Multi-bubble solutions and the geometry of the domains: a survey

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Abstract. In this paper, we consider several types of semilinear elliptic equations with concentration phenomena. We will give a concise survey about the relation between the existence and/or non-existence of solutions with multiple blow up (or concentration) points and the geometry of the domain. This survey is based on a recent joint work of the author [13] with M. Grossi at Università di Roma "La Sapienza".

Keywords: blowing-up solution, Liouville equation, the Robin function.

2010 Mathematics Subject Classifications: 35J60. 35B40, 35J25.

1. Results.

Let Ω be a smooth bounded domain in $\mathbb{R}^N, N \geq 2$. In the following, G will denote the Green function of $-\Delta$ under the Dirichlet boundary condition

$$-\Delta_x G(x,y) = \delta_y(x),\ x \in \Omega, \quad G(x,y) = 0,\ x \in \partial\Omega$$

with a pole $y \in \Omega$, and

$$\Gamma(x,y) = \begin{cases} \frac{1}{2\pi} \log |x-y|^{-1}, & (N=2), \\ \frac{1}{(N-2)\sigma_N} |x-y|^{2-N}, & (N \geq 3) \end{cases}$$

the fundamental solution, where σ_N is a measure of the unit sphere of $\mathbb{R}^N$. Let

$$R(x) = \lim_{y \to x} \left[\Gamma(x,y) - G(x,y) \right]$$

denote the Robin function.

Among semilinear elliptic problems with concentration phenomena, first, we consider the *Liouville equation*

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $\lambda > 0$ is a parameter. The maximum principle implies any solution is positive on Ω. This kind of problem with exponential nonlinearity appears in many fields of mathematics, such as the study of prescribed Gauss curvature equation on a compact Riemann surface, Chern-Simons gauge theories, the vortex theory for the turbulent Euler flow, and so on, and it has attracted many authors for more than decades.

This simple-looking problem is shown to have much richer mathematical structure than expected before, and the following fundamental fact was proved by Nagasaki and Suzuki [16] around 1989, which may be considered as a concrete example of the general principle of concentration-compactness alternatives by P. L. Lions [18] [19] in two-dimensional critical problems.

Proposition 1 (Nagasaki-Suzuki [16]) *Let u_{λ_n} be a solution sequence of (1.1) for $\lambda = \lambda_n \downarrow 0$. Then $\lambda_n \int_\Omega e^{u_{\lambda_n}} dx$ accumulates only on values $8\pi l$ for some $l \in \{0\} \cup \mathbb{N} \cup \{+\infty\}$ (mass quantization). According to these values, the subsequence of solutions $\{u_{\lambda_n}\}$ behaves as follows:*

(a) If $l = 0$, then $\|u_{\lambda_n}\|_{L^\infty(\Omega)} \to 0$.

(b) If $l = +\infty$, then $u_{\lambda_n}(x) \to +\infty$ $(\forall x \in \Omega)$.

(c) If $l \in \mathbb{N}$, then there exists a set of l distinct points $\mathcal{S} = \{a_1, \cdots, a_l\} \subset \Omega$, which is called a blow up set, such that $\|u_{\lambda_n}\|_{L^\infty(K)} = O(1)$ for any compact sets $K \subset \overline{\Omega} \setminus \mathcal{S}$, $\{u_{\lambda_n}(x)\}$ has a limit for any $x \in \overline{\Omega} \setminus \mathcal{S}$, and $u_{\lambda_n}|_{\mathcal{S}} \to +\infty$ (l-points blow up).

Moreover, in the last case, we have

$$u_{\lambda_n} \to 8\pi \sum_{i=1}^{l} G(\cdot, a_i) \quad \text{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}) \quad (n \to \infty)$$

and each $a_i \in \mathcal{S}$ must satisfy

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j) = \vec{0}, \quad (i = 1, 2, \cdots, l). \tag{1.2}$$

Here, G and R denotes the Green function of $-\Delta$ acting on $H_0^1(\Omega)$ and the Robin function, respectively.

For the proof, the authors in [16] used the complex function theory, more precisely, a representation formula of solutions to (1.1), called the Liouville integral formula was a key ingredient. For other proofs of Proposition 1 by using real analysis and PDE theory, see also Brezis-Merle [3] and Ma-Wei [14].

More generally, we consider *the mean field equation*:

$$\begin{cases} -\Delta u = \lambda \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $\lambda > 0$ and V is a given function in $C^2(\overline{\Omega})$. In this case, Ma and Wei [14] proved the following result.

Proposition 2 (Ma-Wei [14]) *Assume $V \in C^2(\overline{\Omega}), \inf_\Omega V > 0$. Let $\{u_\lambda\}$ be a sequence of solutions to (1.3) which is not uniformly bounded from above for λ bounded. Then there exists a subsequence λ_n and a set of l distinct points $\mathcal{S} = \{a_1, \cdots, a_l\}$ such that $\lambda_n \to 8\pi l$, $l \in \mathbb{N}$, and u_{λ_n} blows up at $a_1, \cdots, a_l$ in $\mathcal{S}$, that is,*

$$\lambda_n \frac{V(x)e^{u_{\lambda_n}}}{\int_\Omega V(x)e^{u_{\lambda_n}} dx} \rightharpoonup 8\pi \sum_{i=1}^{l} \delta_{a_i}$$

in the sense of measures on $\overline{\Omega}$ as $n \to \infty$. Moreover, blow up points $\{a_1, \cdots, a_l\}$ must satisfy

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j) - \frac{1}{8\pi}\nabla \log V(a_i) = \vec{0} \tag{1.4}$$

for $i = 1, 2, \cdots, l$.

After the appearance of these results, the existence of blowing-up solutions with multiple blow up points became the next problem to be studied. On this issue, several affirmative results are now available as follows.

Let $l \geq 1$ be an integer. Assume $\Omega' = \{x \in \Omega | V(x) > 0\} \neq \phi$. Set $(\Omega')^l = (\Omega') \times \cdots \times (\Omega')$ (l times) and $\Delta = \{(\xi_1, \cdots, \xi_l) \in (\Omega')^l \mid \xi_i = \xi_j \text{ for some } i \neq j\}$. Now, define the *Hamiltonian* function

$$\mathcal{F}(\xi_1, \cdots, \xi_l) = \sum_{i=1}^{l} R(\xi_i) - \sum_{\substack{i \neq j \\ 1 \leq i,j \leq l}} G(\xi_i, \xi_j) - \frac{1}{4\pi} \sum_{i=1}^{l} \log V(\xi_i) \tag{1.5}$$

on $(\Omega')^l \setminus \Delta$. Note that the former necessary conditions (1.2) or (1.4) for l-distinct points $\{a_1, \cdots, a_l\}$ to be blow up points is nothing more than that $(a_1, \cdots, a_l)$ is a critical point of the Hamiltonian $\mathcal{F}$ on $(\Omega')^l \setminus \Delta$.

We recall some definitions from the critical point theory.

Definition 3 ([17], [8]) *Let $D \subset \mathbb{R}^N$ and $F : D \to \mathbb{R}$ is a C^1 function. A bounded set K of critical points of F is called a C^1-stable critical set of F if for any $\mu > 0$, there exists $\delta > 0$ such that if $G : D \to \mathbb{R}$ is a C^1 function with the property that*

$$\max_{dist(x,K) \leq \mu} (|G(x) - F(x)| + |\nabla G(x) - \nabla F(x)|) \leq \delta,$$

then G has at least one critical point x with $dist(x, K) \leq \mu$.

Definition 4 ([7]) *Let $D \subset \mathbb{R}^N$ and $F : D \to \mathbb{R}$ be a C^1 function. We say that F links in D at critical level c relative to B and B_0 if the followings hold: B, B_0 closed subsets of $\overline{D}$ with B connected, $B_0 \subset B$, and if we set*

$$\begin{aligned}\Gamma = \{\Phi \in C(B, D) | \exists \Psi \in C([0,1] \times B, D) \\ s.t.\ \Psi(0, \cdot) = Id_B, \Psi(1, \cdot) = \Phi, \Psi(t, \cdot)|_{B_0} = Id_{B_0} (\forall t \in [0,1])\}\end{aligned}$$

and

$$c = \inf_{\Phi \in \Gamma} \sup_{y \in B} F(\Phi(y)),$$

then we have $\sup_{y \in B_0} F(y) < c$ and for any $y \in \partial D$ with $F(y) = c$, there exists a vector τ_y tangent to ∂D such that $\nabla F(y) \cdot \tau_y \neq 0$.

Under the circumstances of Definition 4, it is standard to assure that there exists a critical point $y \in D$ such that $F(y) = c$. Therefore the value c is called *a nontrivial critical level* of F in D.

Proposition 5 (Existence of l-blowing up solution) *Assume $\Omega' = \{x \in \Omega | V(x) > 0\} \neq \phi$. If the Hamiltonian $\mathcal{F}$ defined by (1.5) satisfies one of the following assumptions:*

(1) *$\mathcal{F}$ has a nondegenerate critical point $(a_1, \cdots, a_l) \in (\Omega')^l \setminus \Delta$* (Baraket-Pacard [2])*, or*

(2) *there exists a stable critical set K for $\mathcal{F}$ in $(\Omega')^l \setminus \Delta$* (Esposito-Grossi-Pistoia [8])*, or*

(3) *there exists an open set D compactly contained in $(\Omega')^l \setminus \Delta$ where $\mathcal{F}$ has a nontrivial critical level c* (del Pino-Kowalczyk-Musso [7])

then there exists a solution sequence $\{u_\lambda\}$ to (1.3) such that u_λ blows up exactly on $\mathcal{S} = \{a_1, \cdots, a_l\}$.

It is known that a bounded set K of critical points of $\mathcal{F}$ is a stable critical set if K is a set of strict local minimum points of $\mathcal{F}$: $\mathcal{F}(x) = \mathcal{F}(y)$ for any $x, y \in K$ and for some open neighborhood U of K it holds $\mathcal{F}(x) < \mathcal{F}(y)$ for $x \in K$ and $y \in U \setminus K$. Also a strict local maximum set is a stable critical set. Moreover, if the Brower degree $\deg(\nabla\mathcal{F}, U_\varepsilon, 0) \neq 0$ for any $\varepsilon > 0$ small, where U_ε is an ε-neighborhood of K, then K is stable. Furthermore, if $\Omega \subset \mathbb{R}^2$ is not simply-connected, for example, if it has a small hole, then it is proved in [7] that such a set D in which $\mathcal{F}$ has a nontrivial critical level actually exists for any $l \geq 1$. Therefore in this case, we have a blowing-up solution sequence to (1.1) or (1.3), whose blow up set $\mathcal{S}$ consists of l-distinct points for any $l \in \mathbb{N}$.

Even on simply-connected domains, we sometimes have the existence of multi-bubble solutions. To state the next result, we define *l-dumbbell shaped domain* for $l \in \mathbb{N}$. Prepare l smooth bounded domains $\Omega_1, \cdots, \Omega_l$ in $\mathbb{R}^2$ with $\overline{\Omega_i} \cap \overline{\Omega_j} = \phi$ if $i \neq j$. Assume that

$$\Omega_i \subset \{(x, y) \in \mathbb{R}^2 \mid a_i \leq x \leq b_i\}, \quad \Omega_i \cap \{y = 0\} \neq \phi$$

for some $a_i < b_i < a_{i+1} < b_{i+1}, (i = 1, \cdots, l-1)$ and set $\Omega_0 = \Omega_1 \cup \cdots \cup \Omega_l$. Let

$$C_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid |y| \leq \varepsilon, a_1 < x < b_l\}$$

and let Ω_ε be a simply-connected domain such that $\Omega_0 \subset \Omega_\varepsilon \subset \Omega_0 \cup C_\varepsilon$. We call Ω_ε a l-dumbbell shaped domain.

Proposition 6 ([8] l-points blow up solution on dumbbell shaped domains) *Let $l \geq 2$ and $V(x) \equiv 1$. Then there exists l-dumbbell shaped domain (in particular, it is simply connected but not convex) Ω and an l-points set $\mathcal{S} = \{a_1, \cdots, a_l\}$ such that there exists a solutions $\{u_\lambda\}$ to (MFE) satisfying*

$$\lambda \frac{e^{u_\lambda}}{\int_\Omega e^{u_\lambda} dx} \rightharpoonup 8\pi \sum_{i=1}^{l} \delta_{a_i}$$

as $\lambda \to 8\pi l$ on Ω.

However, on *convex* domains, there does not exist any blowing up solutions with multiple blow up points. The nonexistence result for the Liouville equation proved in [13] is the following:

Theorem 7 (Grossi-Takahashi [13]) *Assume Ω is convex. Let $\{u_\lambda\}$ be a solution sequence of (1.1) with $\|u_\lambda\|_{L^\infty(\Omega)} \to +\infty$ as $\lambda \to 0$. Then we have*

$$\lambda \int_\Omega e^{u_\lambda} dx \to 8\pi$$

as $\lambda \to 0$.

Theorem 7 and a direct application of some results in [11] [12] yields

Corollary 8 (Grossi-Takahashi [13]) *Let u_λ and Ω be as in Theorem 7. Then the Morse index of u_λ is exactly 1 for $\lambda > 0$ sufficiently small. Furthermore, u_λ has only one critical point x_λ which is the global maximum point of u_λ, and it holds*

$$(x - x_\lambda) \cdot \nabla u_\lambda(x) < 0, \quad \forall x \in \Omega \setminus \{x_\lambda\}.$$

In particular, the level sets of u_λ are strict star-shaped with respect to x_λ. If $\partial\Omega$ has strictly positive curvature at any point, then the level sets of u_λ have strictly positive curvature at any point different from x_λ for $\lambda > 0$ sufficiently small. In particular, the level sets are strictly convex.

Almost the same argument as in Theorem 7 yields the following:

Theorem 9 (Grossi-Takahashi [13]) *Assume Ω is convex. Let $\{u_\lambda\}$ be a solution sequence of (1.3) with $\|u_\lambda\|_{L^\infty(\Omega)}$ not bounded from above while $\lambda > 0$ bounded. Assume $\inf_\Omega V > 0$ and $R - \frac{1}{4\pi} \log V$ is a convex function on Ω. Then λ accumulates only on 8π. In particular, if $V > 0$ is a concave function on Ω, we have the same conclusion.*

This is a striking contrast with the known existence theorems of multiple-blowing-up solutions on domains which meet some topological conditions, see the results of [2], [8], [7] described in Proposition 5.

We may consider a different type of problem in 2-dimension, which is socalled a *large exponent problem*:

$$\begin{cases} -\Delta u = (u_+)^p \quad \text{in } \Omega \subset \mathbb{R}^2,\ p > 1, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$ is a large exponent.

In [20] [21], the authors showed that least energy solutions u_p to (1.6) (which may be chosen positive on Ω) is bounded from above and below away from zero in L^∞ norm sense uniformly for p large. Also, after taking a subsequence, $p|\nabla u_p|^2 dx \rightharpoonup 8\pi e\delta_a$ in Radon measures, where $a \in \Omega$ is a minimum point of the Robin function R [10]. In this sense, least energy solutions to (1.6) exhibit single point condensation phenomena on any smooth bounded domain in $\mathbb{R}^2$.

Recently, Santra and Wei [23] studied the asymptotic behavior of concentrating solutions to (1.6) with multiple concentration points. Under the assumption

$$p\int_\Omega (u_+)^{p+1} dx = O(1),\ (p \to \infty) \tag{1.7}$$

they obtained the following result.

Proposition 10 (Santra-Wei [23]) *Let u_p be a solution sequence to (E_p) satisfying the assumption (1.7). Then there exists a subsequence $p_n \to \infty$ such that*

$$p_n \int_\Omega ((u_{p_n})_+)^{p_n} dx \to 8\pi\sqrt{e}l, \quad l \in \mathbb{N}$$

holds. Moreover,

(1) $\|u_{p_n}\|_{L^\infty(\Omega)} \to \sqrt{e}$ *as* $p_n \to \infty$,

(2) *there exists l-points set $\mathcal{S} = \{a_1, \cdots, a_l\} \subset \Omega$ such that*

$$p_n u_{p_n} \to 8\pi\sqrt{e}\sum_{i=1}^{l} G(\cdot, a_i) \quad in\ C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}) \quad (p_n \to \infty).$$

(3) *$a_i \in \mathcal{S}$ satisfies*

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j) = \vec{0}, \quad i = 1, 2, \cdots, l. \tag{1.8}$$

Santra and Wei treated the more general problem which includes the polyharmonic operator with the Dirichlet boundary conditions.

On the existence of concentrating solution sequence with multiple concentration points, Esposito, Musso and Pistoia [9] proved the existence of such sequence to the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

when Ω satisfies some topological conditions. In particular, for example, under the assumption that Ω is not simply connected, they proved the existence of solution sequence $\{u_p\}$ which satisfies

$$p|\nabla u_p|^2 dx \rightharpoonup 8\pi e \sum_{j=1}^{l} \delta_{a_j} \text{ weakly in the sense of measures of } \overline{\Omega}$$

as $p \to \infty$ for some l-different concentration points $\{a_j\}_{j=1}^{l} \subset \Omega$, with $\{a_j\}$ satisfying the characterization (1.8).

However, the same argument as in Theorem 7 yields the following nonexistence result.

Theorem 11 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and let $\{u_p\}$ be a solution sequence satisfying the assumption (1.7). Then there exists $a \in \Omega$, for which*

$$\lim_{p\to\infty} p \int_\Omega ((u_p)_+)^p dx = 8\pi\sqrt{e}, \quad pu_p \to 8\pi\sqrt{e}G(\cdot, a) \quad \textit{in } C^2_{loc}(\overline{\Omega} \setminus \{a\})$$

holds true.

Thus the assumption on the domain in [9] is sharp for the construction of multiple concentrating solution.

We may consider the higher-dimensional problem:

$$\begin{cases} -\Delta u = u^{p-\varepsilon} \quad \text{in } \Omega \subset \mathbb{R}^N \ (N \geq 3), \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.9}$$

where $p = (N+2)/(N-2)$ is the critical Sobolev exponent with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a parameter. To describe the result by Bahri, Li and Rey [1] on the blowing-up sequence to (1.9), we prepare some notations.

For $\vec{x} = (x_1, \cdots, x_l) \in \Omega \times \cdots \times \Omega$ (l times), we define $l \times l$ matrix $M(\vec{x}) = (m_{ij})_{1 \leq i,j \leq l}$ as

$$m_{ii} = R(x_i), \quad m_{ij} = -G(x_i, x_j) \ (i \neq j)$$

where R is the Robin function on Ω. Let $\rho(\vec{x})$ denote the least eigenvalue of $M(\vec{x})$, which is known to be simple, and let $r(\vec{x}) \in \mathbb{R}^l$ be the eigenvector associated with $\rho(\vec{x})$. It is proved in [1] that all components of $r(\vec{x})$ may be chosen to be positive. When $\rho(\vec{x}) > 0$, the function

$$F_{\vec{x}}(\Lambda) = \frac{1}{2}\ {}^t\Lambda M(\vec{x})\Lambda - \log \Lambda_1 \cdots \Lambda_l$$

defined for positive vector $\Lambda = {}^t(\Lambda_1, \cdots, \Lambda_l) \in (\mathbb{R}_+)^l$ is strictly convex, so it has a unique minimum point, which is denoted by $\Lambda(\vec{x}) \in (\mathbb{R}_+)^l$.

Bahri-Li-Rey first proved the following proposition when $N \geq 4$. After several years, Rey [22] proved that the same results as Bahri-Li-Rey's hold true even for $N = 3$.

Proposition 12 (Bahri-Li-Rey [1], Rey [22]) *Let $N \geq 3$ and $\{u_\varepsilon\}_{\varepsilon > 0}$ be a sequence of solutions to (1.9) which blows up at $\{a_1, \cdots, a_l\} \subset \overline{\Omega}$ as $\varepsilon \to 0$, in the sense that*

$$|\nabla u_\varepsilon|^2 dx \rightharpoonup S^{N/2} \sum_{i=1}^{l} \delta_{a_i}, \quad u_\varepsilon^{\frac{2N}{N-2}} \rightharpoonup S^{N/2} \sum_{i=1}^{l} \delta_{a_i}$$

where S is the best constant for the Sobolev inequality on $\mathbb{R}^N$. Then

(1) $\vec{a} = (a_1, \cdots, a_l) \in \Omega^l$ *(interior points)*

(2) $\rho(\vec{a}) \geq 0$ *(no collision of blow up points occurs)*

(3) *it holds*

$$\frac{1}{2}\nabla R(a_i)\Lambda_i^2 - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j)\Lambda_i\Lambda_j = \vec{0} \quad (\forall i = 1, 2, \cdots, l)$$

where

$$\Lambda = \ {}^t(\Lambda_1, \cdots, \Lambda_l) = \begin{cases} \Lambda(\vec{a}) & \text{if } \rho(\vec{a}) > 0, \\ r(\vec{a}) & \text{if } \rho(\vec{a}) = 0 \end{cases}$$

As for the existence of multi-peak solutions in higher dimensional case, Musso and Pistoia [15] constructed solutions to (1.9) which blow up and concentrate at l-different points $\{a_1, \cdots, a_l\}$ in Ω, if $\{a_1, \cdots, a_l\}$ satisfies, among other things,

$$\frac{1}{2}\nabla R(a_i)\Lambda_i^2 - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j)\Lambda_i\Lambda_j = \vec{0}, \quad (i = 1, 2, \cdots, l), \tag{1.10}$$

where $\Lambda_i > 0, (i = 1, \cdots, l)$ are some positive constants. We refer to [15] for the precise notion of solutions which "blow up and concentrate at l-different points" and the other assumption imposed on the prescribed blow-up points $\{a_1, \cdots, a_l\}$.

Their method can produce also multispike solutions to the equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

which blow up and concentrate on l-different points satisfying (1.10), when $N \geq 5$. Also they exhibited an example of contractible domains for which the problem (1.9), or (1.11) has a family of solutions which blow up and concentrate at l-different points.

However, like Theorem 7 and Theorem 11, we have the nonexistence results on convex domains.

Theorem 13 ([13]) *Let Ω be a smooth bounded, convex domain in $\mathbb{R}^N, N \geq 3$. Then any solution sequence $\{u_\varepsilon\}$ of the problem*

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

must exhibit the single point blow-up as $\varepsilon \to 0$, i.e.,

$$|\nabla u_\varepsilon|^2 dx \rightharpoonup S^{N/2}\delta_a, \quad u_\varepsilon^{\frac{2N}{N-2}} \rightharpoonup S^{N/2}\delta_a$$

for some $a \in \Omega$, where S is the best constant of the Sobolev inequality.

Theorem 14 *Assume $\Omega \subset \mathbb{R}^N, N \geq 4$ is convex. Then for $l \geq 2$, there does not exist a solution sequence $\{u_\varepsilon\}$ of (1.11), which blows up and concentrate at l-different points $\{a_1, \cdots, a_l\}$ in Ω, those points satisfying (1.10).*

2. Outline of Proof.

All nonexistence results in the former section come from the following Main Theorem.

Main Theorem. *Let Ω be a smooth bounded domain in $\mathbb{R}^N, N \geq 2$ and let $l \geq 2$ be an integer. Set $\Omega^l = \Omega \times \cdots \times \Omega$ (l times), and $\Delta = \{(\xi_1, \cdots, \xi_l) \in \Omega^l \mid \xi_i = \xi_j \text{ for some } i \neq j\}$. For given constants $A, B > 0$ and $\Lambda = (\Lambda_1, \cdots, \Lambda_l)$, $\Lambda_i > 0$, $1 \leq i \leq l$, define a function $\mathcal{F}_\Lambda : \Omega^l \setminus \Delta \to \mathbb{R}$,*

$$\mathcal{F}_\Lambda(\xi_1, \cdots, \xi_l) = A \sum_{i=1}^{l} (R(\xi_i) + K(\xi_i)) \Lambda_i^2 - B \sum_{\substack{i \neq j \\ 1 \leq i,j \leq l}} G(\xi_i, \xi_j)\Lambda_i\Lambda_j,$$

where $K \in C^2(\Omega)$ is such that $R + K$ is a convex function on Ω.

Assume Ω is convex. Then there does not exist any critical point $(a_1, \cdots, a_l)$ of $\mathcal{F}_\Lambda$ in $\Omega^l \setminus \Delta$. That is, there does not exist $(a_1, \cdots, a_l) \in \Omega^l \setminus \Delta$ such that

$$A (\nabla R(a_i) + \nabla K(a_i)) \Lambda_i^2 - B \sum_{j=1, j \neq i}^{l} \nabla_x G(a_i, a_j)\Lambda_i\Lambda_j = \vec{0}$$

for $i = 1, 2, \cdots, l$.

Main Theorem is proved by a contradiction argument, which uses the following two facts:

Theorem 15 (Caffarelli-Friedman [5] ($N = 2$), Cardaliaguet-Tahraoui [6] ($N \geq 3$)) *The Robin function on a domain* Ω *is strictly convex if* Ω *is a smooth bounded convex domain.*

Lemma 16 *Let* $\Omega \subset \mathbb{R}^N, N \geq 2$ *be a smooth bounded domain. For any* $P \in \mathbb{R}^N$ *and* $a, b \in \Omega, a \neq b$*, there holds*

$$\begin{aligned}
&\int_{\partial\Omega} (x-P)\cdot\nu(x)\left(\frac{\partial G(x,a)}{\partial \nu_x}\right)\left(\frac{\partial G(x,b)}{\partial \nu_x}\right) ds_x \\
&= (2-N)G(a,b) + (P-a)\cdot\nabla_x G(a,b) + (P-b)\cdot\nabla_x G(b,a),
\end{aligned}$$

where $\nu(x)$ *is the unit outer normal at* $x \in \partial\Omega$*.*

Note that in Lemma 16, we need not to assume the convexity of Ω.

Proof. We show a formal calculation here for describing the idea of the proof. However, the standard approximating procedure for the delta function as in Brezis and Peletier [4] will yield the rigorous proof. Denote $G_a(x) = G(x,a), G_b(x) = G(x,b)$. For given $P \in \mathbb{R}^N$, define

$$w(x) = (x-P)\cdot\nabla G_a(x).$$

Then we have

$$\begin{aligned}
-\Delta w(x) &= 2\delta_a(x) + (x-P)\cdot\nabla\delta_a(x), \\
-\Delta G_b(x) &= \delta_b(x).
\end{aligned}$$

Multiplying $G_b(x)$, $w(x)$ to these equations respectively, and subtracting, we obtain

$$\begin{aligned}
&\int_\Omega (\Delta G_b(x))\, w(x) - (\Delta w(x))\, G_b(x) dx \\
&= \int_\Omega \left\{2\delta_a(x)G_b(x) + (x-P)\cdot\nabla\delta_a(x) G_b(x) - \delta_b(x) w(x)\right\} dx
\end{aligned}$$

Now, integration by parts gives

$$
\begin{aligned}
LHS &= \int_{\partial\Omega} (x-P)\cdot\nu(x)\left(\frac{\partial G_a(x)}{\partial\nu}\right)\left(\frac{\partial G_b(x)}{\partial\nu}\right) ds_x \\
RHS &= 2G_b(a) - w(b) + \int_\Omega (x-P)\cdot\nabla\delta_a(x)G_b(x)dx \\
&= 2G_b(a) - w(b) + \sum_{i=1}^N \int_\Omega (x_i-P_i)\frac{\partial\delta_a}{\partial x_i}G_b(x)dx \\
&= 2G_b(a) - w(b) - \sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i}\{(x_i-P_i)G_b(x)\}\delta_a(x)dx \\
&= 2G_b(a) - w(b) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left\{(x_i-P_i)G_b(x)\right\}\Big|_{x=a} \\
&= (2-N)G(a,b) + (P-a)\cdot\nabla_x G(a,b) + (P-b)\cdot\nabla_x G(b,a).
\end{aligned}
$$

This proves Lemma 16. □

Proof of Main Theorem

Essential points of the proof can be seen when the function K is constant, so we give a proof for this case. We argue by contradiction and assume that there exists $\{a_1,\cdots,a_l\}\subset\Omega$ ($l\geq 2$) satisfying

$$
\frac{1}{2}A\nabla R(a_i)\Lambda_i^2 - B\sum_{j=1,j\neq i}^{l}\nabla_x G(a_i,a_j)\Lambda_i\Lambda_j = \vec{0} \tag{2.1}
$$

$P\in\Omega$ will be chosen later. Multiplying $P-a_i$ to (2.1) and summing up, we obtain

$$
\begin{aligned}
&\frac{1}{2}A\sum_{i=1}^{l}(P-a_i)\cdot\nabla R(a_i)\Lambda_i^2 \\
&= B\sum_{i=1}^{l}\sum_{j=1,j\neq i}^{l}(P-a_i)\cdot\nabla_x G(a_i,a_j)\Lambda_i\Lambda_j \\
&= B\sum_{1\leq j<k\leq l}\left\{(P-a_j)\cdot\nabla_x G(a_j,a_k) + (P-a_k)\cdot\nabla_x G(a_k,a_j)\right\}\Lambda_j\Lambda_k.
\end{aligned}
$$

By Lemma 16, we see that

$$(P-a_j)\cdot\nabla_x G(a_j,a_k)+(P-a_k)\cdot\nabla_x G(a_k,a_j)$$
$$=\int_{\partial\Omega}(x-P)\cdot\nu(x)\left(\frac{\partial G(x,a_j)}{\partial\nu_x}\right)\left(\frac{\partial G(x,a_k)}{\partial\nu_x}\right)ds_x+(N-2)G(a_j,a_k).$$

The RHS is positive by the convexity of Ω and the positivity of Green's function:

$$(x-P)\cdot\nu(x)>0,\frac{\partial G(x,a_j)}{\partial\nu_x}<0,\ (x\in\partial\Omega),\quad G(a_j,a_k)>0\ (j\neq k).$$

Thus

$$\sum_{i=1}^{l}(a_i-P)\cdot\nabla R(a_i)<0. \tag{2.2}$$

Here, we recall the important fact that the Robin function is strictly convex on a convex domain, see Theorem 15. Thus, all level sets of R is strictly star-shaped with respect to its unique minimum point $P\in\Omega$:

$$(a-P)\cdot\nabla R(a)\geq 0,\quad \forall a\in\Omega\setminus\{P\}.$$

In particular,

$$\sum_{i=1}^{l}(a_i-P)\cdot\nabla R(a_i)\geq 0. \tag{2.3}$$

A contradiction is obvious from (2.2) and (2.3). □

Acknowledgments. The author would like to thank Prof. Yuki Naito for his kind help at the RIMS conference "Mathematical Analysis and Functional Equations: From New Points of View". Part of this work was supported by JSPS Grant-in-Aid for Scientific Research (C), No. 20540216.

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