The existence and non-existence of the non-trivial solutions of the nonlinear Schrödinger equations for one dimensional case

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0. Introduction

In this paper, we consider the one dimensional case of the following nonlinear Schrödinger equations:

$$\begin{aligned} &-u'' + (1+b(x))u = f(u) \quad \text{in } \mathbf{R}, \\ &u \in H^1(\mathbf{R}). \end{aligned} \tag{$*$}$$

Here, we assume that the potential $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies the following assumptions:

(b.1) $1 + b(x) \geq 0$ for all $x \in \mathbf{R}$.

(b.2) $\lim_{|x|\to\infty} b(x) = 0$.

(b.3) There exist $\beta_0 > 2$ and $C_0 > 0$ such that $b(x) \leq C_0 e^{-\beta_0|x|}$ for all $x \in \mathbf{R}$.

We set $F(u) = \int_0^u f(\tau)\,d\tau$ and assume that the nonlinearity $f(u)$ satisfies

(f.1) There exists $\eta_0 > 0$ such that $\lim_{|u|\to\infty} \frac{f(u)}{|u|^{1+\eta_0}} = 0$.

(f.2) There exists $u_0 > 0$ such that

$$\begin{aligned} &F(u) < \frac{1}{2}u^2 \quad \text{for all } u \in (0, u_0), \\ &F(u_0) = \frac{1}{2}u_0^2, \quad f(u_0) > u_0. \end{aligned}$$

(f.3) There exists $\mu_0 > 2$ such that $0 < \mu_0 F(u) \leq u f(u)$ for all $u \neq 0$.

The conditions (f.1) and (f.2) are sufficient conditions for the following equation to have an unique positive solution:

$$-u'' + u = f(u) \quad \text{in } \mathbf{R}, \quad u \in H^1(\mathbf{R}). \tag{0.1}$$

From (b.2), the equation $-u'' + u = f(u)$ appears as a limit when $|x|$ goes to ∞ in $(*)$. The condition (f.3) is so called Ambrosetti-Rabinowitz condition, which guarantees the boundedness of (PS)-sequences for the functional corresponding to the equation $(*)$ and (0.1).

To state an our result about the existence of solutions for $(*)$, we also need the following assumption for $b(x)$.

(b.4) There exists $x_0 \in \mathbf{R}$ such that

$$\overline{\lim_{r\to\infty}} \int_{-r}^{r} b(x - x_0)e^{2|x|}\, dx \in [-\infty, 2).$$

Our first theorem is the following.

Theorem 0.1. *Assume that (b.1)–(b.4) and (f.1)–(f.3) hold. Then $(*)$ has at least a positive solution.*

When we prove Theorem 0.1, it is important to estimate interaction of $\omega(x - R)$ and $\omega(x + R)$ for large $R >> 1$. Here, $\omega(x)$ is an unique solution of (0.1) with $u(0) = \max_{x\in\mathbf{R}} u(x)$. When we estimate interaction of $\omega(x - R)$ and $\omega(x + R)$, we naturally get the conditions (b.4) as a sufficient condition for $(*)$ to have a nontrivial solutions.

We must remark that, for the case function $b(x)$ is contained in nonlinearity or higher dimensional cases, there exist non-trivial solutions without conditions like (b.4). In fact, Bahri-Li [**BaL**] showed that there exists a positive solution of

$$-\Delta u + u = (1 - b(x))|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N), \tag{0.2}$$

where $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies the following conditions:

(b.1)' $1 - b(x) \geq 0$ for all $x \in \mathbf{R}^N$.

(b.2)' $\lim_{|x|\to\infty} b(x) = 0$.

(b.3)' There exist $\beta_0 > 2$ and $C_0 > 0$ such that $b(x) \leq C_0 e^{-\beta_0|x|}$ for all $x \in \mathbf{R}^N$.

For one dimensional case, Spradlin [**Sp**] proved that there exists a positive solution of the equation

$$-u'' + u = (1 - b(x))f(u) \quad \text{in } \mathbf{R}, \qquad u \in H^1(\mathbf{R}). \tag{0.3}$$

They assumed that $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies $1 - b(x) \geq 0$ in $\mathbf{R}$ and (b.2)–(b.3) and $f(u)$ satisfies (f.1)–(f.3) and

(f.4) $\frac{f(u)}{u}$ is an increasing function for all $u > 0$.

Moreover, we can easily apply the computations in [**BaL**] to the following equation which is a higher dimensional version of $(*)$.

$$-\Delta u + (1 + b(x))u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N). \tag{0.4}$$

From this application, we see that (0.4) also has at least a positive solution when $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x)$ satisfies $1 + b(x) \geq 0$ in $\mathbf{R}^N$ and (b.2)'–(b.3)'.

From the above results, it seems that Theorem 0.1 holds without condition (b.4). However (b.4) is an essential assumption for $(*)$ to have non-trivial solutions. In what follows, we will show a result about the non-existence of nontrivial solutions for $(*)$.

In next our result, we will assume that $b(x)$ satisfies the following condition:

(b.5) There exist $\mu > 0$ and $m_2 \geq m_1 > 0$ such that

$$m_1 \mu e^{-\mu|x|} \leq b(x) \leq m_2 \mu e^{-\mu|x|} \quad \text{for all } x \in \mathbf{R}\,.$$

Here, we remark that, if (b.5) holds for $\mu > 2$, then $b(x)$ satisfies (b.1)–(b.3) and

$$\frac{2\mu}{\mu - 2} m_1 \leq \int_{-\infty}^{\infty} b(x) e^{2|x|}\, dx \leq \frac{2\mu}{\mu - 2} m_2.$$

Thus, when $m_2 < 1$ and μ is very large, the condition (b.4) also holds.

Our second result is the following:

Theorem 0.2. *Assume that (b.5) holds and* $f(u) = |u|^{p-1}u$ $(p > 1)$.

(i) If $m_1 > 1$, *there exists* $\mu_1 > 0$ *such that* $(*)$ *does not have non-trivial solution for all* $\mu \geq \mu_1$.

(ii) If $m_2 < 1$, *there exists* $\mu_2 > 0$ *such that* $(*)$ *has at least a non-trivial solution for all* $\mu \geq \mu_2$.

(iii) There exists $\mu_3 > 0$ *such that* $(*)$ *does not have sign-changing solutions for all* $\mu \geq \mu_3$.

From Theorem 0.2, we see that Theorem 0.1 does not hold except for condition (b.4). This is a drastically different situation from the higher dimensional cases. This is one of the interesting points in our results.

We remark that the condition (b.4) implies $\overline{\lim}_{r\to\infty} \int_{-r}^{r} b(x)\, dx < 2$ and the assumption of (ii) of Theorem 0.2 also means $\int_{-\infty}^{\infty} b(x)\, dx < 2$. Thus we expect that the difference from existence and non-existence of non-trivial solutions of $(*)$ depends on the quantity of integrate of $b(x)$.

We can obtain this expectation from another viewpoint, which is a perturbation problem. Setting $b_\mu(x) = m\mu e^{-\mu|x|}$, $b_\mu(x)$ satisfies (b.5) and, when $\mu \to \infty$, $b_\mu(x)$ converges to the delta function $2m\delta_0$ in distribution sense. Thus $(*)$ approaches to the equation

$$-u'' + (1 + 2m\delta_0)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad u \in H^1(\mathbf{R}), \tag{0.5}$$

in distribution sense. Here, if u is a solution of (0.5) in distribution sense, we can see that u is of C^2-function in $\mathbf{R}\setminus\{0\}$ and continuous in $\mathbf{R}$ and u satisfies

$$u'(+0) - u'(-0) = 2mu(0). \tag{0.6}$$

Moreover, since u is a homoclinic orbit of $-u''+u = f(u)$ in $(-\infty, 0)$ or $(0, \infty)$, respectively, u satisfies

$$-\frac{1}{2}u'(x)^2 + \frac{1}{2}u(x)^2 - \frac{1}{p+1}|u(x)|^{p+1} = 0 \quad \text{for} \quad x \neq 0. \tag{0.7}$$

When $x \to \pm 0$ in (0.7), from (f.1), we find

$$u'(-0) = -u'(+0), \quad |u'(\pm 0)| < |u(0)|. \tag{0.8}$$

Thus, from (0.6) and (0.8), it easily see that (0.5) has an unique positive solution when $|m| < 1$ and (0.5) has no non-trivial solutions when $|m| \geq 1$. Therefore we can regard Theorem 0.2 as results of a perturbation problem of (0.5).

To prove Theorem 0.2, we develop the shooting arguments which used in [**BE**]. Bianchi and Egnell [**BE**] argued about the existence and non-existence of radial solutions for

$$-\Delta u = K(|x|)|u|^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in } \mathbf{R}^N, \quad u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty. \tag{0.9}$$

Here $N \geq 3$ and $K(|x|)$ is a radial continuous function. Roughly speaking their approach, by setting $u(r) = u(|x|)$, they reduce (0.9) to an ordinary differential equation and considered solutions of two initial value problems of that ordinary differential equation which have initial conditions $u(0) = \lambda$ and $\lim_{r\to\infty} r^{N-2}u(r) = \lambda$. And, examining whether those solutions have suitable matchings at $r = 1$, they argued about the existence and non-existence of radial solutions. In this paper, we also consider two initial value problems for initial conditons $\lim_{x\to-\infty} e^{-x}u(x) = \lambda$ and $\lim_{x\to\infty} e^{x}u(x) = \lambda$. In order to prove Theorem 0.2, we examine whether those solutions has suitable matchings at $x = 0$.

We devote the next three sections to proofs of our theorems. In Section 1, we give a proof of Theorem 0.1. To prove it, we use a variational approach and it is important to estimate interaction of $\omega(x - R)$ and $\omega(x + R)$ for large $R >> 1$. Since the computation of this estimate is slightly complicated, we compute it in Section 2. In Section 3, we prove Theorem 0.2 by a shooting argument. (ii) of Theorem 0.2 directly follows from Theorem 0.1 but we can also get it as a by-product of our shooting argument.

1. The existence result

In this section, we state the proof of Theorem 0.1. We will developed a variational approach which was used in [**BaL**] and [**Sp**].

In what follows, since we seek positive solutions of $(*)$, without loss of generalities, we assume $f(u) = 0$ for $u < 0$. To prove Theorem 0.1, we seek non-trivial critical points of the functional

$$I(u) = \frac{1}{2}||u||^2_{H^1(\mathbf{R})} + \frac{1}{2}\int_{-\infty}^{\infty} b(x)u^2\,dx - \int_{-\infty}^{\infty} F(u)\,dx \in C^1(H^1(\mathbf{R}),\mathbf{R}),$$

whose critical points are positive solutions of $(*)$. Here we use the following notations:

$$||u||^2_{H^1(\mathbf{R})} = ||u'||^2_{L^2(\mathbf{R})} + ||u||^2_{L^2(\mathbf{R})},$$
$$||u||^p_{L^p(\mathbf{R})} = \int_{\mathbf{R}} |u|^p\,dx \quad \text{for} \quad p > 1.$$

From (f.1)–(f.2), we can see that $I(u)$ satisfies a mountain pass geometry, that is, $I(u)$ satisfies

(i) $I(0) = 0$.
(ii) There exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for all $||u||_{H^1(\mathbf{R})} = \rho$.
(iii) There exists $u_0 \in H^1(\mathbf{R})$ such that $I(u_0) < 0$ and $||u_0||_{H^1(\mathbf{R})} > \rho$.

From the mountain pass geometry (i)–(iii), we can define a standard minimax value $c > 0$ by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \tag{1.1}$$
$$\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbf{R}))\,|\,\gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

And, by a standard way, we can construct $(PS)_c$-sequence $(u_n)_{n=1}^\infty$, that is, $(u_n)_{n=1}^\infty$ satisfies

$$I(u_n) \to c \qquad (n \to \infty),$$
$$I'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbf{R}) \qquad (n \to \infty).$$

Moreover, since $(u_n)_{n=1}^\infty$ is bounded in $H^1(\mathbf{R})$ from (f.3), $(u_n)_{n=1}^\infty$ has a subsequence $(u_{n_j})_{j=1}^\infty$ which weakly converges to some u_0 in $H^1(\mathbf{R})$. If $(u_{n_j})_{j=1}^\infty$ strongly converges to u_0 in $H^1(\mathbf{R})$, c is a non-trivial critical value of $I(u)$ and our proof is completed. However, since the embedding $L^p(\mathbf{R}) \subset H^1(\mathbf{R})$ $(p > 1)$ is not compact, there may not exist a subsequence $(u_{n_j})_{j=1}^{\infty}$ which strongly converges in $H^1(\mathbf{R})$. Therefore, in our situation, we don't know c is a critical value.

In our situation, from the lack of the compactness mentioned the above, we must use the concentration-compactness approach as [**BaL**] and [**Sp**]. In the concentration-compactness approach, we examine in detail what happens in bounded (PS)-sequences. When we state the concentration-compactness argument for the (PS)-sequences of $I(u)$, the limit problem (0.1) plays an important role. Setting

$$J(u) = \frac{1}{2}||u||^2_{H^1(\mathbf{R})} - \int_{-\infty}^{\infty} F(u)\, dx \in C^1(H^1(\mathbf{R}), \mathbf{R}),$$

the critical points of $I_0(u)$ correspond to the solutions of limit problem (0.1). The equation (0.1) has an unique positive solution, identifying ones which obtain by translations. Thus let $\omega(x)$ be an unique positive solution of (0.1) with $\max_{x\in\mathbf{R}} \omega(x) = \omega(0)$ and we set $c_0 = J(\omega)$. Since J also satisfies the mountain pass geometry (i)–(iii), we see $c_0 > 0$ and c_0 is an unique non-trivial critical value.

For the bounded (PS)-sequences of $I(u)$, we have the following:

Proposition 1.1. *Suppose (b.1)–(b.2) and (f.1)–(f.2) holds. If* $(u_n)_{n=1}^{\infty}$ *is a bounded (PS)-sequence of* $I(u)$*, then there exist a subsequence* $n_j \to \infty$*,* $k \in \mathbf{N} \cup \{0\}$*, k-sequences* $(x_j^1)_{j=1}^{\infty}, \cdots, (x_j^k)_{j=1}^{\infty} \subset \mathbf{R}$*, and a critical point* u_0 *of* $I(u)$ *such that*

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$\left|\left| u_{n_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x_j^\ell) \right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

Proof. See [**JT1**].

If the minimax value c satisfies $c \in (0, c_0)$, from Proposition 1.1, we see that $I(u)$ has at least a non-trivial critical point. In fact, let $(u_n)_{n=1}^{\infty}$ be a bounded $(\mathrm{PS})_c$-sequence of $I(u)$, from Proposition 1.1, there exists a subsequence $n_j \to \infty$, $k \in \mathbf{N} \cup \{0\}$ and a critical point u_0 of $I(u)$ such that

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty).$$

Here, if $u_0 = 0$, we get $I(u_{n_j}) \to kc_0$ as $j \to \infty$. However this contradicts to the fact that $I(u_n) \to c \in (0, c_0)$ as $n \to \infty$. Thus $u_0 \neq 0$ and u_0 is a non-trivial critical point of $I(u)$. From the above argument, we have the following corollary.

Corollary 1.2. *Suppose $I(u)$ has no non-trivial critical points and let $(u_n)_{n=1}^{\infty}$ be a (PS)-sequence of $I(u)$. Then, only kc_0's ($k \in \mathbf{N} \cup \{0\}$) can be limit points of $\{I(u_n) \,|\, n \in \mathbf{N}\}$.*

Remark 1.3. Corollary 1.2 essentially depends on the uniqueness of the positive solution of (0.1).

As mentioned the above, when $c \in (0, c_0)$, $I(u)$ has at least a non-trivial critical point. However, unfortunately, under the condition (b.1)–(b.4), it may be $c = c_0$. Thus we need consider another minimax value. To define another minimax value, we use a path $\gamma_0(t) \in C(\mathbf{R}, H^1(\mathbf{R}))$ which is defined as follows: for small $\epsilon_0 > 0$, we set

$$h(x) = \begin{cases} \omega(x) & x \in [0, \infty], \\ x^4 + u_0 & x \in [-\epsilon_0, 0), \\ \epsilon_0^4 + u_0 & x \in (-\infty, -\epsilon_0), \end{cases}$$

$$\gamma_0(t)(x) = \begin{cases} h(x-t) & x \geq 0, \\ h(-x-t) & x < 0. \end{cases}$$

Here, we remark that u_0 was given in (f.2) and $u_0 = \max_{x \in \mathbf{R}} u(x) = u(0)$. This path $\gamma_0(t)$ was introduced in [**JT2**]. Choosing a proper $\epsilon_0 > 0$ sufficiently small, $\gamma_0(t)$ achieves the mountain pass value of $I_0(u)$ and satisfies the followings:

Lemma 1.4. *Suppose (f.1)–(f.2) hold. Then $\gamma_0(t)$ satisfies*

(i) $\gamma_0(0)(x) = \omega(x)$.

(ii) $J(\gamma_0(t)) < J(\omega) = c_0$ *for all* $t \neq 0$.

(iii) $\lim_{t \to -\infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = 0$, $\lim_{t \to \infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = \infty$.

Proof. See [**JT2**].

Now, for $R > 0$, we consider a path $\gamma_R \in C(\mathbf{R}^2, H^1(\mathbf{R}))$ which is defined by

$$\gamma_R(s,t)(x) = \max\{\gamma_0(s)(x+R),\ \gamma_0(t)(x-R)\}.$$

In our proof of Theorem 0.1, the following proposition is a key proposition.

Proposition 1.5. *Suppose (b.1)–(b.3) and (f.1)–(f.2) hold. Then, for any $L > 0$, we have*

$$\lim_{R \to \infty} e^{2R} \left\{ \max_{(s,t) \in [-L,L]^2} I(\gamma_R(s,t)) - 2c_0 \right\} \leq \frac{\lambda_0^2}{2} \left(\overline{\lim_{r \to \infty}} \int_{-r}^{r} b(x) e^{2|x|}\, dx - 2 \right). \tag{1.2}$$

Here $\lambda_0 = \lim_{|x| \to \infty} \omega(x) e^{|x|}$.

We will give the proof of Proposition 1.5 in next Section 2. By using a translation, without loss of generalities, we assume $x_0 = 0$ in (b.4). If (b.4) with $x_0 = 0$ holds, from Proposition 1.5, for any $L > 0$, there exists $R_0 > 0$ such that

$$\max_{(s,t) \in [-L,L]^2} I(\gamma_{R_0}(s,t)) < 2c_0.$$

To prove the Theorem 0.1, we also need a map $m : H^1(\mathbf{R})\setminus\{0\} \to \mathbf{R}$ which is defined by the following: for any $u \in H^1(\mathbf{R})\setminus\{0\}$, a function

$$T_u(s) = \int_{-\infty}^{\infty} \tan^{-1}(x-s)|u(x)|^2\,dx : \mathbf{R} \to \mathbf{R}$$

is strictly decreasing and $\lim_{s\to\infty} T_u(s) = -||u||^2_{L^2(\mathbf{R})} < 0$ and $\lim_{s\to-\infty} T_u(s) = ||u||^2_{L^2(\mathbf{R})} > 0$. Thus, from the theorem of the intermediate value, $T_u(s)$ has an unique $s = m(u)$ such that $T_u(m(u)) = 0$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u,s) \mapsto T_u(s)$. The map $m(u)$ was introduced in [**Sp**]. We remark that $m(u)$ is regarded as a kind of center of mass of $|u(x)|^2$ and we can check the followings.

Lemma 1.6. *We have*

(i) $m(\gamma_0(t)) = 0$ *for all* $t \in \mathbf{R}$.

(ii) $m(\gamma_R(s,t)) > 0$ *for all* $-R < s < t < R$.

(iii) $m(\gamma_R(s,t)) < 0$ *for all* $-R < t < s < R$.

Proof. Since $\gamma_0(t)(x)$ is a even function, we have (i). We remark that

$$\gamma_R(s,t)(x) = \begin{cases} \gamma_0(s)(x+R) & \text{for } x \in (-\infty, \frac{s-t}{2}], \\ \gamma_0(t)(x-R) & \text{for } x \in (\frac{s-t}{2}, \infty). \end{cases} \tag{1.3}$$

Since $\gamma_R(s,s)(x)$ is also a even function, we have

$$m(\gamma_R(s,s)) = 0 \quad \text{for all} \quad s \in \mathbf{R},$$

and we get (ii)–(iii). ∎

In what follows, we will complete the proof of Theorem 0.1.

Proof of Theorem 0.1. First of all, we defined a minimax value $c_1 > 0$ by

$$c_1 = \inf_{\gamma\in\Gamma_1} \max_{t\in[0,1]} I(\gamma(t)),$$
$$\Gamma_1 = \{\gamma(t) \in C([0,1], H^1(\mathbf{R}))\,|\,\gamma(0) = 0,\ I(\gamma(1)) < 0,\ |m(\gamma(t))| < 1\}.$$

Noting $\Gamma_1 \subset \Gamma$, we have

$$0 < c \le c_1.$$

Since Γ_1 is not invariant by standard deformation flows of $I(u)$, c_1 may not be a critical point of $I(u)$. We will use c_1 to divide the case. We divide the case into the following three cases:

(i) $c_1 < c_0$.

(ii) $c_1 = c_0$.

(iii) $c_1 > c_0$.

Proof of Theorem 0.1 for the case (i). Since the inequality $c_1 < c_0$ implies $0 < c < c_0$, from Corollary 1.2, we can see $I(u)$ has at least a non-trivial critical point. ∎

Proof of Theorem 0.1 for the case (ii). In this case, if $c < c_1 = c_0$, then $I(u)$ has at least a non-trivial critical point from Corollary 1.2. Thus we may consider the case $c = c_1 = c_0$. In this case, for any $\epsilon > 0$, there exists $\gamma_\epsilon(t) \in \Gamma_1$ such that

$$c \leq \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon.$$

Since $\gamma_\epsilon \in \Gamma_1 \subset \Gamma$ and Γ is an invariant set by standard deformation flows of $I(u)$, by a standard Ekland principle, there exists $u_\epsilon \in H^1(\mathbf{R})$ such that

$$\begin{aligned}
&c \leq I(u_\epsilon) \leq \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon, \\
&||I'(u_\epsilon)|| < 2\sqrt{\epsilon}, \\
&\inf_{t\in[0,1]} ||u_\epsilon - \gamma_\epsilon(t)||_{H^1(\mathbf{R})} < \epsilon.
\end{aligned} \tag{1.4}$$

Then, from Proposition 1.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbf{N} \cup \{0\}$, k-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}$, and a critical point u_0 of $I(u)$ such that

$$\begin{aligned}
&I(u_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty), \qquad (1.5)\\
&\left|\left| u_{\epsilon_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x_j^\ell) \right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty), \\
&|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'), \\
&|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).
\end{aligned}$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (1.5), it must be $k = 1$. Thus, we have

$$\begin{aligned}
&\left|\left| u_{\epsilon_j}(x) - \omega(x - x_j^1) \right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty). \\
&|x_j^1| \to \infty \quad (j \to \infty).
\end{aligned} \tag{1.6}$$

On the other hand, we remark that, since $m(\omega) = 0$ and m is of continuous, there exists $\delta > 0$ such that

$$|m(u)| < 1 \quad \text{for all} \quad u \in B_\delta(\omega) = \{v \in H^1(\mathbf{R}) \,|\, ||v - \omega||_{H^1(\mathbf{R})} < \delta\}.$$

Thus, from (1.4) and (1.6), for some $\epsilon_0 \in (0, \frac{\delta}{2})$ and $t_0 \in [0,1]$, we have

$$|m(\gamma_{\epsilon_0}(t_0)) - x_j^1| < 1.$$

This contradicts to $\gamma_{\epsilon_0} \in \Gamma_1$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. ∎

Proof of the Theorem 0.1 for the case (iii). First of all, we set $\delta = \frac{c_1 - c_0}{2} > 0$ and choose $L_0 > 0$ such that

$$\max_{(s,t)\in \overline{D_{2L_0}\setminus D_{L_0}}} I(\gamma_R(s,t)) < c_0 + \delta < c_1 \quad \text{for all} \quad R > 3L_0. \tag{1.7}$$

Here we set $D_L = [L, L]\times[L, L] \subset \mathbf{R}^2$. Next, from Proposition 1.5, we can choose $R_0 > 3L_0$ such that

$$\max_{(s,t)\in D_{L_0}} I(\gamma_{R_0}(s,t)) < 2c_0. \tag{1.8}$$

Here we fix $\gamma_{R_0}(s,t)$ and define the following minimax value:

$$c_2 = \inf_{\gamma\in\Gamma_2} \max_{(s,t)\in D_{2L_0}} I(\gamma(s,t)),$$
$$\Gamma_2 = \{\gamma(s,t) \in C(D_{2L_0}, H^1(\mathbf{R})) \,|\, \gamma(s,t) = \gamma_{R_0}(s,t) \text{ for all } (s,t) \in D_{2L_0} \setminus D_{L_0}\}.$$

Then we have the following lemma.

Lemma 1.7. *We have*

$$0 < c_0 < c_1 \leq c_2 < 2c_0.$$

We postpone the proof of Lemma 1.7 to end of this section. If Lemma 1.7 is true, then Γ_2 is an invariant set by the deformation flows of $I(u)$. Thus $I(u)$ has a (PS)-sequence $(u_n)_{n=1}^{\infty}$ such that

$$I(u_n) \to c_2 \in (c_0, 2c_0) \quad (n \to \infty).$$

From Corollary 1.2, we can see that $I(u)$ must have at least a non-trivial critical point. Combining the proofs of the cases (i)–(iii), we complete a proof of Theorem 0.1. ∎

Finally we show Lemma 1.7.

Proof of Lemma 1.7. The inequality $c_0 < c_1$ is an assumption of the case (iii). From $\gamma_{R_0} \in \Gamma_2$ and (1.7)–(1.8), $c_2 < 2c_0$ is obvious. Thus we show $c_1 \leq c_2$. For any $\gamma(s,t) \in \Gamma_2$, we have

$$m(\gamma(s,t)) > 0 \quad \text{for all} \quad (s,t) \in D_1, \tag{1.9}$$
$$m(\gamma(s,t)) < 0 \quad \text{for all} \quad (s,t) \in D_2. \tag{1.10}$$

Here we set $D_1 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s < t\}$ and $D_2 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s > t\}$. From (1.9)–(1.10), a set $\{(s,t) \in D_{2L_0} \,|\, |m(\gamma(s,t))| < 1\}$ have a connected component which contains a path joining two points $\gamma_{R_0}(-2L_0,-2L_0)$ and $\gamma_{R_0}(2L_0,2L_0)$. Thus we construct a path $\gamma_1(t) \in \Gamma_1$ such that

$$\{\gamma_1(t) \,|\, t \in [1/3,2/3]\} \subset \{\gamma(s,t) \,|\, (s,t) \in D_{2L_0}\},$$
$$\max_{t\in[0,1/3]\cup[2/3,1]} I(\gamma_1(t)) \le c_0.$$

Thus we see

$$\begin{aligned} c_1 &\le \max_{t\in[0,1]} I(\gamma_1(t)) \\ &\le \max_{(s,t)\in D_{2L_0}} I(\gamma(s,t)). \end{aligned} \tag{1.11}$$

Since $\gamma(s,t) \in \Gamma_2$ is arbitrary, from (1.11), we have

$$c_1 \le c_2.$$

Thus we get Lemma 1.7. ∎

2. The proof of Proposition 1.5.

In this section, we fix a $L > 0$ and give a proof of Proposition 1.5. To prove Proposition 1.5, we need estimate $I(\gamma_R(s,t))$ for $(s,t) \in [-L,L]^2$ and large $R > 0$. We use the following notation: for an interval $(a,b) \subset \mathbf{R}$, we set

$$J_{(a,b)}(u) = \frac{1}{2}||u||^2_{H^1(a,b)} - \int_a^b F(u)\,dx.$$

Then, we note that

$$I(u) = J(u) + \frac{1}{2}\int_{-\infty}^{\infty} b(x)u^2\,dx, \tag{2.1}$$

$$J_{(-\infty,\ell)}(\gamma_0(s)(x+R)) = J_{(-\infty,R+\ell)}(\gamma_0(s)) = J(\gamma_0(s)) - J_{(R+\ell-s,\infty)}(\omega)$$
$$\text{for all } (s,\ell) \in [-L,L]^2 \text{ and } R > 2L. \tag{2.2}$$

Now, from (2.1)–(2.2) and (1.3), $I(\gamma_R(s,t))$ is written as follows:

$$\begin{aligned} I(\gamma_R(s,t)) &= J(\gamma_R(s,t)) + \frac{1}{2}\int_{-\infty}^{\infty} b(x)\gamma_R(s,t)^2\,dx \\ &= J_{(-\infty,\frac{s-t}{2})}(\gamma_0(s)(x+R)) + J_{(\frac{s-t}{2},\infty)}(\gamma_0(t)(x-R)) \\ &\quad + \frac{1}{2}\int_{-\infty}^{\frac{s-t}{2}} b(x)\gamma_0(s)(x+R)^2\,dx + \frac{1}{2}\int_{\frac{s-t}{2}}^{\infty} b(x)\gamma_0(t)(x-R)^2\,dx \\ &= J(\gamma_0(s)) + J(\gamma_0(t)) - 2J_{(R-\frac{s+t}{2},\infty)}(\omega) \\ &\quad + \frac{1}{2}\int_{-\infty}^{\frac{s-t}{2}} b(x)\gamma_0(s)(x+R)^2\,dx + \frac{1}{2}\int_{\frac{s-t}{2}}^{\infty} b(x)\gamma_0(t)(x-R)^2\,dx. \end{aligned} \tag{2.3}$$

Thus Proposition 1.5 can be obtained from the following three lemmas.

Lemma 2.1. *There exists $a_1 > 0$ such that*

$$J(\gamma_0(t)) \le c_0 - a_1|t|^5 \quad \textit{for all} \quad |t| \le 1.$$

Proof. We set $g(t) = J(\gamma_0(t))$. From Lemma 1.4, we remark that $g(t) < c_0$ $(t \ne 0)$ and

$$g(t) = \begin{cases} c_0 - \int_t^{-t} \frac{1}{2}|\omega'|^2 + \frac{1}{2}|\omega|^2 - F(\omega)\, dx & -1 \ll t < 0, \\ c_0 + \int_{-t}^{t} \frac{1}{2}|4x^3|^2 + \frac{1}{2}|u_0 + x^4|^2 - F(u_0 + x^4)\, dx & 0 < t \ll 1. \end{cases}$$

When $-1 \ll t < 0$, by directly differentiating $g(t)$, we see that

$$\lim_{t \to -0} g'(t) = \lim_{t \to -0} g''(t) = 0 \quad \text{and} \quad \lim_{t \to -0} g'''(t) = 4(u_0 - f(u_0))^2 > 0. \tag{2.4}$$

When $0 < t \ll 1$, from the mean value theorem, for some $\theta = \theta_x \in (0,1)$, we have

$$g(t) = c_0 + \int_{-t}^{t} 8x^6 + \frac{1}{2}x^8 + x^4(u_0 - f(u_0 + \theta x^4))\, dx. \tag{2.5}$$

Noting $f(u_0) > u_0$, from (2.4)–(2.5), we obtain Lemma 2.1. ∎

For the unique solution $\omega(x)$ of (0.1), we set

$$\lambda_0 = \lim_{|x| \to \infty} \omega(x) e^{|x|}. \tag{2.6}$$

Then $\lambda_0 > 0$ and we also have

$$\pm\lambda_0 = \lim_{x \to \pm\infty} \omega'(x) e^{|x|}. \tag{2.7}$$

Lemma 2.2. *For $\ell \in [-L, L]$, we have*

$$\lim_{R \to \infty} e^{2R} J_{(R+\ell,\infty)}(\omega) = \frac{\lambda_0^2}{2} e^{-2\ell}. \tag{2.8}$$

Moreover, the convergence as $R \to \infty$ in (2.8) is uniformly with respect to $\ell \in [-L, L]$.

Proof. Firstly, we show

$$\lim_{R \to \infty} e^{2R} \int_{R+\ell}^{\infty} \omega(x)^2\, dx = \frac{\lambda_0^2}{2} e^{-2\ell}. \tag{2.9}$$

From (2.6), for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$(\lambda_0 - \epsilon)^2 e^{-2x} \le \omega(x)^2 \le (\lambda_0 + \epsilon)^2 e^{-2x} \quad \text{for all} \quad x \ge R_\epsilon.$$

Thus, for $R > R_\epsilon + L$, integrating the above inequality from $R+\ell$ to ∞, we have

$$\frac{1}{2}(\lambda_0-\epsilon)^2 e^{-2(R+\ell)} \le \int_{R+\ell}^{\infty} \omega(x)^2\,dx \le \frac{1}{2}(\lambda_0+\epsilon)^2 e^{-2(R+\ell)}.$$

Since $\epsilon > 0$ is arbitrary, we get (2.9). By similar computations, we find

$$\lim_{R\to\infty} e^{2R}\int_{R+\ell}^{\infty} |\omega'(x)|^2\,dx = \frac{\lambda_0^2}{2}e^{-2\ell}, \tag{2.10}$$

$$\lim_{R\to\infty} e^{2R}\int_{R+\ell}^{\infty} F(\omega(x))\,dx = 0. \tag{2.11}$$

Combining (2.9)–(2.11), we get Lemma 2.2. ∎

Lemma 2.3. *For $(s,\ell)\in[-L,L]^2$, we have*

$$\lim_{R\to\infty} e^{2R}\int_{-\infty}^{\ell} b(x)\gamma_0(s)(x+R)^2\,dx = \lambda_0^2 e^{2s}\overline{\lim_{r\to\infty}}\int_{-r}^{\ell} b(x)e^{-2x}\,dx. \tag{2.12}$$

Moreover, the convergence as $R\to\infty$ in (2.12) is uniformly with respect to $(s,\ell)\in[-L,L]^2$.

We postpone the proof of Lemma 2.3 to end of this section. By using the above three lemmas, we can prove Proposition 1.5.

Proof of Proposition 1.5. From (2.3) and Lemma 2.1, for any $(s,t)\in[-L,L]^2$ and $R > 2L$, we have

$$\begin{aligned} I(\gamma_R(s,t)) - 2c_0 \le{}& -a_1(|s|^5+|t|^5) - 2J_{(R-\frac{s+t}{2},\infty)}(\omega) \\ &+\frac{1}{2}\int_{-\infty}^{\frac{s-t}{2}} b(x)\gamma_0(s)(x+R)^2\,dx + \frac{1}{2}\int_{\frac{s-t}{2}}^{\infty} b(x)\gamma_0(t)(x-R)^2\,dx. \end{aligned} \tag{2.13}$$

Here, let $(s_R,t_R)\in[-L,L]^2$ be a maximum point of $(s,t)\mapsto I(\gamma_R(s,t))$. If $\underline{\lim}_{R\to\infty}(|s_R|+|t_R|) > 0$, then, from Lemma 2.2, Lemma 2.3 and (2.13), we get

$$\lim_{R\to\infty} e^{2R}\{I(\gamma_R(s_R,t_R)) - 2c_0\} = -\infty.$$

Thus (1.2) holds. On the other hand, if $\underline{\lim}_{R\to\infty}(|s_R|+|t_R|) = 0$, then, we get

$$\lim_{R\to\infty} e^{2R}\{I(\gamma_R(s_R,t_R)) - 2c_0\} \le \frac{\lambda_0^2}{2}\left(\overline{\lim_{r\to\infty}}\int_{-r}^{r} b(x)e^{2|x|}\,dx - 2\right).$$

Therefore (1.2) holds and we completed a proof of Proposition 1.5. ∎

Finally we show Lemma 2.3.

Proof of Lemma 2.3. We fix a $r \in [L, \infty)$. From (2.6), for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\left| \omega(x)^2 e^{2|x|} - \lambda_0^2 \right| < \epsilon \quad \text{for all} \quad |x| \geq R_\epsilon. \tag{2.14}$$

For $R > R_\epsilon + L + r$ and $(s, \ell) \in [-L, L]^2$, we divide the integration into two ones as follows:

$$\begin{aligned} & e^{2R} \int_{-\infty}^{\ell} b(x)\gamma_0(s)(x+R)^2 \, dx \\ & \quad = e^{2R} \int_{-r}^{\ell} b(x)\omega(x+R-s)^2 \, dx + e^{2R} \int_{-\infty}^{-r} b(x)\gamma_0(s)(x+R)^2 \, dx \\ & \quad = (I) + (II). \end{aligned}$$

For (I), from (2.14), we have

$$\begin{aligned} \left| (I) - \lambda_0^2 e^{2s} \int_{-r}^{\ell} b(x) e^{-2x} \, dx \right| &= \left| \int_{-r}^{\ell} b(x) \left[(\omega(x+R-s)^2 e^{2x+2R-2s} - \lambda_0^2 \right] e^{-2x+2s} \, dx \right| \\ &\leq \epsilon e^{4L} ||b||_{L^1(-r,\ell)}. \end{aligned} \tag{2.15}$$

We remark that, from the definition of $\gamma_0(s)$ and (2.6), there exists $C_1 > 0$ such that

$$|\gamma_0(s)(x)| \leq C_1 e^{-|x|} \quad \text{for all} \quad x \in \mathbf{R}, \ s \in [-L, L]. \tag{2.16}$$

For (II), from (2.16) and (b.3), we find

$$\begin{aligned} |(II)| &\leq e^{2R} \int_{-\infty}^{-r} C_0 e^{\beta_0 x} C_1^2 e^{-2x-2R} \, dx \\ &= \frac{C_0 C_1^2}{\beta_0 - 2} e^{-(\beta_0 - 2)r}. \end{aligned} \tag{2.17}$$

Thus, from (2.15) and (2.17), we get

$$\left| \lim_{R \to \infty} e^{2R} \int_{-\infty}^{\ell} b(x)\gamma_0(s)(x+R)^2 \, dx - \lambda_0^2 e^{2s} \int_{-r}^{\ell} b(x) e^{-2x} \, dx \right| \leq \frac{C_0 C_1^2}{\beta_0 - 2} e^{-(\beta_0 - 2)r}. \tag{2.18}$$

Since (2.18) holds for any $r \in [L, \infty)$, we obtain Lemma 2.3 as $r \to \infty$. ∎

3. The non-existence result.

In this section, we assume $f(u) = |u|^{p-1}u$ $(p > 1)$ and give a proof of Theorem 0.2. To prove Theorem 0.2, we develop a shooting argument which used in [**BE**]. We remark that (ii) of Theorem 0.2 directly follows from Theorem 0.1. Thus we may show only (i) and (iii) of Theorem 0.2 but we can get (ii) as a by-product of our shooting argument. The framework of our shooting argument is available if $b(x)$ satisfies the following condition.

(b.6) There exists $\delta > 0$ such that $\lim_{|x| \to \infty} b(x) e^{\delta |x|} = 0$.

First of all, to consider our shooting argument, we show the following propositon.

Proposition 3.1. *Suppose (b.6) holds and let $u(x)$ be a solution of $(*)$. Then there exist $\lambda_-, \lambda_+ > 0$ such that*

$$\lim_{x\to-\infty} u(x)e^{-x} = \lambda_-, \quad \lim_{x\to\infty} u(x)e^{x} = \lambda_+, \tag{3.1}$$

$$\lim_{x\to-\infty} u'(x)e^{-x} = \lambda_-, \quad \lim_{x\to\infty} u'(x)e^{x} = -\lambda_+. \tag{3.2}$$

Proof. Firstly we show (3.1). For any $\beta_1 \in (\max\{\frac{1}{1+\eta_0}, 1-\delta\}, 1)$, by the comparison theorem, we see that, for some $C_1 > 0$, $u(x)$ satisfies

$$|u(x)| \le C_1 e^{-\beta_1|x|} \quad \text{for all} \quad x \in \mathbf{R}.$$

Thus, for $\beta_2 = \min\{(1+\eta_0)\beta_1, \delta+\beta_1\} > 1$ and some $C_2 > 0$, $u(x)$ satisfies

$$\begin{aligned} &-u'' + u = f(u) - b(x)u \quad \text{in}\, \mathbf{R}, \\ &|f(u) - b(x)u| \le C_2 e^{-\beta_2|x|} \quad \text{for all} \quad x \in \mathbf{R}. \end{aligned}$$

Therefore, using again the comparison theorem, we get (3.1). Next we show (3.2). We remark that the solution $u(x)$ of $(*)$ satisfies

$$-\frac{1}{2}u'(x)^2 + \frac{1}{2}u(x)^2 - F(u(x)) = -\int_{-\infty}^{x} b(t)u(t)u'(t)\,dx = \int_{x}^{\infty} b(t)u(t)u'(t)\,dx. \tag{3.3}$$

Noting $u \in H^1(\mathbf{R})$, from (3.1) and (3.3), for $\beta_3 = \min\{1+\delta, 2\} \in (1,2]$ and some $C_3 > 0$, $u'(x)$ satisfies

$$|u'(x)|^2 \le C_3 e^{-\beta_3|x|} \quad \text{for all} \in \mathbf{R}. \tag{3.4}$$

Again using (3.1), (3.3) and (3.4), for $\beta_4 = \min\{1+\delta+\frac{\beta_3}{2}, 2\} \in (1,2]$ and some $C_4 > 0$, $u'(x)$ satisfies

$$|u'(x)|^2 \le C_4 e^{-\beta_4|x|} \quad \text{for all} \in \mathbf{R}.$$

Thus by a iteration argument, for some $C_5 > 0$, $u'(x)$ satisfies

$$|u'(x)| \le C_5 e^{-|x|} \quad \text{for all} \in \mathbf{R}. \tag{3.5}$$

Now (3.2) follows from (3.1), (3.3) and (3.5). Therefore we get Proposition 3.1. ∎

In what follows, for $\lambda \in \mathbf{R}$, we would like to consider the following two initial value problems:

$$\begin{aligned} &-u'' + (1+b(x))u_- = |u|^{p-1}u, \\ &\lim_{x\to-\infty} e^{-x}u(x) = \lim_{x\to-\infty} e^{-x}u'(x) = \lambda, \end{aligned} \tag{3.6}$$

and

$$
\begin{aligned}
&-u'' + (1 + b(x))u = |u|^{p-1}u, \\
&\lim_{x\to\infty} e^x u(x) = -\lim_{x\to\infty} e^x u(x) = \lambda.
\end{aligned}
\tag{3.7}
$$

To justify the above initial value problems, we need the existence and uniqueness of solutions of (3.6) and (3.7). Moerover, to use a shooting argument, we need the continuity of solutions with respect to initial value λ. Since we are not used to treating the problems for the initial condition of (3.6) or (3.7), we change (3.6) and (3.7) into another problems. For (3.6), setting

$$
w(y) = \frac{1}{y} u(\log y), \tag{3.8}
$$

then (3.6) becomes to the following initial value problem:

$$
\begin{aligned}
&-y^3 w'' - 3y^2 w' + b_-(y) y w = y^p |w|^{p-1} w_{\pm}, \\
&w(0) = \lambda, \quad \lim_{y\to+0} y w'(y) = 0.
\end{aligned}
\tag{3.9}
$$

Here we set $b_-(y) = b(\log y)$. By a similar way, for (3.7), setting $w(y) = \frac{1}{y} u(-\log y)$, the problem (3.7) changes to the initial value problem

$$
\begin{aligned}
&-y^3 w'' - 3y^2 w' + b_+(y) y w = y^p |w|^{p-1} w, \\
&w(0) = \lambda, \quad \lim_{y\to+0} y w'(y) = 0.
\end{aligned}
\tag{3.10}
$$

Here we set $b_+(y) = b(-\log y)$. We remark that (b.6) implies

$$
\lim_{y\to+0} y^{-\delta} b_{\pm}(y) = 0. \tag{3.11}
$$

Remarking (3.11), if (b.6) holds, then we can easily find that the initial values problem (3.9) and (3.11) are equivalent to the following integral equations respectively:

$$
w(y) = T_{\pm}(w)(y) = \lambda + \int_0^y \frac{1}{s^3} \int_0^s b_{\pm}(t) t w(t) - t^p |w|^{p-1} w \, dt ds. \tag{3.12}
$$

Then, by the following proposition, we can use a suitable shooting argument.

Proposition 3.2. *Suppose (b.6) holds. Then for any* $\lambda \in \mathbf{R}$, (3.9) *and* (3.10) *have an unique solution* $w_-(y;\lambda)$ *and* $w_+(y;\lambda)$ *in* $[0,1]$ *respectively. Moreover* $\lambda \mapsto w_{\pm}(x;\lambda) : \mathbf{R} \to C([0,1]) \cap C^1((0,1])$ *are of continuous.*

Proof. We set

$$
X = \{ w \in C([0,\epsilon]) \mid |w(y) - \lambda| \le 1 \text{ for all } y \in [0,\epsilon] \}.
$$

Then, for sufficient small $\epsilon > 0$, $T_{\pm}$ are contraction maps on X. Thus (3.9) and (3.10) have an unique solution in $[0,\epsilon]$. By a standard way, we can extend those solutions to $[0,1]$ and get the continuity with respect to initial values. ∎

From Proposition 3.2, we have the following corollary.

Corollary 3.3. *Suppose (b.6) holds. Then for any $\lambda \in \mathbf{R}$,* (3.6) *and* (3.7) *have an unique solution $u_-(x;\lambda)$ in $(-\infty, 0]$ and $u_+(x;\lambda)$ in $[0,\infty)$ respectively. Moreover*

(i) The maps $\lambda \mapsto u_-(x;\lambda) : \mathbf{R} \to C^1((-\infty,0])$ and $\lambda \mapsto u_+(x;\lambda) : \mathbf{R} \to C^1([0,\infty))$ are of continuous.

(ii) For $\lambda \in \mathbf{R}$, we have

$$u_-(0;\lambda) = w_-(1;\lambda), \quad u_-'(0;\lambda) = w_-'(1;\lambda) + w_-(1;\lambda), \tag{3.13}$$

$$u_+(0;\lambda) = w_+(1;\lambda), \quad u_+'(0;\lambda) = -w_+'(1;\lambda) - w_+(1;\lambda). \tag{3.14}$$

Proof. Remarking that (3.6) and (3.7) are equivalent to (3.9) and (3.10) respectively, Corollary 3.3 follows from Proposition 3.2. ∎

Now, we can construct a shooting argument. We set

$$\Gamma_- = \{(u_-(0;\lambda), u_-'(0;\lambda)) \in \mathbf{R}^2 \mid \lambda \neq 0\},$$
$$\Gamma_+ = \{(u_+(0;\lambda), u_+'(0;\lambda)) \in \mathbf{R}^2 \mid \lambda \neq 0\}.$$

Then whether Γ_- and Γ_+ have intersections is equivalent to whether $(*)$ has non-trivial solutions. Thus to prove Theorem 0.2, we may investigate intersections of Γ_- and Γ_+.

Remark 3.4. When $b(x) \equiv 0$, we can explicitly write the solution $\omega(x)$ of (0.1) with $\max_{x\in\mathbf{R}} \omega(x) = \omega(0)$ by

$$\omega(x) = [2(p+1)]^{\frac{1}{p-1}} \left[e^{\frac{p-1}{2}x} + e^{-\frac{p-1}{2}x} \right]^{-\frac{2}{p-2}}.$$

Thus, letting $u_{0-}(x;\lambda)$ and $u_{0+}(x;\lambda)$ be solutions of (3.6) and (3.7) with $b(x) \equiv 0$ respectively, we can also explicitly write as follows:

$$u_{0-}(x;\lambda) = u_{0+}(-x;\lambda) = \lambda e^x \left[\frac{\lambda^{p-1}}{2(p+1)} e^{(p-1)x} + 1 \right]^{-\frac{2}{p-1}} \quad \text{for all} \quad x \in \mathbf{R},\ \lambda > 0.$$

Here we setting

$$\varphi(\lambda) = u_{0-}(0;\lambda) = u_{0+}(0;\lambda),$$
$$\psi(\lambda) = u_{0-}'(0;\lambda) = -u_{0+}'(0;\lambda),$$

we also see that

$$\begin{aligned}\Gamma_- = \Gamma_+ &= \{(\varphi(\lambda), \psi(\lambda)) \in \mathbf{R}^2 \mid \lambda \neq 0\} \\ &= \left\{ (u,v) \in \mathbf{R}^2 \setminus \{(0,0)\} \,\middle|\, -\frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{1}{p+1}|u|^{p+1} = 0 \right\}.\end{aligned}$$

We remark that $\Gamma_- = \Gamma_+$ corresponds to the fact that (0.1) has infinite many solution $\{\omega(x+\ell) \mid \ell \in \mathbf{R}\}$.

To investigate $\Gamma_- \cap \Gamma_+$, the following propositon is a key.

Proposition 3.5. *Assume that (b.5) holds. Then for any $\epsilon > 0$ there exists $\mu_3 > 0$ such that for all $\mu \geq \mu_3$, $\lambda > 0$ and $\sigma \in \{-,+\}$ we have*

$$\varphi(\lambda) \leq u_\sigma(0;\lambda) \leq (1-\epsilon)\varphi(\lambda) \tag{3.15}$$

$$(m_1-\epsilon)\varphi(\lambda) + \psi(\lambda) \leq -\sigma u_\sigma'(0;\lambda) \leq \psi(\lambda) + (m_2-\epsilon)\varphi(\lambda) \tag{3.16}$$

Here $\varphi(\lambda)$ and $\psi(\lambda)$ were given in Remark 3.4.

From Proposition 3.5, we immediately get the following corollary.

Corollary 3.6. *Assume that (b.5) holds.*

(i) If $m_1 > 1$, there exists $\mu_1 > 0$ such that $\Gamma_- \cap \Gamma_+ = \emptyset$ for all $\mu \geq \mu_1$.

(ii) If $m_2 < 1$, there exists $\mu_2 > 0$ such that $\Gamma_- \cap \Gamma_+ \neq \emptyset$ for all $\mu \geq \mu_2$.

Proof. Firsty, we show (i). We remark that by direct caluculations, we have

$$\psi(\lambda) = \varphi(\lambda)\frac{2(p+1)-\lambda^{p-1}}{2(p+1)+\lambda^{p-1}} \quad \text{for all} \quad \lambda > 0. \tag{3.17}$$

If $m_1 > 0$, from (3.17) and Proposition 3.5, there exists $\mu_1 > 0$ such that for all $\mu \geq \mu_1$, we have

$$u_-(0;\lambda) > 0, \quad u_-'(0;\lambda) > 0 \quad \text{for all} \quad \lambda > 0$$
$$u_+(0;\lambda) > 0, \quad u_+'(0;\lambda) < 0 \quad \text{for all} \quad \lambda > 0$$

Noting $u_\sigma(x;-\lambda) = -u_\sigma(x;\lambda)$ ($\sigma \in \{-,+\}$), this implies $\Gamma_- \cap \Gamma_+ = \emptyset$. Next we show (ii). We remark that

$$\lim_{\lambda\to+0}(\varphi(\lambda),\psi(\lambda)) = (+0,+0), \quad \lim_{\lambda\to\infty}(\varphi(\lambda),\psi(\lambda)) = (+0,-0). \tag{3.18}$$

and

$$\lim_{\lambda\to+0}\frac{\psi(\lambda)}{\varphi(\lambda)} = 1, \quad \lim_{\lambda\to\infty}\frac{\psi(\lambda)}{\varphi(\lambda)} = -1. \tag{3.19}$$

If $m_2 < 1$, from (3.18)–(3.19) and Proposition 3.5, there exists $\mu_2 > 0$ such that for all $\mu \geq \mu_2$, we have

$$\liminf_{\lambda\to+0}\frac{u_-'(0;\lambda)}{u_-(0;\lambda)} \in (1,\infty) \quad \limsup_{\lambda\to\infty}\frac{u_-'(0;\lambda)}{u_-(0;\lambda)} \in (-1,0),$$
$$\limsup_{\lambda\to+0}\frac{u_+'(0;\lambda)}{u_+(0;\lambda)} \in (-\infty,-1) \quad \liminf_{\lambda\to\infty}\frac{u_-'(0;\lambda)}{u_-(0;\lambda)} \in (0,1).$$

These imply $\Gamma_- \cap \Gamma_+ \neq \emptyset$.

We remark that (i)–(ii) of Theorem 0.2 are directly conclutions of Corollary 3.6. In the remaining parts of this section, we give proofs of Proposition 3.5. (iii) of Theorem 0.2 will also immediately follow from those proofs. We will show (3.15)–(3.16) for $u_-(0;\lambda)$, because we can show it for $u_+(0;\lambda)$ by a similar way. We use the following function:

$$w_0(y;\lambda) = \lambda\left[\frac{\lambda^{p-1}}{2(p+1)}y^{p-1}+1\right]^{-\frac{2}{p-1}} \quad \text{for} \quad y\in[0,1],\ \lambda>0. \tag{3.20}$$

The $w_0(y;\lambda)$ satisfies the initial value problem

$$\begin{aligned} &-y^3w''-3y^2w' = y^p|w|^{p-1}w,\\ &w(0)=\lambda, \quad \lim_{y\to+0} yw'(y)=0. \end{aligned} \tag{3.21}$$

We remark that (3.21) comes from (3.9) when $b(x)=0$.

Lemma 3.7. *$w_0(y;\lambda)$ satisfies*

(i) $0<y^2w_0(y;\lambda)\le w_0(1;\lambda)\le w_0(y;\lambda)$ *for all* $y\in[0,1]$, $\lambda>0$.

(ii) $yw_0(y;\lambda)\le\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}$ *for all* $y\in[0,1]$, $\lambda>0$.

(iii) $w_0(1;\lambda)=\varphi(\lambda)$ *and* $w_0'(1;\lambda)=\psi(\lambda)-\varphi(\lambda)$ *for all* $\lambda>0$.

Proof. These are conclusions from direct calculations for (3.20). ∎

Here we set

$$h(y;\lambda)=w_-(y;\lambda)-w_0(y;\lambda) \tag{3.22}$$

and show the following lemma.

Lemma 3.8. *Assume that (b.5) holds. Then for any $\epsilon>0$, there exists $\mu_3>0$ such that for all $\mu\ge\mu_3$ and $\lambda>0$ we have*

$$0<h(1;\lambda)\le\epsilon\varphi(\lambda), \tag{3.23}$$

$$(m_1-\epsilon)\varphi(\lambda)\le h'(1;\lambda)\le(m_2+\epsilon)\varphi(\lambda). \tag{3.24}$$

We can get (3.15) from Lemma 3.8.

Proof of Proposition 3.5. We will show only for $u_-(0;\lambda)$. From (3.13)–(3.14) and (3.22), we have

$$\begin{aligned} u_-(0;\lambda) &= w_0(1;\lambda)+h(1;\lambda)\\ u_-'(0;\lambda) &= w_0'(1;\lambda)+w_0(1;\lambda)+h(1;\lambda)+h'(1;\lambda). \end{aligned}$$

Thus (3.15)–(3.16) follow from (3.23)–(3.24) and (iii) of Lemma 3.7. Therefore Proposition 3.5 holds from Lemma 3.8. ∎

Now, we show Lemma 3.8.

Proof of Lemma 3.8. Since $w_-(y;\lambda)$ and $w_0(y;\lambda)$ satisfy (3.9) and (3.21) respectively, $h(y;\lambda)$ satisfies

$$\begin{gathered} -y^3h'' - 3y^2h' + b_-(y)(h+w_0) = y^p|h+w_0|^{p-1}(h+w_0) - y^pw_0^p \quad \text{in} \quad (0,1], \\ h(0)=0, \qquad \lim_{y\to+0} yh'(y) = 0. \end{gathered} \tag{3.25}$$

The differential equation (3.25) is equivalent to the integral equation

$$h(x) = S(h)(y) = \int_0^y \frac{1}{s^3}\int_0^s b_-(t)t(h+w_0) - t^p\left[|h+w_0|^{p-1}(h+w_0) - w_0^p\right]\,dtds. \tag{3.26}$$

We remark that, from Proposition 3.2, (3.26) has an unique solution $h(y;\lambda)$ in $(0,1]$. For $\epsilon > 0$, we set

$$\begin{aligned} &X = \left\{h \in C([0,1]) \;\middle|\; |h(y)| \le \epsilon\varphi(\lambda)y^{\mu-2}\right\}, \\ &X_\epsilon = \left\{h \in C^1([0,1]) \;\middle|\; 0 < h(y) \le \epsilon\varphi(\lambda)y^{\mu-2},\ (m_1-\epsilon)\varphi(\lambda) \le h'(1) \le (m_2+\epsilon)\varphi(\lambda)\right\}. \end{aligned}$$

Since X is a closed convex set and S is a compact operator on X, if S is a operator from X to $X_\epsilon \subset X$, then by the Schauder's fix point theorem we see that $h(y;\lambda) \in X_\epsilon$ which implies (3.23)–(3.24). Thus we may show that, for large $\mu > 0$, $S(h) \in X_\epsilon$ for all $h \in X$. We remark that (b.5) means

$$\mu m_1 y^\mu \le b_-(y) \le \mu m_2 y^\mu \quad \text{for all} \quad y \in [0,1]. \tag{3.27}$$

From (i), (iii) of Lemma 3.7 and (3.27), we have

$$\varphi(\lambda)\frac{m_1}{\mu+2}y^\mu \le \int_0^y \frac{1}{s^3}\int_0^s b_-(t)tw_0\,dtds \le \varphi(\lambda)\frac{m_2}{\mu-2}y^{\mu-2}. \tag{3.28}$$

From (3.27), there exists $M_1 > 0$ such that for $h \in X$, we have

$$\left|\int_0^y \frac{1}{s^3}\int_0^s b_-(t)th\,dtds\right| \le \epsilon\varphi(\lambda)\frac{M_1}{\mu^2}y^{2\mu-2}. \tag{3.29}$$

From (ii) of Lemma 3.7 and (3.27), there exists $M_2 > 0$ such that for $h \in X$, we have

$$\begin{aligned} &\left|\int_0^x \frac{1}{s^3}\int_0^s t^p\left[|h+w_0|^{p-1}(h+w_0) - w_0^p\right]\,dtds\right| \\ &\qquad \le p2^{p-1}\int_0^y \frac{1}{s^3}\int_0^s t^p\left[|h|^{p-1} + |w_0|^{p-1}\right]h\,dtds \\ &\qquad \le \epsilon\varphi(\lambda)\frac{M_2}{\mu^2}y^\mu. \end{aligned} \tag{3.30}$$

From (3.28)–(3.30), we see that

$$\varphi(\lambda)\left[\frac{m_1}{\mu+2}-\frac{\epsilon(M_1+M_2)}{\mu^2}\right]y^{\mu}\le S(h)(y)\le\varphi(\lambda)\left[\frac{m_2}{\mu-2}+\frac{\epsilon(M_1+M_2)}{\mu^2}\right]y^{\mu-2}. \quad (3.31)$$

Next we remark that

$$S(h)'(y)=\frac{1}{y^3}\int_0^y b_-(t)t(h+w_0)-t^p\left[|h+w_0|^{p-1}(h+w_0)-w_0^p\right]dtds.$$

By a similar computation, for some $M_3>0$, we get

$$\varphi(\lambda)\left[\frac{\mu m_1}{\mu+2}-\frac{\epsilon M_3}{\mu}\right]y^{\mu-1}\le S(h)'(y)\le\varphi(\lambda)\left[m_2+\frac{\epsilon M_3}{\mu}\right]y^{\mu-3}. \quad (3.32)$$

From (3.31)–(3.32), for large $\mu>0$ we see that $S(h)\in X_\epsilon$ for all $h\in X$. Therefore we completed a proof of Lemma 3.8. ∎

Proof of Theorem 0.2. Firstly, (i)–(ii) follow from Corollary 3.6. In the above proof in Lemma 3.8, $h(y)\in X_\epsilon$ implies $u_\sigma(x;\lambda)>0$ for all $\lambda>0$, $x\in\mathbf{R}$ and $\sigma\in\{-,+\}$. Thus we also see that (iii) holds. ∎

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