Variable Lebesgue norm estimates for BMO functions *

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Abstract

In this paper, we are going to obtain characterizations of the space $BMO(\mathbb{R}^n)$ through variable Lebesgue spaces.

1 Introduction

One of the most interesting problems on spaces with variable exponent is the boundedness of the Hardy–Littlewood maximal operator. The important sufficient conditions called "log-Hölder" have been obtained by Cruz-Uribe, Fiorenza, and Neugebauer [2] and Diening [3]. Under the conditions many results on spaces with variable exponent have been obtained now.

The aim of this paper is to obtain characterizations of $BMO(\mathbb{R}^n)$. Recently an attempt has been made to characterize $BMO(\mathbb{R}^n)$ through various function spaces. Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function for a measurable set $S \subset \mathbb{R}^n$. All cubes are assumed to have their sides parallel to the coordinate axes. Given a function f and a a measurable set S, f_S denotes the mean value of f on S, namely

$$f_S := \frac{1}{|S|} \int_S f(x)\, dx.$$

Definition 1.1. *The space $BMO(\mathbb{R}^n)$ consists of all measurable functions b satisfying*

$$\|b\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx < \infty, \tag{1}$$

where the supremum is taken over all cubes Q.

Recently, given a Banach function space X, we have been asking ourselves the following problem.

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Problem 1.2. *The norm $\|b\|_{BMO(\mathbb{R}^n)}$ is equivalent to*

$$\|b\|_X^* = \sup_{Q:\text{cube}} \frac{1}{\|\chi_Q\|_X} \|\chi_Q(b - b_Q)\|_X.$$

Here is a series of affirmative results concerning Problem 1.2.

1. $X = L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. This is well-known as the John–Nirenberg inequality (See Lemma 3.1 to follow).
2. X is a rearrangement invariant function space [7]. By rearrangement invariant we mean that the X-norm of a function f depends only upon the function $t \in (0, \infty) \mapsto |\{|f| > t\}| \in (0, \infty)$.
3. X is a quasi-rearrangement invariant Banach function space with $p \le p_Y \le q_Y < \infty$ ([8]).

The aim of this paper is to show that this is the case even when X is not rearrangement invariant. First, we consider the case when X is a Morrey space.

Theorem 1.3. *Let $1 \le q \le p < \infty$. If we define the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ by*

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} = \sup_{Q:\text{cube}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, dx \right)^{1/q},$$

then Problem 1.2 is true for $X = \mathcal{M}^p_q(\mathbb{R}^n)$.

The second (and main) spaces we take up in this paper are variable Lebesgue spaces. A measurable function $p(\,\cdot\,) : \mathbb{R}^n \to [1, \infty]$ is called a variable exponent. A variable exponent space showed up around 1990s [11]. After 2005 the theory which are fundamental in harmonic analysis is established very rapidly. For more details we refer to the recent book [5]. Here is a precise definition.

Definition 1.4. *Given a variable exponent $p(\,\cdot\,)$, one denotes*

$$\Omega_{\infty,p} := \{x \in \mathbb{R}^n \,:\, p(x) = \infty\} = p^{-1}(\infty)$$
$$\rho_p(f) := \int_{\mathbb{R}^n \setminus \Omega_{\infty,p}} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_{\infty,p})}.$$

The variable Lebesgue space is defined by

$$L^{p(\,\cdot\,)}(\mathbb{R}^n) := \{f \text{ is measurable } : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

The variable Lebesgue space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} := \inf\{\lambda > 0 \,:\, \rho_p(f/\lambda) < \infty\}.$$

This is a special case of the theory developed by Luxemburg and Nakano [13, 14, 15]. We additionally set

$$p_- := \operatorname{ess\,inf}\{p(x) \,:\, x \in \mathbb{R}^n\}, \quad p_+ := \operatorname{ess\,sup}\{p(x) \,:\, x \in \mathbb{R}^n\}.$$

Theorem 1.5. *If a variable exponent $p(\cdot)$ satisfies $1 \le p_- \le p_+ < \infty$ and the estimates*

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le -\frac{C_1}{\log|x-y|} \qquad \left(|x-y| \ge \frac{1}{2}\right)$$

and

$$\left|\frac{1}{p(x)} - \frac{1}{p(\infty)}\right| \le \frac{C_2}{\log(e+|x|)} \quad (x \in \mathbb{R}^n)$$

holds for some $C_1, C_2, p(\infty) > 0$, then Problem 1.2 is true for $X = L^{p(\cdot)}(\mathbb{R}^n)$, that is,

$$C^{-1}\|b\|_{BMO(\mathbb{R}^n)} \le \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\|b\|_{BMO(\mathbb{R}^n)}$$

holds for all $b \in BMO(\mathbb{R}^n)$.

Needless to say, $L^{p(\cdot)}(\mathbb{R}^n)$ is not rearrangement invariant. Examples in [17] show that $\mathcal{M}^p_q(\mathbb{R}^n)$ is rearrangement invariant only when $p = q$.

Theorem 1.3 is considerably easy to prove. Indeed, from the definition of the Morrey norm, we have

$$\begin{aligned}\frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^q(\mathbb{R}^n)} &\le \frac{1}{\|\chi_Q\|_{\mathcal{M}^p_q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \\ &\le \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

So the matters are reduced to the case when $X = L^p(\mathbb{R}^n)$.

However, a similar argument does not seem to work for Theorem 1.5. Especially the estimate which corresponds to

$$\frac{1}{\|\chi_Q\|_{L^q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{L^q(\mathbb{R}^n)} \le \frac{1}{\|\chi_Q\|_{\mathcal{M}^p_q(\mathbb{R}^n)}}\|\chi_Q(b-b_Q)\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$$

is hard to obtain.

We organize the remaining part of this paper as follows: Section 2 intends as an review of variable Lebesgue spaces. We prove Theorem 1.5 in Section 3. Section 4 contains another characterization of $BMO(\mathbb{R}^n)$ related to the variable exponent Lebesgue norms.

Finally we give a convention which we use throughout the rest of this paper. A symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2 Some basic facts on variable Lebesgue spaces

Given a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x.

One of the key developments of the theory of variable Lebesgue spaces is that we obtained a good criterion of the boundedness of the Hardy–Littlewood maximal operators [3, 4, 5].

Definition 2.1. *Let $r(\,\cdot\,) : \mathbb{R}^n \to (0, \infty)$ be a measurable function.*

1. *The function $r(\,\cdot\,)$ is said to be locally log-Hölder continuous if*

$$|r(x) - r(y)| \le \frac{C}{-\log(|x-y|)} \quad (|x-y| \le 1/2) \tag{2}$$

holds. The set LH_0 consists of all locally log-Hölder continuous functions.

2. *The function $r(\,\cdot\,)$ is said to be log-Hölder continuous at infinity if there exists a constant $r(\infty)$ such that*

$$|r(x) - r(\infty)| \le \frac{C}{\log(e + |x|)}. \tag{3}$$

The set LH_∞ consists of all log-Hölder continuous at infinity functions.

3. *Define $LH := LH_0 \cap LH_\infty$ and say that each function belonging to LH is globally log-Hölder continuous.*

The next proposition is initially proved by Cruz-Uribe et al. [2], when $p_+ < \infty$. Later Cruz-Uribe et al. [1] and Diening et al. [5] have independently extended the result even to the case of $p_+ = \infty$.

Proposition 2.2. *Suppose that a variable exponent $p(\,\cdot\,)$ satisfies $1 < p_- \le p_+ \le \infty$ and $1/p(\,\cdot\,) \in LH$. Then M is bounded on $L^{p(\,\cdot\,)}(\mathbb{R}^n)$, namely*

$$\|Mf\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\, \|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \tag{4}$$

holds for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$.

We note that $p(\,\cdot\,)$ always satisfies $p_- > 1$ whenever (4) is true ([5]). In the case of $p_- = 1$, the weak $(p(\,\cdot\,), p(\,\cdot\,))$ type inequality for M holds. The following has been also proved by Cruz-Uribe et al. [1].

Proposition 2.3. *If a variable exponent $p(\,\cdot\,)$ satisfies $1 = p_- \le p_+ \le \infty$ and $1/p(\,\cdot\,) \in LH$, then we have that for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$,*

$$\sup_{t>0} t \left\| \chi_{\{Mf(x)>t\}} \right\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\, \|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}. \tag{5}$$

We will need the following two lemmas in order to get the main results.

Lemma 2.4. *If a variable exponent $p(\,\cdot\,)$ satisfies the weak $(p(\,\cdot\,), p(\,\cdot\,))$ type inequality (5) for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$, then*

$$|f|_Q \|\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\, \|f\, \chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}$$

holds for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$ and all cubes Q.

Proof. Take $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and a cube Q arbitrarily. We may assume $|f|_Q > 0$. Let $t = |f|_Q/2$. Now that $|f|_Q \chi_Q(x) \leq M(f\chi_Q)(x)$, we obtain $M(f\chi_Q)(x) > t$ whenever $x \in Q$. Thus we have

$$\begin{aligned}
|f|_Q \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq |f|_Q \left\| \chi_{\{M(f\chi_Q)(x)>t\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq |f|_Q \cdot Ct^{-1} \|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&= C \|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark 2.5. Lerner [12] has proved the converse of Lemma 2.4, provided that $p(\cdot)$ is radial decreasing and satisfies $p_- > 1$.

The next lemma is due to Diening [4, Lemma 5.5].

Lemma 2.6. *If a variable exponent $p(\cdot)$ satisfies $1 < p_- \leq p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a constant $0 < \delta_1 < 1$ such that for all $0 < \delta < \delta_1$, all families of pairwise disjoint cubes Y, all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $|f|_Q > 0$ ($Q \in Y$) and all $t_Q > 0$ ($Q \in Y$),*

$$\left\| \sum_{Q\in Y} t_Q \left| \frac{f}{f_Q} \right|^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \sum_{Q\in Y} t_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

In particular

$$\left\| f^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \, |f_Q|^\delta \, \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{6}$$

holds.

3 Main results

We describe some known facts before we state the main results.

Lemma 3.1. *If $1 \leq q < \infty$, then we have that for all $b \in BMO(\mathbb{R}^n)$,*

$$\|b\|_{BMO(\mathbb{R}^n)} \leq \sup_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q \, dx \right)^{1/q} \leq C_0 q \, \|b\|_{BMO(\mathbb{R}^n)}, \tag{7}$$

where $C_0 > 0$ is a constant independent of q.

The left hand-side inequality of (7) directly follows from the Hölder inequality. The right one is a famous consequence of an application of the John–Nirenberg inequality (cf. [10]).

Proposition 3.2. *There exist two positive constants C_1, C_2 depending only on n such that for all $b \in BMO(\mathbb{R}^n)$, all cubes Q and all $t \geq 0$,*

$$|\{x \in Q \, : \, |b(x) - b_Q| > t\}| \leq C_1 \, |Q| \exp\left(-C_2 t / \|b\|_{BMO(\mathbb{R}^n)}\right).$$

Lemma 3.1 can additionally be generalized to the case of variable exponents. Now we are going to prove Theorem 1.5. Recall that we announced that we are going to prove;

> If a variable exponent $p(\,\cdot\,)$ satisfies $1 < p_- \le p_+ < \infty$ and M is bounded on $L^{p(\,\cdot\,)}(\mathbb{R}^n)$, then we have that for all $b \in BMO(\mathbb{R}^n)$,

$$C^{-1}\|b\|_{BMO(\mathbb{R}^n)} \le \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}} \|(b-b_Q)\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\,\|b\|_{BMO(\mathbb{R}^n)} \tag{8}$$

The author [9] has initially proved Theorem **??**. Later we will give an another proof of it.

In view of Lemma 3.1, it may be a natural question to prove (8) for the case of $p_- = 1$.

Now we shall prove Theorem 1.5.

Proof of Theorem 1.5. Take a cube Q and $b \in BMO(\mathbb{R}^n)$ arbitrarily. By virtue of Lemma 2.4 we have

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q|\,dx \cdot \|\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\,\|(b-b_Q)\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}\,.$$

This gives us the left hand side inequality of the theorem. Next we shall prove the right hand side one. Let us fix a number r so that $rp_- > 1$ and write $u(\,\cdot\,) := rp(\,\cdot\,)$. Then the variable exponent $u(\,\cdot\,)$ satisfies $1 < u_-$ and $1/u(\,\cdot\,) \in LH$. Hence the boundedness of M on $L^{u(\,\cdot\,)}(\mathbb{R}^n)$ holds by Proposition 2.2. Using Lemma 2.6, we can take a constant $\delta \in (0, 1/r)$ so that

$$\left\|f^\delta \chi_Q\right\|_{L^{u(\,\cdot\,)}(\mathbb{R}^n)} \le C\,|f_Q|^\delta\,\|\chi_Q\|_{L^{u(\,\cdot\,)}(\mathbb{R}^n)}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Now we obtain

$$\begin{aligned}\left\|f^{r\delta}\chi_Q\right\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} &= \left\|f^\delta\chi_Q\right\|^r_{L^{u(\,\cdot\,)}(\mathbb{R}^n)}\\ &\le C\,|f_Q|^{r\delta}\,\|\chi_Q\|^r_{L^{u(\,\cdot\,)}(\mathbb{R}^n)}\\ &= C\,|f_Q|^{r\delta}\,\|\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}\,. \end{aligned} \tag{9}$$

If we put $f := |b-b_Q|^{1/(r\delta)}$ and apply Lemma 3.1 with $q = 1/(r\delta) > 1$, then we get

$$|f_Q|^{r\delta} = \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{1/(r\delta)}\,dx\right)^{r\delta} \le C\,\|b\|_{BMO(\mathbb{R}^n)}. \tag{10}$$

Combing (9) and (10) we obtain

$$\|(b-b_Q)\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\,\|b\|_{BMO(\mathbb{R}^n)}\,\|\chi_Q\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}\,.$$

This leads us to the desired inequality and completes the proof. □

Proof of Theorem **??**. We have only to follow the same argument as the proof of Theorem 1.5 with $r = 1$. □

4 Related inequalities

According to Lemma 3.1, we have

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^q\,dx\right)^{1/q} \le C_0 q\|b\|_{BMO(\mathbb{R}^n)},$$

where $C_0 > 0$ is independent of $q \in [1,\infty)$. This can be rephrased as

$$\frac{1}{|Q|}\int_Q \left(\frac{|b(x)-b_Q|}{C_0 q\|b\|_{BMO(\mathbb{R}^n)}}\right)^q dx \le 1$$

for all cubes Q. Observe that the estimate above is uniform over $1 \le q < \infty$. Therefore, the following inequality seems to hold;

$$\frac{1}{|Q|}\int_Q \left(\frac{|b(x)-b_Q|}{C_0 p(x)\|b\|_{BMO(\mathbb{R}^n)}}\right)^{p(x)} dx \le 1$$

Suppose that $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a variable exponent which is not necessarily continuous or bounded. Then define

$$\|b\|^{\dagger}_{p(\cdot)} = \sup_Q \left(\inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \left(\frac{|b(x)-b_Q|}{p(x)\lambda}\right)^{p(x)} dx \le 1\right\}\right)$$

for measurable functions b. Now we are going to prove;

Theorem 4.1. *If a variable exponent $p(\cdot)$ satisfies $p(x) < \infty$ for almost every $x \in \mathbb{R}^n$, then we have*

$$\|b\|^{\dagger}_{p(\cdot)} \le C\|b\|_{BMO(\mathbb{R}^n)}.$$

Furthermore, if $p(\cdot)$ is bounded, then the norms $\|\cdot\|^{\dagger}_{p(\cdot)}$ and $\|\cdot\|_{BMO(\mathbb{R}^n)}$ are mutually equivalent.

Proof. According to the John-Nirenberg inequality, we have

$$\frac{1}{|Q|}\int_Q \left\{\exp\left(\frac{\lambda|b(x)-b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}}\right) - 1\right\} dx \le 1$$

for some $\lambda > 0$. Since

$$\begin{aligned}\left(\frac{\lambda|b(x)-b_Q|}{3p(x)\|b\|_{BMO(\mathbb{R}^n)}}\right)^{p(x)} &= \left(\frac{1}{3p(x)}\right)^{p(x)} \left(\frac{\lambda|b(x)-b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}}\right)^{p(x)} \\ &\le \min\left\{\left(\frac{1}{[p(x)]}\right)^{[p(x)]}, \left(\frac{1}{[p(x)+1]}\right)^{[p(x)+1]}\right\} \left(\frac{\lambda|b(x)-b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}}\right)^{p(x)} \\ &\le \exp\left(\frac{\lambda|b(x)-b_Q|}{\|b\|_{BMO(\mathbb{R}^n)}}\right) - 1.\end{aligned}$$

Hence it follows that

$$\|b\|^{\dagger}_{p(\cdot)} \le 3\lambda^{-1}\|b\|_{BMO(\mathbb{R}^n)}.$$

If $p(\cdot)$ is bounded, then

$$\begin{aligned}\|b\|^{\dagger}_{p(\cdot)} &\geq \sup_Q \left(\inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \left(\frac{|b(x) - b_Q|}{p_+ \lambda} \right)^{p(x)} dx \leq 1 \right\} \right) \\ &= \sup_Q \left(\inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{|b(x) - b_Q|}{p_+ \lambda} \right)^{p(x)} \right\} dx \leq 1 \right\} \right) \\ &= \sup_Q \left(\inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \frac{|b(x) - b_Q|}{2p_+ \lambda} dx \leq 1 \right\} \right) \\ &= (2p_+)^{-1} \|b\|_{BMO(\mathbb{R}^n)}.\end{aligned}$$

Therefore, these norms are mutually equivalent. □

Remark 4.2. Let Φ be a Young function. Namely, $\Phi : [0, \infty) \to [0, \infty)$ is a homeomorphism which is convex. If we assume that $\Phi(t) \leq t^a$ $(t \geq 2)$ for some $a > 1$ and define

$$\|b\|^{\dagger}_{\Phi} = \sup_Q \left(\inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \Phi \left(\frac{|b(x) - b_Q|}{p(x)\lambda} \right) dx \leq 1 \right\} \right)$$

for measurable functions b, then $\|b\|^{\dagger}_{\Phi}$ is equivalent to $\|b\|_{\mathrm{BMO}}$. Indeed, as we have shown in [16], the norm $\|b\|^{\dagger}_{\Phi}$ remains unchanged if we redefine $\Phi(t) = \Phi(2)(t/2)^a$ for $0 \leq t \leq 2$. Therefore, $\|b\|^{\dagger}_{\Phi} \leq C\|b\|_{\mathrm{BMO}}$ by virtue of Lemma 3.1. The reverse inequality is also clear since we have $\Phi(t) \geq \Phi(1)t$ for $t \geq 1$.

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