Cook-hats and Crowns

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This paper is dedicated to Professor Clifford Earle.

Abstract. We will study hyperbolic structures on a torus with a hole (named as a "cook-hat"), and on a thrice-punctured sphere with a hole (named as a "crown"). See Figure 1 below. Both of them have three simple closed geodesics called canonical triples, whose hyperbolic lengths and the hyperbolic length of the boundary geodesic define homogeneous coordinates of the Teichmüller space for each cases. We will show that their Teichmüller spaces are realized as convex polyhedra in the three-dimensional real projective space $P(\mathbb{R}^4)$, by means of the canonical isomorphism between them.

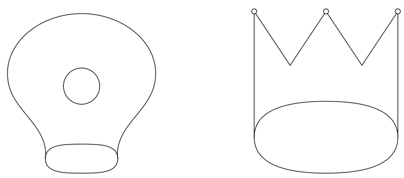

Figure 1. a cook-hat and a crown

1. Introduction

Let X be an orientable surface of genus g with n punctures whose Euler number is negative, $\chi(X) := 2-2g-n < 0$. Then the Teichmüller space $\mathcal{T}(X)$ is the space of isotopy classes of hyperbolic metrics on X which has a metric space structure homeomorphic to the real affine space $\mathbb{R}^{6g-6+2n}$.

By using hyperbolic lengths of simple closed geodesics we can embed $\mathcal{T}(X)$ into the infinite-dimensional real affine space as follows: Let $\mathcal{S}$ be the non-trivial and non-peripheral free homotopy classes of simple closed curves on X. For any hyperbolic structure $m \in \mathcal{T}(X)$ and any free homotopy class $\alpha \in \mathcal{S}$, we denote the hyperbolic length of a unique simple closed geodesic belonging to α by $l(m,\alpha)$. Then the mapping $l_* : T_{g,n} \to \mathbb{R}_+^{\mathcal{S}}$ defined by $l_*(m) = (l(m,\alpha))_{\alpha\in\mathcal{S}}$ is injective.

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In practice we can embed $\mathcal{T}(X)$ into $\mathbb{R}^{9g-9+3n}$: Fix a pants decomposition $\mathcal{P}$ on X, i.e. a multicurve such that $X \setminus \mathcal{P}$ is homeomorphic to the disjoint union of thrice punctured spheres. $\mathcal{P}$ consists of $3g-3+n$ numbers of disjoint simple close curves. The Fenchel-Nielsen coordinates associate to each $m \in \mathcal{T}(X)$ the length and the twist of each components of $\mathcal{P}$, which is a diffeomorphism from $\mathcal{T}(X)$ onto $\mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$ (see [**IT**]). On the other hand the twist of each components of $\mathcal{P}$ can be determined by the lengths of two more curves for each components so that $\mathcal{T}(X)$ can be embedded into $\mathbb{R}^{9g-9+3n}$ by length functions of $9g-9+3n$ number of simple closed geodesics. It should be remarked that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}(X)$ is equal to $dim_{\mathbb{R}}\mathcal{T}(X)+1 = 6g-5+2n$ (see [**S1**]).

Let π be the projection from $\mathbb{R}^{\mathcal{S}} \setminus \{0\}$ to the infinite-dimensional real projective space $P(\mathbb{R}^{\mathcal{S}})$. In Proposition 6 of Exposé 7 [**FLP**] Kerckhoff showed that the composition map $\pi \circ l_* : \mathcal{T}(X) \to P(\mathbb{R}^{\mathcal{S}})$ is also injective: In his argument, it is essential that the surface X has at least one handle, because he used the fact that for the case $g \geq 1$ we can find two simple closed curves γ_1 and γ_2 whose intersection number is equal to one. Then simple closed curves γ_3 and γ_4 which are freely homotopic to $\gamma_1 \cdot \gamma_2$ and $\gamma_1^{-1} \cdot \gamma_2$ respectively satisfy the key identity for his proof:

$$\cosh(\frac{l_1+l_2}{2}) + \cosh(\frac{l_1-l_2}{2}) = \cosh(\frac{l_3}{2}) + \cosh(\frac{l_4}{2}).$$

where $l_i := l(m, [\gamma_i])$ for $m \in \mathcal{T}(X)$ and $i = 1,2,3,4$. Hence for the case $g = 0$, we should look for other ideas to claim that the composition map $\pi \circ l_* : T_{g,n} \to P(\mathbb{R}^{\mathcal{S}})$ is also injective (see Corollary 3.7 in Section 3).

The composition map $\pi \circ l_* : T_{g,n} \to P(\mathbb{R}^{\mathcal{S}})$ is the basic ingredient for the Thurston compactification of $\mathcal{T}(X)$: The image $\pi \circ l_*(\mathcal{T}(X))$ is relatively compact in $P(\mathbb{R}^{\mathcal{S}})$ and its compactification $\overline{\pi \circ l_*(\mathcal{T}(X))}$ in $P(\mathbb{R}^{\mathcal{S}})$ is homeomorphic to the closed ball of dimension $6g-6+2n$. The relative boundary of $\pi \circ l_*(\mathcal{T}(X))$ coincides with $\mathcal{PMF}(X)$ the projective image of the space of measured foliations on X under the intersection number functions, which has a PL-manifold structure homeomorphic to the sphere of dimension $6g-7+2n$ (see Exposé 8 [**FLP**]).

Now we have the following natural question:

> Can we find $dim_{\mathbb{R}}\mathcal{T}(X)+1$-number of simple closed geodesics whose hyperbolic lengths embed $\mathcal{T}(X)$ into the finite dimensional real projective space $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}(X)+1})$?

Because of the PL-Structure of the Thurston boundary, we might expect that the image should be the interior of some convex polyhedron in $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}(X)+1})$.

For this question, Schmutz proved affirmatively for the case $(g,n) = (2,0)$ (see [**S2**]). Hamenstädt also consider the similar question by using non-simple geodesics for the case $n \geq 1$ (see [**H**]). Gendulphe and the author solved this question affirmatively for non-orientable genus 3 surfaces (see [**GK**]). They also showed that the image of the Teichmüller space in $P(\mathbb{R}^4)$ becomes a convex polyhedron.

To attack this question in general, in this paper we will consider the case of a torus with a hole in section 2, and the case of a thrice punctured sphere with a hole in section 3, since any surface X contains one of these surface as an essential subsurface. In practice we will answer this question for surfaces with at least one hole with few exceptional cases (see Corollaries 2.7 and 3.8). In section 2 we will show that the Teichmüller space of a torus with a hole can be realized as a convex polyhedron in $P(\mathbb{R}^4)$ via hyperbolic length functions, which is a key idea for the main results of [**GK**]. Then in section 3 we will show the geometric bijection between the Teichmüller space of a torus with a hole and the Teichmüller space of a thrice punctured sphere with a hole, which itself seems interesting. By means of this bijection we can also realize the Teichmüller space of a thrice punctured sphere with a hole as a convex polyhedron in $P(\mathbb{R}^4)$. And as an application of this result, we will prove that the composition map $\pi \circ l_* : T_{g,n} \to P(\mathbb{R}^{\mathcal{S}})$ is injective also for the case $g = 0$.

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2. Cook-hats

In this section we will consider complete hyperbolic structures on a torus with a hole. We call a hyperbolic torus with a hole a **cook-hat**.

Definition 2.1. Three simple closed geodesics (α, β, γ) on a cook-hat is called a **canonical triple** if each pair of them has the intersection number equal to one.

We remark that the hyperbolic lengths of a canonical triple (α, β, γ) satisfy triangle inequalities.

For the hyperbolic lengths of a canonical triple (α, β, γ) and the boundary geodesic δ on a cook-hat, we have the following equality and inequality.

Proposition 2.2. *For any cook-hat with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following equality and inequality:*

$$\cosh^2 \frac{l(\delta)}{4} = (\cosh \frac{l(\beta)+l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta)-l(\gamma)}{2}). \tag{2.1}$$

$$l(\alpha) + l(\beta) + l(\gamma) > l(\delta). \tag{2.2}$$

Proof. We uniformize a cook-hat by a Fuchsian group $\Gamma \subset SL(2, \mathbb{R})$, and denote the traces of elements representing α, β, γ and δ by $t(\alpha), t(\beta), t(\gamma)$ and $t(\delta)$. Then they satisfy

$$t(\delta) - 2 = t(\alpha)t(\beta)t(\gamma) - (t(\alpha)^2 + t(\beta)^2 + t(\gamma)^2). \tag{2.3}$$

By means of the relation between trace functions and length functions

$$|t(\alpha)| = 2 \cosh \frac{l(\alpha)}{2} \tag{2.4}$$

and the equality

$$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y),$$

we can rewrite (2.3) in terms of length functions

$$\begin{aligned}
& 2\cosh\frac{l(\delta)}{2}-2=t(\delta)-2\\
= \ & t(\alpha)t(\beta)t(\gamma)-(t(\alpha)^2+t(\beta)^2+t(\gamma)^2)\\
= \ & 4(2\cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)}{2}\cosh\frac{l(\gamma)}{2}-\cosh^2\frac{l(\alpha)}{2}-\cosh^2\frac{l(\beta)}{2}-\cosh^2\frac{l(\gamma)}{2})\\
= \ & 4(\cosh\frac{l(\beta)+l(\gamma)}{2}-\cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2}-\cosh\frac{l(\beta)-l(\gamma)}{2})-4.
\end{aligned}$$

Therefore

$$\begin{aligned}
\cosh^2\frac{l(\delta)}{4} &= \frac{1}{2}(\cosh\frac{l(\delta)}{2}+1)\\
&= (\cosh\frac{l(\beta)+l(\gamma)}{2}-\cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2}-\cosh\frac{l(\beta)-l(\gamma)}{2})
\end{aligned}$$

which is the equality (2.1).

Since $\cosh x$, hence $\cosh^2 x$ is monotonely increasing function of x, the equality (2.1) implies that it is enough to show that

$$(\cosh\frac{l(\beta)+l(\gamma)}{2}-\cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2}-\cosh\frac{l(\beta)-l(\gamma)}{2})<\cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4}$$

for the proof of the inequality (2.2). In practice

$$\begin{aligned}
& \cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4}\\
& -(\cosh\frac{l(\beta)+l(\gamma)}{2}-\cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2}-\cosh\frac{l(\beta)-l(\gamma)}{2})\\
= \ & \cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4}+\cosh^2\frac{l(\alpha)}{2}+\cosh\frac{l(\beta)+l(\gamma)}{2}\cosh\frac{l(\beta)-l(\gamma)}{2}\\
& -\cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)+l(\gamma)}{2}-\cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)-l(\gamma)}{2}\\
= \ & \frac{1}{4}\{(e^{l(\alpha)}-e^{\frac{l(\alpha)+l(\beta)-l(\gamma)}{2}})+(e^{l(\beta)}-e^{\frac{l(\beta)+l(\gamma)-l(\alpha)}{2}})+(e^{l(\gamma)}-e^{\frac{l(\gamma)+l(\alpha)-l(\beta)}{2}})\\
& +(1-e^{\frac{l(\alpha)-l(\beta)-l(\gamma)}{2}})+(1-e^{\frac{l(\beta)-l(\gamma)-l(\alpha)}{2}})+(1-e^{\frac{l(\gamma)-l(\alpha)-l(\beta)}{2}})\\
& +e^{-l(\alpha)}+e^{-l(\beta)}+e^{-l(\gamma)}+1\}>0.
\end{aligned}$$

□

REMARK 2.3. (1) The equality (2.1) also follows from the plane hyperbolic geometry of the right angled hexagon which is the symmetric half of the pair of pants $T\setminus\alpha$.

(2) The inequality (2.2) also comes from the fact that the curve $\alpha\cup\beta\cup\gamma$ is freely homotopic to the geodesic δ.

By means of the equality (2.1) in Proposition 2.2, we can embed the Teichmüller space $\mathcal{T}(T)$ of a torus with a hole into the 3-dimensional real projective space $P(\mathbb{R}^4)$.

THEOREM 2.4. *For a cook hat with a canonical triple* (α,β,γ) *and the boundary geodesic* δ*, their hyperbolic lengths* $l(\alpha), l(\beta), l(\gamma)$ *and* $l(\delta)$ *satisfy*

$$\cosh^2\frac{sl(\delta)}{4}<(\cosh\frac{sl(\beta)+sl(\gamma)}{2}-\cosh\frac{sl(\alpha)}{2})(\cosh\frac{sl(\alpha)}{2}-\cosh\frac{sl(\beta)-sl(\gamma)}{2})$$

for any $s > 1$. In particular the system of length functions $L := (l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $\mathcal{T}(T)$ of a torus with a hole into $P(\mathbb{R}^4)$.

Proof. For simplicity we will write

$$a = l(\alpha), b = l(\beta), c = l(\gamma), d = l(\delta).$$

Then our claim is rewritten as

$$\frac{d}{4}s < \cosh^{-1}\sqrt{f(s)}, \ \ \forall s > 1$$

where

$$f(s) := (\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s)(\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s),$$

for which it is enough to show that

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} > \frac{d}{4}, \ \ \forall s > 1.$$

By the inequality (2.2), it is enough to show that

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} > \frac{a+b+c}{4}, \ \ \forall s > 1.$$

By the following simple estimation

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} = \frac{f'(s)}{2\sqrt{f(s)}\sqrt{f(s)-1}} > \frac{f'(s)}{2f(s)}$$

we will show that

$$\frac{f'(s)}{f(s)} > \frac{a+b+c}{2}, \ \ \forall s > 1.$$

In practice

$$\begin{aligned}\frac{f'(s)}{f(s)} &= \frac{\frac{d}{ds}(\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s)}{\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s} + \frac{\frac{d}{ds}(\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s)}{\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s} \\ &> \frac{b+c}{2} + \frac{a}{2} = \frac{a+b+c}{2}.\end{aligned}$$

Here we use the following lemma:

Lemma 2.5. *For $0 < p < q$,*

$$g(s) := \frac{\frac{d}{ds}(\cosh qs - \cosh ps)}{\cosh qs - \cosh ps} = \frac{q\sinh qs - p\sinh ps}{\cosh qs - \cosh ps} > q, \ \ \forall s > 1.$$

Proof. It is enough to show that the derivative of $g(s)$ is negative for $\forall s > 1$, since

$$\lim_{s\to\infty} g(s) = \lim_{s\to\infty}\frac{q\sinh qs - p\sinh ps}{\cosh qs - \cosh ps} = q.$$

Hence we will show the negativity of the numerator of $g'(s)$:

$$g'(s) = \frac{(q^2\cosh qs - p^2\cosh ps)(\cosh qs - \cosh ps) - (q\sinh qs - p\sinh ps)^2}{(\cosh qs - \cosh ps)^2}.$$

In practice

$$\begin{aligned}
& (q^2\cosh qs - p^2\cosh ps)(\cosh qs - \cosh ps) - (q\sinh qs - p\sinh ps)^2 \\
= \;& q^2\cosh^2 qs + p^2\cosh^2 ps - (q^2+p^2)\cosh qs\cosh ps \\
& -q^2\sinh^2 qs - p^2\sinh^2 ps + 2pq\sinh qs\sinh ps \\
= \;& q^2+p^2 - \frac{1}{2}(q+p)^2\cosh(q-p)s - \frac{1}{2}(q-p)^2\cosh(q+p)s \\
< \;& q^2+p^2 - \frac{1}{2}(q+p)^2 - \frac{1}{2}(q-p)^2 = 0.
\end{aligned}$$

□

□

By means of the triangle inequalities of $l(\alpha), l(\beta), l(\gamma)$ and the inequality (2.2) in Proposition 2.2, we can determine the image of $\mathcal{T}(T)$ in $\mathcal{P}(\mathbb{R}^4)$ as follows.

THEOREM 2.6. *The image of $\mathcal{T}(T)$ the Teichmüller space of a cook-hat under the map $L := (l(\alpha) : l(\beta) : l(\gamma) : l(\delta))$ is the convex polyhedron Δ in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\begin{aligned}
\Delta \;:=\; & \{(a:b:c:d) \in \mathcal{P}(\mathbb{R}^4) \mid a>0,\ b>0,\ c>0,\ d>0, \\
& a<b+c,\ b<c+a,\ c<a+b,\ d<a+b+c\}.
\end{aligned}$$

PROOF. By means of the inequality (2.2) in Proposition 2.2, we have $L(T) \subset \Delta$. Hence we will prove that $\Delta \subset L(T)$. Take any point $p \in \Delta$ and four positive real numbers $(a,b,c,d) \in \mathbb{R}^4_+$ satisfying $p = (a:b:c:d)$. Then there exist $s>0$ and a hyperbolic structure $m \in \mathcal{T}(T)$ such that

$$(l(\alpha), l(\beta), l(\gamma), l(\delta)) = (as, bs, cs, d_s)$$

where $l(\alpha) = l(m,\alpha)$ and $d_s > 0$ is defined by

$$d_s := 4\cosh^{-1}\sqrt{(\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2})}.$$

To conclude that $L(m) = p$, It is enough to show that there is $s>0$ such that $d_s = sd$. We will show that d_s/s takes any value between 0 and $a+b+c$ when s varies. In practice d_s/s is a continuous function on s and

$$(\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2}) \to 1$$

when s decreases, hence $d_s/s \to 0$. On the other hand,

$$\begin{aligned}
& (\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2}) \\
= \;& e^{\frac{(a+b+c)s}{2}} O(1), \quad s\to\infty
\end{aligned}$$

and

$$\cosh\frac{d_s}{4} = e^{\frac{d_s}{4}} O(1), \quad s\to\infty$$

imply that $\lim_{s\to\infty} d_s/s = a+b+c$. Hence d_s/s takes any value between 0 and $a+b+c$. □

As an application, let us consider the Teichmüller space $\mathcal{T}(Y)$ of a orientable surface Y of genus g with n punctures and r holes. Assuming that $r \geq 1$, then it is known that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}(Y)$ is $dim_{\mathbb{R}}\mathcal{T}(X) = 6g - 6 + 2n + 3r$ (see [**S1**]). Moreover suppose that Y has at least one handle i.e. $g \geq 1$, there is a subsurface homeomorphic to a torus with a hole. Therefore Theorem 2.4 implies the affirmative answer to our question in section 1 for the case that $r \geq 1$ and $g \geq 1$:

Corollary 2.7. *Assume that $r \geq 1$ and $g \geq 1$. Then via length functions of simple closed geodesics, the Teichmüller space $\mathcal{T}(Y)$ of a orientable surface Y of genus g with n punctures and r holes can be embedded into $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}(X)+1})$.*

3. Crowns

In this section we will consider complete hyperbolic structures on a thrice-punctured sphere with a hole. We call a hyperbolic thrice-punctured sphere with a hole a **crown**.

Definition 3.1. Three simple closed geodesics (α, β, γ) on a crown is called a **canonical triple** if each pair of them has the intersection number equal to two.

We will show that similar results in section 2 also hold for $\mathcal{T}(S)$ the Teichmüller space of a thrice-punctured sphere with a hole with the help of the geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$ explained below. For this purpose we realize $\mathcal{T}(T)$ and $\mathcal{T}(S)$ as hypersurfaces in $\mathbb{R}^4$ in terms of trace functions:

Theorem 3.2. (Theorem 2 of [**L**] and Proposition 3.1 of [**NN**])

(1) *We uniformize a cook-hat $m \in \mathcal{T}(T)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_T : \mathcal{T}(T) \to \mathbb{R}^4$ defined by $\varphi_T(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_T(\mathcal{T}(T))$ is described as follows:*

$$\{(a,b,c,d) \in \mathbb{R}^4 \quad | \quad a > 2, b > 2, c > 2, d > 2, \\ abc - a^2 - b^2 - c^2 + 2 = d\}.$$

(2) *We uniformize a crown $m \in \mathcal{T}(S)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_S : \mathcal{T}(S) \to \mathbb{R}^4$ defined by $\varphi_S(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_S(\mathcal{T}(S))$ is described as follows:*

$$\{(p,q,r,s) \in \mathbb{R}^4 \quad | \quad p > 2, q > 2, r > 2, s > 2, s^2 + 2(p+q+r+4)s \\ +4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 = 0\}.$$

Than by means of trace functions, we have the following geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$:

Theorem 3.3. *There is a bijection from $\mathcal{T}(T)$ to $\mathcal{T}(S)$ which sends a cook-hat T with the lengths of a canonical triple and the boundary geodesic equal to (l_1, l_2, l_3, l_4) to a crown S with the lengths of a canonical triple and the boundary geodesic equal to $(2l_1, 2l_2, 2l_3, l_4)$.*

PROOF. When we substitute (a^2-2, b^2-2, c^2-2, d) for (p, q, r, s), the equation $s^2 + 2(p+q+r+4)s + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8$ factorizes as

$$\begin{aligned} & d^2 + 2(p+q+r+4)d + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 \\ = \; & (d - (abc - a^2 - b^2 - c^2 + 2))(d - (-abc - a^2 - b^2 - c^2 + 2)). \end{aligned}$$

Hence the map $\Psi : \varphi_T(\mathcal{T}(T)) \to \varphi_S(\mathcal{T}(S))$ defined by $\Psi(a, b, c, d) := (a^2 - 2, b^2 - 2, c^2 - 2, d)$ is bijective. Also the relation between trace functions and length functions

$$|t(\alpha)| = 2 \cosh \frac{l(\alpha)}{2}$$

tells us the length relations between $m \in \mathcal{T}(T)$ and $\varphi_S^{-1} \circ \Psi \circ \varphi_T(m) \in \mathcal{T}(S)$. □

REMARK 3.4. For the limiting case $l(\delta) = 0$, this bijection reduces to the well-known correspondence between punctured tori and forth-punctured spheres, which follows from the commensurability of uniformizing Fuchsian groups (see [**ASWY**]).

This bijection induces the next corollaries: The following inequality is the counterpart of the inequality (2.2) in Proposition 2.2 for crowns.

COROLLARY 3.5. *For any crown with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following inequality:*

$$l(\alpha) + l(\beta) + l(\gamma) > 2l(\delta).$$

Next result is the counterpart of Theorem 2.4 and 2.6 for crowns.

COROLLARY 3.6. *For a crown with a canonical triple (α, β, γ) and the boundary geodesic δ, the system of length functions $(l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $\mathcal{T}(S)$ into $P(\mathbb{R}^4)$. The image of $\mathcal{T}(S)$ is the convex polyhedron in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\begin{aligned} & \{(a : b : c : d) \in P(\mathbb{R}^4) \mid a > 0,\ b > 0,\ c > 0,\ d > 0, \\ & a < b + c,\ b < c + a,\ c < a + b,\ 2d < a + b + c\}. \end{aligned}$$

As an application of Corollary 3.6,

COROLLARY 3.7. *The composition map $\pi \circ l_* : \mathcal{T}(X) \to P(\mathbb{R}^{\mathcal{S}})$ is also injective for the case $g = 0$.*

For the final application of Corollary 3.6, let us consider the Teichmüller space $\mathcal{T}(Y)$ of a orientable surface Y of genus g with n punctures and r holes for the case that $g = 0, n \geq 3$ and $r \geq 1$. Then there is a subsurface homeomorphic to a thrice-punctured sphere with a hole, hence Corollary 3.6 implies the affirmative answer to our question in section 1 for this case also:

COROLLARY 3.8. *Assume that $g = 0, n \geq 3$ and $r \geq 1$. Then via length functions of simple closed geodesics, the Teichmüller space $\mathcal{T}(Y)$ of a orientable surface Y of genus g with n punctures and r holes can be embedded into $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}(X)+1})$.*

For a sphere (i.e., $g = 0$) with holes (i.e., $r \geq 1$), this question is still open for the cases $n = 0, 1, 2$.

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