{n}{p}}(x)$ for all $0<|x|<\delta_0$. Then for any R with $0<R<\frac{\delta_0}{2}$, we see

$$\begin{aligned}
&\int_{\{|x|>R\}} G_{\frac{n}{p}}(2x)^{p'}dx \geq \int_{\{R<|x|<\frac{\delta_0}{2}\}} G_{\frac{n}{p}}(2x)^{p'}dx \geq \int_{\{R<|x|<\frac{\delta_0}{2}\}}\left(\frac{1}{2}I_{\frac{n}{p}}(2x)\right)^{p'}dx\\
&= C\int_{\{R<|x|<\frac{\delta_0}{2}\}}|x|^{-n}\,dx = C\log\left(\frac{\delta_0}{2R}\right).
\end{aligned}$$

Thus combining the above estimates from below, we have for $0<R<\frac{\delta_0}{2}$,

$$\left(\int_{\{|x|<R\}}(w_r\varphi)(x)\,dx\right)^{\frac{1}{(r-1)p'}}\left(\int_{\{|x|>R\}} G_{\frac{n}{p}}(2x)^{p'}dx\right)^{\frac{1}{p'}}$$

$$
\begin{aligned}
&\geq C\left(\int_0^R \left[\log\left(\frac{1}{t}\right)\right]^{-r} \tilde{\varphi}(t)\,\frac{dt}{t}\right)^{\frac{1}{(r-1)p'}} \left[\log\left(\frac{\delta_0}{2R}\right)\right]^{\frac{1}{p'}} \\
&= C\left(g(R)\left[\log\left(\frac{1}{R}\right)\right]^{-(r-1)}\right)^{\frac{1}{(r-1)p'}} \left[\log\left(\frac{\delta_0}{2R}\right)\right]^{\frac{1}{p'}} \\
&= Cg(R)^{\frac{1}{(r-1)p'}}\left(1+\frac{\log(\frac{\delta_0}{2})}{\log(\frac{1}{R})}\right)^{\frac{1}{p'}} \to \infty \quad \text{as} \;\; R \downarrow 0,
\end{aligned}
$$

which implies (3.3). Thus Theorem 3.1 is proved. □

4 Acknowledgments

The first author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, and the second author is supported by the Fujukai Foundation.

References

[1] P. Drábek, H. Heinig and A. Kufner, *Higher-dimensional Hardy inequality*, General Inequalities 7, 3–16, Internat. Ser. Numer. Math. **123**, Birkhäuser, Basel, 1997.

[2] S. Nagayasu and H. Wadade, *Characterization of the critical Sobolev space on the optimal singularity at the origin*, J. Funct. Anal. **258** (2010), 3725–3757.

[3] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton U. Press, Princeton, N.J., 1970.

[4] R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060.

Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
nagayasu@math.sci.hokudai.ac.jp

Advanced Mathematical Institute
Osaka City University
Osaka 558-8585, Japan
hidemitsuwadade@gmail.com