Some identities of Green's function for the polyharmonic operator with the Navier boundary conditions and its applications

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Abstract

We prove several integral identities for Green's function of the polyharmonic operator $(-\Delta)^p, p \in \mathbb{N}$ under the Navier boundary conditions. As an application, we prove the nondegeneracy of the critical point of the Robin function associated to the Green function on some symmetric domains.

1 Introduction.

Recently, many authors have been interested in the study of linear and nonlinear elliptic partial differential equations involving the higher-order differential operator, see for example, the recent book [5] and the reference therein.

2010 Mathematics Subject Classification: 35J08, 35J40.
Key words: Green's function, polyharmonic operator, Navier boundary condition.

In the following, we fix $p \in \mathbb{N}$ and let $G = G(x,y)$ denote the Green function of $(-\Delta)^p$ under the Navier boundary condition:

$$\begin{cases} (-\Delta)^p G(\cdot, y) = \delta_y \quad \text{in } \Omega, \\ G(\cdot, y) = (-\Delta)^j G(\cdot, y) = 0 \quad \text{on } \partial\Omega \ (j = 1, \cdots p-1), \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2p$. Decompose G as $G(x,y) = \Gamma(x,y) - H(x,y)$, where $\Gamma(x,y)$ is the fundamental solution of $(-\Delta)^p$ on $\mathbb{R}^N$ defined as

$$\Gamma(x,y) = \begin{cases} C_{N,p}\, |x-y|^{2p-N}, & N > 2p, \\ C_p \log \frac{1}{|x-y|}, & N = 2p, \end{cases}$$

where

$$C_{N,p} = \frac{2\Gamma(\frac{N}{2} - p)}{2^{2p}(p-1)!\Gamma(\frac{N}{2})\sigma_N}, \quad N > 2p, \tag{1.1}$$

$$C_p = \frac{1}{\{2^{p-1}(p-1)!\}^2 \sigma_{2p}}, \quad N = 2p, \tag{1.2}$$

and $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$. $H = H(x,y) \in C^\infty(\Omega \times \Omega)$ is called the regular part of the Green function, and satisfies

$$\begin{cases} (-\Delta)^p H(\cdot, y) = 0 \quad \text{in } \Omega, \\ (-\Delta)^j H(\cdot, y) = (-\Delta)^j \Gamma(\cdot, y) \quad \text{on } \partial\Omega \ (j = 0, 1, \cdots p-1). \end{cases}$$

Note that the coefficients in the expression of $\Gamma(x,y)$ are determined by the formula

$$(-1)^p \int_{\partial B_r(0)} \frac{\partial \Delta^{p-1}\Gamma(x,0)}{\partial \nu_x} ds_x = 1, \tag{1.3}$$

here ν is the unit normal vector to $\partial B_r(0)$. Finally, let $R(y) = H(y,y)$ denote the Robin function of the Green function of $(-\Delta)^p$ with the Navier boundary condition.

In this paper, we prove the nondegeneracy of critical points of the Robin function on some symmetric domains. More precisely, let Ω be a smooth bounded domain in $\mathbb{R}^N, N \geq 2p$, which is symmetric with respect to hyperplanes $\{x_i = 0\}$ and convex in x_i-directions for $i = 1, \cdots, N$. This kind of domains are sometimes called Gidas-Ni-Nirenberg domains (GNN domains for short) after the famous paper [6]. We will prove the Hessian matrix of the Robin function associated to the Green function of $(-\Delta)^p$ under the Navier boundary condition computed at the origin is diagonal and all diagonal elements are strictly positive. For the second order case ($p = 1$), this result was former proved by M. Grossi [7]. Basically our strategy of the proof is to follow his argument faithfully. However, in the course of the proof, we need to generalize some integral identities relating boundary integrations of the Green function to the Robin function, which, in turn, originate from the paper by Brezis and Peletier [1].

As for the second order case, it is known that the Robin function of $-\Delta$ with the Dirichlet boundary condition is strictly convex and has a unique nondegenerate critical point (global minimum point) on a bounded convex domain. This important fact was first proved by Caffarelli-Friedman [2] when $N = 2$, and later extended to $N \geq 3$ by Cardaliaguet-Tahraoui [3]. Whether the same result holds true for the Robin function of $(-\Delta)^p$ with the Navier boundary condition is completely open. We hope the theorem mentioned above could shed light on this subject.

This paper is organized as follows. In §2, we recall some well-known facts on the Green function of $(-\Delta)^p$ under the Navier boundary conditions. §3 will be devoted to the proof of integral identities mentioned above, and we hope that this part would be useful in itself for some readers. In §4, we will prove the nondegeneracy of the critical point of the Robin function on GNN domains. C will denote various constants from line to line until otherwise stated.

2 Preliminaries.

In this section, we recall some elementary facts that are useful later. First, we recall Green's 2nd identity

$$\begin{aligned}
&\int_\Omega [(\Delta^p f)g - f(\Delta^p g)]\, dx \\
&= \sum_{k=1}^{p} \int_{\partial\Omega} \left[\left(\frac{\partial \Delta^{k-1} f}{\partial \nu_x} \right) \left(\Delta^{p-k} g \right) - \left(\Delta^{k-1} f \right) \left(\frac{\partial \Delta^{p-k} g}{\partial \nu_x} \right) \right] ds_x, \qquad (2.1)
\end{aligned}$$

which holds for smooth f, g.

In the following, we set $\overline{G}_j(x,y) = (-\Delta_x)^j G(x,y)$ for $j = 0, 1, \cdots, p-1$. Then $\overline{G}_j$ satisfies

$$\begin{cases} -\Delta_x \overline{G}_j = \overline{G}_{j+1} \quad \text{in } \Omega, \ (j = 0, 1, \cdots, p-2), \\ -\Delta_x \overline{G}_{p-1} = \delta_y \quad \text{in } \Omega, \\ \overline{G}_j > 0 \quad \text{in } \Omega, \ (j = 0, 1, \cdots, p-1), \\ \overline{G}_j = 0 \quad \text{on } \partial\Omega, \ (j = 0, 1, \cdots, p-1). \end{cases} \tag{2.2}$$

Note that $\overline{G}_{p-1}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By using these symbols, the well-known Green's representation formula for the unique solution to the linear problem

$$\begin{cases} (-\Delta)^p u = f \quad \text{in } \Omega, \\ (-\Delta)^j u = g_j \quad \text{on } \partial\Omega \ (j = 0, 1, \cdots p-1), \end{cases}$$

where f and g_j are smooth functions, can be written as follows:

$$u(y) = \int_\Omega G(x,y) f(x) dx - \sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x} \right) g_{p-k}(x) ds_x \tag{2.3}$$

for $y \in \Omega$.

Also we need a version of Pohozaev identity for the polyharmonic equation

$$(-\Delta)^p u = f(u) \text{ in } \Omega, \quad u \in C^{2p}(\overline{\Omega}) \tag{2.4}$$

without boundary conditions.

Lemma 2.1 *Assume $f \in C^1(\mathbb{R}, \mathbb{R})$. Then*

$$\int_\Omega \left[NF(u) - \left(\frac{N-2p}{2} \right) uf(u) \right] dx = \int_{\partial\Omega} (x \cdot \nu) \left(F(u) - \frac{1}{2} uf(u) \right) ds_x$$
$$+ \frac{(-1)^{p-1}}{2} \sum_{k=1}^{p} \int_{\partial\Omega} \left[\left(\frac{\partial \Delta^{k-1} u}{\partial \nu} \right) \Delta^{p-k}(x \cdot \nabla u) - \left(\Delta^{k-1} u \right) \frac{\partial \Delta^{p-k}(x \cdot \nabla u)}{\partial \nu} \right] ds_x \tag{2.5}$$

holds for a solution $u \in C^{2p}(\overline{\Omega})$ to (2.4), where $F(u) = \int_0^u f(s) ds$.

More general version is known, see [8], [9] and so on. We show a proof of the above lemma in order to make this paper self-contained.

Proof. By Green's 2nd identity (2.1) with $f = u, g = (x \cdot \nabla u)$, we have

$$\int_\Omega [(\Delta^p u)(x \cdot \nabla u) - u\Delta^p (x \cdot \nabla u)]\, dx$$
$$= \sum_{k=1}^p \int_{\partial\Omega} \left[\left(\frac{\partial \Delta^{k-1} u}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla u) - \left(\Delta^{k-1} u\right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla u)}{\partial \nu_x} \right) \right] ds_x$$
$$=: \sum_{k=1}^p B_k.$$

Note also that

$$\Delta^j (x \cdot \nabla u) = 2j\Delta^j u + (x \cdot \nabla \Delta^j u), \quad (j = 0, 1, 2, \cdots)$$

which is easily shown by induction. Thus,

$$\int_\Omega u\Delta^p (x \cdot \nabla u) dx = 2p \int_\Omega u\Delta^p u dx + \int_\Omega u(x \cdot \nabla \Delta^p u) dx$$
$$= 2p(-1)^p \int_\Omega uf(u) dx + (-1)^p \int_\Omega u(x \cdot \nabla f(u)) dx$$
$$= (-1)^p \left\{ 2p \int_\Omega uf(u) dx + \int_\Omega \mathrm{div}\, (x(uf(u) - F(u)))\, dx + \int_\Omega N(F(u) - uf(u)) dx \right\}$$
$$= (-1)^p \left\{ \int_\Omega \{NF(u) - (N - 2p)uf(u)\}\, dx + \int_{\partial\Omega} (x \cdot \nu)(uf(u) - F(u)) ds_x \right\},$$

where we have used (2.4) and the formula $u(x \cdot \nabla f(u)) = \mathrm{div}\, (x(uf(u) - F(u))) + N(F(u) - uf(u))$.

On the other hand,

$$\int_\Omega (\Delta^p u)(x \cdot \nabla u) dx = (-1)^p \int_\Omega f(u)(x \cdot \nabla u) dx$$
$$= (-1)^p \int_\Omega \{\mathrm{div}(xF(u)) - NF(u)\}\, dx = (-1)^p \left\{ \int_{\partial\Omega} (x \cdot \nu) F(u) ds_x - \int_\Omega NF(u) dx \right\}.$$

Combining all together, we have

$$(-1)^p \left\{ \int_{\partial\Omega} (x \cdot \nu) F(u) ds_x - \int_\Omega NF(u) dx \right\} - \sum_{k=1}^p B_k$$
$$= (-1)^p \left\{ \int_\Omega NF(u) - (N - 2p)uf(u) dx + \int_{\partial\Omega} (x \cdot \nu)(uf(u) - F(u)) ds_x \right\},$$

which implies the lemma. □

3 Integral identities for Green's function of $(-\Delta)^p$ with the Navier boundary conditions.

In this section, we will prove some identities for the Green function of $(-\Delta)^p$ under the Navier boundary conditions. Part of these formulas were former proved by Brezis and Peletier [1] when $p = 1, N > 2$, Ren and Wei [10] when $p = 1, N = 2$, Chou and Geng [4] when $p = 2, N > 4$, and Takahashi [11] when $p = 2, N = 4$.

Theorem 3.1 *For any $y \in \Omega$, we have*

(1)

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x}\right)\left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial \nu_x}\right) ds_x = (N-2p)R(y) \tag{3.1}$$

when $N > 2p$, and

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x}\right)\left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial \nu_x}\right) ds_x = C_p \tag{3.2}$$

when $N = 2p$, where C_p is defined in (1.2).

(2)

$$\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x}\right)\left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial \nu_x}\right) \nu_i(x)\, ds_x = \frac{\partial R}{\partial y_i}(y) \tag{3.3}$$

for $i = 1, \cdots, N$ when $N \geq 2p$.

(3)

$$\begin{aligned} &2\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial x_i}\right)\frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial \nu_x}\right) ds_x = \\ &2\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial x_i}\right)\frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x}\right) ds_x = \frac{\partial^2 R}{\partial y_i \partial y_j}(y) \end{aligned} \tag{3.4}$$

for $1 \leq i, j \leq N$, $N \geq 2p$.

Here $\nu = \nu(x)$ is the outer unit normal at $x \in \partial\Omega$.

Proof. First we prove (3.1). We may assume $y = 0$ and choose $r > 0$ small such that $B_r := B_r(0) \subset\subset \Omega$. We apply the Pohozaev identity (2.5) in Lemma 2.1 to

$$\begin{cases} (-\Delta)^p G(\cdot, 0) = 0 \quad \text{in } \Omega \setminus B_r, \\ G(\cdot, 0) = (-\Delta)^j G(\cdot, 0) = 0 \quad \text{on } \partial\Omega \ (j = 1, \cdots p-1). \end{cases}$$

Thus we obtain

$$\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \Delta^{k-1} G}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla G) ds_x$$
$$= \sum_{k=1}^{p} \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{k-1} G}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla G) - \left(\Delta^{k-1} G \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla G)}{\partial \nu_x} \right) \right] ds_x, \tag{3.5}$$

where $G = G(x, 0)$. Since $\Delta^{p-k}(x \cdot \nabla G) = 2(p-k)\Delta^{p-k} G + (x \cdot \nabla \Delta^{p-k} G) = \left(\frac{\partial \Delta^{p-k} G}{\partial \nu} \right) (x \cdot \nu)$ on $\partial\Omega$, we have

$$\text{LHS of (3.5)} = \sum_{k=1}^{p} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial \Delta^{k-1} G(x,0)}{\partial \nu} \right) \left(\frac{\partial \Delta^{p-k} G(x,0)}{\partial \nu} \right) ds_x. \tag{3.6}$$

On the other hand, inputting $G(x, 0) = \Gamma(x) - g(x)$ where $\Gamma(x) = \Gamma(x, 0) = C_{N,p}|x|^{2p-N}$, $g(x) = H(x, 0)$, we see

$$\text{RHS of (3.5)} = \sum_{k=1}^{p} \left(I_{1,k} - I_{2,k} - I_{3,k} + I_{4,k} \right), \tag{3.7}$$

where

$$I_{1,k} = \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{k-1} \Gamma}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla \Gamma) - \left(\Delta^{k-1} \Gamma \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla \Gamma)}{\partial \nu_x} \right) \right] ds_x,$$
$$I_{2,k} = \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{k-1} \Gamma}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla g) - \left(\Delta^{k-1} \Gamma \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla g)}{\partial \nu_x} \right) \right] ds_x,$$
$$I_{3,k} = \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{k-1} g}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla \Gamma) - \left(\Delta^{k-1} g \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla \Gamma)}{\partial \nu_x} \right) \right] ds_x,$$
$$I_{4,k} = \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{k-1} g}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla g) - \left(\Delta^{k-1} g \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla g)}{\partial \nu_x} \right) \right] ds_x.$$

First, we easily see that $I_{4,k} = o(1)$ as $r \to 0$ for any $k = 1, 2, \cdots, p$.

Next, we set $\Gamma(r) = C_{N,p} r^{2p-N}$ for $r = |x|$. Then by induction, we have

$$\begin{aligned} \Delta^l \Gamma(r) &= C_{N,p} \prod_{i=0}^{l-1} (2p - N - 2i) \prod_{j=1}^{l} (2p - 2j) r^{2p-N-2l} \\ &=: A_l \, r^{2p-N-2l} \quad (l = 0, 1, 2, \cdots,) \end{aligned} \tag{3.8}$$

where in this formula, we agree the convention that $\prod_{i=0}^{i=-1}(\cdots) = \prod_{j=1}^{j=0}(\cdots) = 1$. Also, on ∂B_r, we see that

$$\begin{aligned} &\left(\frac{\partial \Delta^l \Gamma}{\partial \nu_x} \right) = \left(\Delta^l \Gamma \right)' (r) = (2p - N - 2l) A_l r^{2p-N-2l-1}, \\ &\left(x \cdot \nabla \Delta^l \Gamma \right)(x) = r \left(\Delta^l \Gamma \right)' (r) = (2p - N - 2l) A_l r^{2p-N-2l}, \\ &\Delta^l \left(x \cdot \nabla \Gamma \right) = 2l \Delta^l \Gamma + \left(x \cdot \nabla \Delta^l \Gamma \right) = (2p - N) A_l r^{2p-N-2l}, \\ &\frac{\Delta^l \left(x \cdot \nabla \Gamma \right)}{\partial \nu_x} = \left(\Delta^l \left(x \cdot \nabla \Gamma \right) \right)' (r) = (2p - N)(2p - N - 2l) A_l r^{2p-N-2l-1} \end{aligned} \tag{3.9}$$

holds for $l = 0, 1, 2, \cdots$. Note that, by the formula (1.3), we have

$$(-1)^p = \int_{\partial B_r(0)} \frac{\partial \Delta^{p-1} \Gamma(x, 0)}{\partial \nu_x} ds_x = (2 - N) A_{p-1} \sigma_N. \tag{3.10}$$

Therefore by (3.8) and (3.9), the integrand of $I_{1,k}$ is

$$\begin{aligned} &\left(\frac{\partial \Delta^{k-1} \Gamma}{\partial \nu_x} \right) \Delta^{p-k} (x \cdot \nabla \Gamma) - \left(\Delta^{k-1} \Gamma \right) \left(\frac{\partial \Delta^{p-k} (x \cdot \nabla \Gamma)}{\partial \nu_x} \right) \\ &= A_{k-1} (2p - N - 2k + 2) r^{2p-N-2k+1} \cdot (2p - N) A_{p-k} r^{2k-N} \\ &- A_{k-1} r^{2p-N-2k+2} \cdot A_{p-k} (2p - N)(2k - N) r^{2k-N-1} \\ &= 2(2p - N) r^{2p-2N+1} A_{k-1} A_{p-k} (p - 2k + 1), \end{aligned}$$

thus

$$I_{1,k} = 2\sigma_N (2p - N) r^{2p-N} A_{k-1} A_{p-k} (p - 2k + 1).$$

Since we easily check that

$$\sum_{k=1}^{p} A_{k-1} A_{p-k} (p - 2k + 1) = 0$$

for any $p \in \mathbb{N}$, we obtain that $\sum_{k=1}^{p} I_{1,k} = 0$.

For $I_{2,k}$, by (3.9) we see that

$$\int_{\partial B_r} \left(\frac{\partial \Delta^{k-1}\Gamma}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla g) ds_x = Cr^{2p-2k} \cdot r^{1-N} \int_{\partial B_r} (\text{smooth function})\, ds_x$$
$$= o(1) \quad \text{as } r \to 0,\ k \in \{1, 2, \cdots, p-1\}.$$

Also when $k = p$,

$$\int_{\partial B_r} \left(\frac{\partial \Delta^{p-1}\Gamma}{\partial \nu_x} \right) (x \cdot \nabla g) ds_x = Cr^{1-N} \int_{\partial B_r} r \left(\frac{\partial g}{\partial \nu} \right) ds_x \to 0 \quad \text{as } r \to 0.$$

Similarly, we have

$$\int_{\partial B_r} \left(\Delta^{k-1}\Gamma \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla g)}{\partial \nu_x} \right) ds_x = Cr^{2p-2k+1} \cdot r^{1-N} \int_{\partial B_r} (\text{smooth function})\, ds_x$$
$$= o(1) \quad \text{as } r \to 0,\ k \in \{1, 2, \cdots, p\}.$$

Combining these, we obtain $I_{2,k} = o(1)$ as $r \to 0$ for all $k = 1, \cdots, p$.

For $I_{3,k}$, we see by (3.9),

$$\int_{\partial B_r} \left(\frac{\partial \Delta^{k-1} g}{\partial \nu_x} \right) \Delta^{p-k}(x \cdot \nabla \Gamma) ds_x = Cr^{2k-1} \cdot r^{1-N} \int_{\partial B_r} (\text{smooth function})\, ds_x$$
$$= o(1) \quad \text{as } r \to 0,\ k \in \{1, 2, \cdots, p\}.$$
$$\int_{\partial B_r} \left(\Delta^{k-1} g \right) \left(\frac{\partial \Delta^{p-k}(x \cdot \nabla \Gamma)}{\partial \nu_x} \right) ds_x = Cr^{2k-2} \cdot r^{1-N} \int_{\partial B_r} \left(\Delta^{k-1} g \right) ds_x$$
$$= o(1) \quad \text{as } r \to 0,\ k \in \{2, 3, \cdots, p\}.$$

On the other hand, when $k = 1$,

$$\int_{\partial B_r} g(x) \left(\frac{\partial \Delta^{p-1}(x \cdot \nabla \Gamma)}{\partial \nu_x} \right) ds_x = (2p - N)(2 - N) A_{p-1} \cdot r^{1-N} \int_{\partial B_r} g(x) ds_x$$
$$\to (2p - N)(2 - N) A_{p-1} \cdot \sigma_N g(0)$$
$$= (2p - N)(-1)^p g(0)$$

as $r \to 0$, here we have used (3.10). Combining these, we obtain that

$$-\sum_{k=1}^{p} I_k^3 = \int_{\partial B_r} g(x) \left(\frac{\partial \Delta^{p-1}(x \cdot \nabla \Gamma)}{\partial \nu_x} \right) ds_x + o(1) = (N - 2p)(-1)^{p-1} g(0) + o(1)$$

as $r \to 0$.

Returning to (3.6), (3.7) with these estimates, we obtain

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial \Delta^{k-1} G(x,0)}{\partial \nu} \right) \left(\frac{\partial \Delta^{p-k} G(x,0)}{\partial \nu} \right) ds_x = (-1)^{p-1}(N-2p)g(0),$$

which leads to (3.1) when $N > 2p$.

Next, we prove (3.2) when $N = 2p$. We treat the case when $p \geq 2$ only, since the formula for $p = 1$ ($N = 2$):

$$\int_{\partial\Omega} (x-y) \cdot \nu(x) \left(\frac{\partial G(x,y)}{\partial \nu_x} \right)^2 ds_x = \frac{1}{2\pi} \ (= C_1)$$

holds for any $y \in \Omega$ similarly.

Now, $\Gamma(r) = -C_p \log r$ where C_p is defined as (1.2), therefore, we have on ∂B_r,

$$\begin{aligned} &\Delta^l \Gamma(r) = 2^{l-1}(l-1)! C_p \prod_{i=1}^{l} (2i-2p) r^{-2l} =: B_l \, r^{-2l}, \\ &\left(\frac{\partial \Delta^l \Gamma}{\partial \nu_x} \right) = \left(\Delta^l \Gamma \right)'(r) = (-2l) B_l \, r^{-2l-1}, \\ &\left(x \cdot \nabla \Delta^l \Gamma \right)(x) = r \left(\Delta^l \Gamma \right)'(r) = (-2l) B_l \, r^{-2l}, \\ &\Delta^l \left(x \cdot \nabla \Gamma \right) = 2l \Delta^l \Gamma + \left(x \cdot \nabla \Delta^l \Gamma \right) = 2l B_l r^{-2l} + (-2l) B_l r^{-2l} = 0, \\ &\frac{\Delta^l \left(x \cdot \nabla \Gamma \right)}{\partial \nu_x} = \left(\Delta^l \left(x \cdot \nabla \Gamma \right) \right)'(r) = 0 \qquad (3.11) \end{aligned}$$

for $l = 1, 2, \cdots, p$. Note that $\Delta^p \Gamma(r) = 0$. Just as before, we have (3.5), (3.6), (3.7), and $I_{4,k} = o(1)$ as $r \to 0$ for $k = 1, \cdots, p$.

For $I_{1,k}$, since $\Delta^{p-k}(x \cdot \nabla \Gamma) = \frac{\Delta^{p-k}(x \cdot \nabla \Gamma)}{\partial \nu_x} = 0$ for $k \neq p$, we have $I_{1,k} = 0$ for $k \in \{1, 2, \cdots, p-1\}$. On the other hand, since

$$\begin{aligned} &x \cdot \nabla \Gamma = r \left(\Gamma(r) \right)' = -C_p, \\ &\frac{\partial \Delta^{p-1} \Gamma}{\partial \nu} = \left(\Delta^{p-1} \Gamma \right)'(r) = B_{p-1}(2-2p) r^{1-2p}, \quad (p \geq 2), \end{aligned}$$

we see

$$\begin{aligned} I_{1,p} &= \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{p-1} \Gamma}{\partial \nu_x} \right) (x \cdot \nabla \Gamma) - \left(\Delta^{p-1} \Gamma \right) \left(\frac{\partial (x \cdot \nabla \Gamma)}{\partial \nu_x} \right) \right] ds_x \\ &= \int_{\partial B_r} (-C_p) B_{p-1}(2-2p) r^{1-2p} ds_x = B_{p-1} C_p 2(p-1) \sigma_{2p}. \end{aligned}$$

Again we check that $2(p-1)B_{p-1}\sigma_{2p} = (-1)^{p-1}$ by (1.3). Thus we have

$$I_{1,k} = \begin{cases} 0, & k \in \{1, 2, \cdots, p-1\}, \\ (-1)^{p-1}C_p, & k = p. \end{cases}$$

Also, since $\Delta^{k-1}\Gamma(r) = B_{p-1}r^{-2(k-1)}$, $\frac{\partial \Delta^{k-1}\Gamma}{\partial \nu}(r) = B_{k-1}(2-2k)r^{1-2k}$ for $k \in \{2, 3, \cdots, p\}$, we easily check that $I_{2,k} = o(1)$ as $r \to 0$ just as before for $k \in \{2, 3, \cdots, p\}$. For $I_{2,1}$, we see

$$\begin{aligned} I_{2,1} &= \int_{\partial B_r} \left[\left(\frac{\partial \Gamma}{\partial \nu_x} \right) \Delta^{p-1}(x \cdot \nabla g) - \Gamma \left(\frac{\partial \Delta^{p-1}(x \cdot \nabla g)}{\partial \nu_x} \right) \right] ds_x \\ &= \int_{\partial B_r} (-C_p) \left(\frac{1}{r} \right) \Delta^{p-1}(x \cdot \nabla g) ds_x + C_p \int_{\partial B_r} \log r \left(\frac{\partial \Delta^{p-1}(x \cdot \nabla g)}{\partial \nu_x} \right) ds_x \\ &= \left(r^{2p-2} + r^{2p-1} \log r \right) \int_{S^{2p-1}} (\text{smooth function})\, d\omega \\ &= o(1) \end{aligned}$$

as $r \to 0$ when $p \geq 2$. Thus we have $I_{2,k} = o(1)$ for all $k \in \{1, 2, 3, \cdots, p\}$.

For $I_{3,k}$, since $\Delta^{p-k}(x \cdot \Gamma) = 0$ for $k \neq p$, we see $I_{3,k} = 0$ for $k \neq p$. Also, since $x \cdot \nabla \Gamma = -C_p$, we see

$$\begin{aligned} I_{3,p} &= \int_{\partial B_r} \left[\left(\frac{\partial \Delta^{p-1} g}{\partial \nu_x} \right) (x \cdot \nabla \Gamma) - \left(\Delta^{p-1} g \right) \left(\frac{\partial (x \cdot \nabla \Gamma)}{\partial \nu_x} \right) \right] ds_x \\ &= (-C_p) \int_{\partial B_r} \left(\frac{\partial \Delta^{p-1} g}{\partial \nu_x} \right) ds_x = o(1) \end{aligned}$$

as $r \to 0$.

Returning to (3.6), (3.7), we obtain

$$\sum_{k=1}^{p} \int_{\partial \Omega} (x \cdot \nu) \left(\frac{\partial \Delta^{k-1} G(x,0)}{\partial \nu} \right) \left(\frac{\partial \Delta^{p-k} G(x,0)}{\partial \nu} \right) ds_x = (-1)^{p-1} C_p,$$

which ends the proof of (3.2).

To prove (3.3), we apply Green's 2nd identity (2.1) for $f = G = G(x,0)$, $g = G_{x_i}$ on $\Omega \setminus B_r(0)$. Since $\Delta^p G = (\Delta^p G)_{x_i} \equiv 0$ on $\Omega \setminus B_r(0)$, we get

$$0 = \sum_{k=1}^{p} \int_{\partial(\Omega \setminus B_r)} \left[\left(\frac{\partial \Delta^{k-1} G}{\partial \nu_x} \right) \left(\Delta^{p-k} G \right)_{x_i} - \left(\Delta^{k-1} G \right) \left(\frac{\partial \left(\Delta^{p-k} G \right)_{x_i}}{\partial \nu_x} \right) \right] ds_x,$$

which leads to

$$\sum_{k=1}^{p}\int_{\partial\Omega}\left(\frac{\partial\Delta^{k-1}G}{\partial\nu_x}\right)\left(\Delta^{p-k}G\right)_{x_i}ds_x$$
$$=\sum_{k=1}^{p}\int_{\partial B_r}\left[\left(\frac{\partial\Delta^{k-1}G}{\partial\nu_x}\right)\left(\Delta^{p-k}G\right)_{x_i}-\left(\Delta^{k-1}G\right)\left(\frac{\partial\left(\Delta^{p-k}G\right)_{x_i}}{\partial\nu_x}\right)\right]ds_x. \tag{3.12}$$

Since $\left(\Delta^{p-k}G\right)_{x_i}=\left(\frac{\partial\Delta^{p-k}G}{\partial\nu_x}\right)\nu_i(x)$ on $\partial\Omega$, we see

$$\text{LHS of (3.12)}=\sum_{k=1}^{p}\int_{\partial\Omega}\left(\frac{\partial\Delta^{k-1}G(x,0)}{\partial\nu_x}\right)\left(\frac{\partial\Delta^{p-k}G(x,0)}{\partial\nu_x}\right)\nu_i(x)ds_x. \tag{3.13}$$

On the other hand, inputting $G(x,0)=\Gamma(x)-g(x)$, $g(x)=H(x,0)$ as before, we obtain

$$\text{RHS of (3.5)}=\sum_{k=1}^{p}\left(J_{1,k}-J_{2,k}-J_{3,k}+J_{4,k}\right), \tag{3.14}$$

where

$$J_{1,k}=\int_{\partial B_r}\left[\left(\frac{\partial\Delta^{k-1}\Gamma}{\partial\nu_x}\right)\left(\Delta^{p-k}\Gamma\right)_{x_i}-\left(\Delta^{k-1}\Gamma\right)\left(\frac{\partial\left(\Delta^{p-k}\Gamma\right)_{x_i}}{\partial\nu_x}\right)\right]ds_x,$$
$$J_{2,k}=\int_{\partial B_r}\left[\left(\frac{\partial\Delta^{k-1}\Gamma}{\partial\nu_x}\right)\left(\Delta^{p-k}g\right)_{x_i}-\left(\Delta^{k-1}\Gamma\right)\left(\frac{\partial\left(\Delta^{p-k}g\right)_{x_i}}{\partial\nu_x}\right)\right]ds_x,$$
$$J_{3,k}=\int_{\partial B_r}\left[\left(\frac{\partial\Delta^{k-1}g}{\partial\nu_x}\right)\left(\Delta^{p-k}\Gamma\right)_{x_i}-\left(\Delta^{k-1}g\right)\left(\frac{\partial\left(\Delta^{p-k}\Gamma\right)_{x_i}}{\partial\nu_x}\right)\right]ds_x,$$
$$J_{4,k}=\int_{\partial B_r}\left[\left(\frac{\partial\Delta^{k-1}g}{\partial\nu_x}\right)\left(\Delta^{p-k}g\right)_{x_i}-\left(\Delta^{k-1}g\right)\left(\frac{\partial\left(\Delta^{p-k}g\right)_{x_i}}{\partial\nu_x}\right)\right]ds_x.$$

Again, we see that $J_{4,k}=o(1)$ as $r\to 0$ for any $k=1,2,\cdots,p$, $N\geq 2p$.

Now, we treat the case $N > 2p$. In this case, since $\Delta^l \Gamma = A_l r^{2p-N-2l}$ by (3.8), we have

$$
\begin{aligned}
&\left(\Delta^l \Gamma\right)_{x_i} = A_l(2p-N-2l) r^{2p-N-2l-1}\nu_i(x), \\
&\frac{\partial \left(\Delta^l \Gamma\right)_{x_i}}{\partial \nu_x} = \frac{x}{r}\cdot \nabla \left(\Delta^l \Gamma\right)_{x_i} \\
&= \frac{x}{r}\cdot A_l(2p-N-2l)\{(2p-N-2l-2)r^{2p-N-2l-3}x_i\frac{x}{r} + r^{2p-N-2l-2}e_i\} \\
&= A_l(2p-N-2l)(2p-N-2l-1)r^{2p-N-2l-2}\nu_i(x) \qquad (3.15)
\end{aligned}
$$

for $l = 0, 1, 2, \cdots$ on ∂B_r, here $e_i = \nabla x_i$ and we have used $\nu_i(x) = \frac{x_i}{r}$ on ∂B_r for $i = 1, 2, \cdots, N$. By (3.8) and (3.15), we have

$$
\begin{aligned}
J_{1,k} &= \int_{\partial B_r}\left[\left(\frac{\partial \Delta^{k-1}\Gamma}{\partial \nu_x}\right)\left(\Delta^{p-k}\Gamma\right)_{x_i} - \Delta^{k-1}\Gamma\left(\frac{\partial \left(\Delta^{p-k}\Gamma\right)_{x_i}}{\partial \nu_x}\right)\right] ds_x \\
&= C\int_{\partial B_r} r^{2p-N-2k+1}\cdot r^{2k-N-1}\nu_i(x) ds_x - C'\int_{\partial B_r} r^{2p-N-2k+2}\cdot r^{2k-N-2}\nu_i(x) ds_x \\
&= C'' r^{2(p-N)}\int_{\partial B_r}\nu_i(x) ds_x = 0, \quad k \in \{1, 2\cdots, p\},
\end{aligned}
$$

here we have used $\int_{\partial B_r}\nu_i(x) ds_x = 0$.

As for the estimate of $J_{2,k}$, as in the proof of (3.1), we see

$$
\begin{aligned}
\int_{\partial B_r}\left(\frac{\partial \Delta^{k-1}\Gamma}{\partial \nu_x}\right)\left(\Delta^{p-k}g\right)_{x_i} ds_x &= C r^{2(p-k)}\cdot r^{1-N}\int_{\partial B_r}(\text{smooth function})\, ds_x \\
&= o(1), \quad (k = 1, 2, \cdots, p-1) \\
\int_{\partial B_r}\left(\frac{\partial \Delta^{p-1}\Gamma}{\partial \nu_x}\right) g_{x_i} ds_x &= A_{p-1}(2-N) r^{1-N}\int_{\partial B_r} g_{x_i} ds_x \\
&\to A_{p-1}(2-N)\sigma_N g_{x_i}(0) = (-1)^p g_{x_i}(0), \\
\int_{\partial B_r}\left(\Delta^{k-1}\Gamma\right)\left(\frac{\partial \left(\Delta^{p-k}g\right)_{x_i}}{\partial \nu_x}\right) ds_x &= C r^{2p-2k+1}\cdot r^{1-N}\int_{\partial B_r}(\text{smooth function})\, ds_x \\
&= o(1) \quad (k = 1, 2, \cdots, p).
\end{aligned}
$$

Thus we have

$$
J_{2,k} = \begin{cases} o(1), & k \in \{1, 2, \cdots, p-1\}, \\ (-1)^p g_{x_i}(0) + o(1), & k = p, \end{cases}
$$

as $r \to 0$.

As for the estimate of $J_{3,k}$, we see

$$\begin{aligned}\int_{\partial B_r} \left(\frac{\partial \Delta^{k-1} g}{\partial \nu_x}\right) \left(\Delta^{p-k}\Gamma\right)_{x_i} ds_x &= Cr^{2k-2} \cdot r^{1-N} \int_{\partial B_r} \left(\frac{\partial \Delta^{k-1} g}{\partial \nu_x}\right) \nu_i(x) ds_x \\ &= o(1), \quad (k = 2, 3 \cdots, p), \\ \int_{\partial B_r} \left(\Delta^{k-1} g\right) \left(\frac{\partial \left(\Delta^{p-k}\Gamma\right)_{x_i}}{\partial \nu_x}\right) ds_x &= Cr^{2k-3} \cdot r^{1-N} \int_{\partial B_r} \left(\Delta^{k-1} g\right) \nu_i(x) ds_x \\ &= o(1), \quad (k = 2, 3 \cdots, p),\end{aligned}$$

thus we have $J_{3,k} = o(1)$ for $k \in \{2, 3, \cdots, p\}$. Now, we estimate $J_{3,1}$. For smooth g, we have

$$\begin{aligned}&\int_{\partial B_r} \left(\frac{\partial g}{\partial \nu_x}\right) \nu_i(x) ds_x = \int_{\partial B_r} \left(\sum_{j=1}^{N} \frac{\partial g}{\partial x_j} \nu_j(x)\right) \nu_i(x) ds_x \\ &= \sum_{j=1}^{N} \int_{\partial B_r} \left(\frac{\partial g}{\partial x_j}(x) - \frac{\partial g}{\partial x_j}(0)\right) \nu_j \nu_i ds_x + \sum_{j=1}^{N} \int_{\partial B_r} \frac{\partial g}{\partial x_j}(0) \nu_j \nu_i ds_x \\ &= O(r) \cdot O(r^{N-1}) + \frac{\partial g}{\partial x_i}(0) \frac{\sigma_N}{N} r^{N-1},\end{aligned}$$

where we have used

$$\int_{\partial B_r} \nu_i \nu_j ds_x = \begin{cases} 0, & (i \neq j), \\ \frac{\sigma_N}{N} r^{N-1}, & (i = j). \end{cases}$$

Thus we obtain

$$\begin{aligned}\int_{\partial B_r} \left(\frac{\partial g}{\partial \nu_x}\right) \left(\Delta^{p-1}\Gamma\right)_{x_i} ds_x &= A_{p-1}(2-N) r^{1-N} \int_{\partial B_r} \left(\frac{\partial g}{\partial \nu_x}\right) \nu_i(x) ds_x \\ &\to A_{p-1}(2-N) \frac{\sigma_N}{N} g_{x_i}(0), \quad \text{as } r \to 0.\end{aligned}$$

Also by Taylor expansion, we have

$$\begin{aligned}&\int_{\partial B_r} g \left(\frac{\partial \left(\Delta^{p-1}\Gamma\right)_{x_i}}{\partial \nu_x}\right) ds_x \\ &= A_{p-1}(2-N)(1-N) r^{-N} \int_{\partial B_r} \left(g(0) + \nabla g(0) \cdot x + O(|x|^2)\right) \nu_i(x) ds_x.\end{aligned}$$

Since

$$
\begin{aligned}
&r^{-N}\int_{\partial B_r} g(0)\nu_i(x)ds_x = 0,\\
&r^{-N}\int_{\partial B_r} O(|x|^2)\nu_i(x)ds_x = O(r^{2-N})\times O(r^{1-N}) = O(r)\to 0,\\
&r^{-N}\int_{\partial B_r} (\nabla g(0)\cdot x)\,\nu_i(x)ds_x = r^{1-N}\sum_{j=1}^N \frac{\partial g}{\partial x_j}(0)\int_{\partial B_r}\nu_j\nu_i ds_x = \frac{\sigma_N}{N}\frac{\partial g}{\partial x_i}(0),
\end{aligned}
$$

we obtain that

$$
\int_{\partial B_r} g\left(\frac{\partial\left(\Delta^{p-1}\Gamma\right)_{x_i}}{\partial\nu_x}\right)ds_x \to A_{p-1}(2-N)(1-N)\frac{\sigma_N}{N}\frac{\partial g}{\partial x_i}(0)
$$

as $r\to 0$. Thus, by (3.10), we have

$$
\begin{aligned}
J_{3,1} &\to A_{p-1}(2-N)\frac{\sigma_N}{N}g_{x_i}(0) - A_{p-1}(2-N)(1-N)\frac{\sigma_N}{N}g_{x_i}(0)\\
&= A_{p-1}(2-N)\sigma_N g_{x_i}(0) = (-1)^p g_{x_i}(0).
\end{aligned}
$$

Returning to (3.13), (3.14) with the above estimates, we have

$$
\begin{aligned}
&\sum_{k=1}^p \int_{\partial\Omega}\left(\frac{\partial\Delta^{k-1}G(x,0)}{\partial\nu_x}\right)\left(\frac{\partial\Delta^{p-k}G(x,0)}{\partial\nu_x}\right)(x,y)\nu_i(x)ds_x\\
&= 0 - J_{2,p} - J_{3,1} + o(1) = -(-1)^p g_{x_i}(0) - (-1)^p g_{x_i}(0) + o(1)\\
&= (-1)^{p-1}R_{x_i}(0) + o(1)
\end{aligned}
$$

where we have used that $g_{x_i}(0) = \frac{\partial}{\partial x_i}H(x,0)\Big|_{x=0} = \frac{1}{2}R_{x_i}(0)$. Letting $r\to 0$, we have (3.3) when $N>2p$.

Next, we prove (3.3) when $N=2p$. The argument is almost the same as before, so we should be brief. Again we only treat the case $p\geq 2$, since the formula was proved in [10] when $p=1$. Recall $\Gamma(r) = -C_p\log r$ and $\Delta^l\Gamma = B_l r^{-2l}$ for $l = 1,2,\cdots$, on ∂B_r, here B_l is defined in (3.11). Note that $B_l = 0$ for $l\geq p$. Thus if we put

$$
\tilde{B}_l := (-2l)B_l = -C_p 2^l l!\prod_{i=1}^l (2i-2p)
$$

and we agree the convention that $\tilde{B}_0 = -C_p$, we have

$$\left(\Delta^l \Gamma\right)_{x_i} = \tilde{B}_l r^{-2l-1} \nu_i(x),$$
$$\frac{\partial \left(\Delta^l \Gamma\right)_{x_i}}{\partial \nu_x} = -(2l+1)\tilde{B}_l r^{-2l-2} \nu_i(x) \tag{3.16}$$

for $l = 0, 1, 2, \cdots$, on ∂B_r. By using (3.11) and (3.16), we obtain, as before,

$$J_{1,k} = 0, \quad k \in \{1, 2, \cdots, p\},$$
$$J_{2,k} = \begin{cases} o(1), & k \in \{1, 2, \cdots, p-1\}, \\ \tilde{B}_{p-1}\sigma_N g_{x_i}(0) + o(1), & k = p, \end{cases}$$
$$J_{3,k} = o(1), \quad k \in \{2, 3, \cdots, p\},$$
$$J_{3,1} = \tilde{B}_{p-1}\frac{\sigma_N}{N} g_{x_i}(0) - \left\{-(2p-1)\tilde{B}_{p-1}\frac{\sigma_N}{N} g_{x_i}(0)\right\} + o(1)$$
$$= \tilde{B}_{p-1}\sigma_N g_{x_i}(0) + o(1)$$

as $r \to 0$, where $\tilde{B}_{p-1}\sigma_N = (-1)^p$ by (1.3). Thus returning to (3.13), (3.14), we obtain (3.3) when $N = 2p$.

Finally, we prove (3.4). Differentiating (3.3) with respect to y_j, we obtain

$$\frac{\partial^2 R}{\partial y_i \partial y_j}(y)$$
$$= \sum_{k=1}^{p} \int_{\partial\Omega} \left[\left\{ \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{k-1}}{\partial \nu_x} \right) \right\} \left(\frac{\partial \overline{G}_{p-k}}{\partial \nu_x} \right) \nu_i(x) + \left(\frac{\partial \overline{G}_{k-1}}{\partial \nu_x} \right) \nu_i(x) \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{p-k}}{\partial \nu_x} \right) \right] ds_x.$$

Note that $\left(\frac{\partial \overline{G}_j}{\partial \nu_x}(x,y)\right)\nu_i(x) = \frac{\partial \overline{G}_j}{\partial x_i}(x,y)$ for any $j = 0, 1, \cdots, p-1$ on $\partial\Omega$ since $\overline{G}_j = 0$ on $\partial\Omega$. Thus we have

$$\frac{\partial^2 R}{\partial y_i \partial y_j}(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \left[\left\{ \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{k-1}}{\partial \nu_x} \right) \right\} \left(\frac{\partial \overline{G}_{p-k}}{\partial x_i} \right) + \left(\frac{\partial \overline{G}_{k-1}}{\partial x_i} \right) \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{p-k}}{\partial \nu_x} \right) \right] ds_x$$
$$= 2\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial x_i} \right) \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial \nu_x} \right) ds_x$$
$$= 2\sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{k-1}(x,y)}{\partial x_i} \right) \frac{\partial}{\partial y_j}\left(\frac{\partial \overline{G}_{p-k}(x,y)}{\partial \nu_x} \right) ds_x.$$

□

4 Nondegeneracy of critical points of the Robin function on symmetric domains.

In this section, we prove the nondegeneracy of the critical point of the Robin function associated to the Green function of $(-\Delta)^p$ with the Navier boundary conditions on some symmetric domains. Let $\Omega \subset \mathbb{R}^N, (N \geq 2p)$ be a smooth bounded domain. We call Ω a *GNN domain*, if the followings hold.

(H1) Ω is symmetric with respect to hyperplanes $\{x_i = 0\}$ $(i = 1, \cdots, N)$.

(H2) Ω is convex with respect to x_i-directions $\quad (i = 1, \cdots, N)$.

See [6]. Note that a GNN domain need not be convex.

In this section we prove the following theorem, which extends the result obtained by Grossi [7] when $p = 1$ to the general case $p \in \mathbb{N}$.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^N, (N \geq 2p)$ be a smooth bounded domain with (H1), (H2). Let $R = R(y)$ be the Robin function of $(-\Delta)^p$ under the Navier boundary condition. Then we have*

$$\nabla_y R(0) = 0, \qquad \frac{\partial^2 R}{\partial y_i \partial y_j}(0) = \begin{cases} 0 \quad (i \neq j), \\ a_i > 0 \quad (i = j) \end{cases}$$

holds true.

We proceed as in [7]. First, we prepare some lemmas. In the following, let us denote $x = (x_1, x') \in \Omega$, $x' = (x_2, \cdots, x_N) \in \mathbb{R}^{N-1}$.

Lemma 4.2 *Assume Ω is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and set $\Omega_0 = \Omega \cap \{x_1 = 0\}$. Then for any $y_0 \in \Omega_0$, we have*

$$\overline{G}_k((x_1, x'), y_0) = \overline{G}_k((-x_1, x'), y_0), \quad \forall k = 1, 2, \cdots, p-1.$$

Proof. By Lemma 2.1 of [7], we know that $\overline{G}_{p-1}(x, y_0)$ is even with respect to x_1-variable. Let us fix any $\phi \in C_0^\infty(\Omega)$. By (2.2), $-\Delta \overline{G}_{p-2}(x, y_0) = \overline{G}_{p-1}(x, y_0)$ for $x \in \Omega$. Since Ω is symmetric with respect to the plane $\{x_1 = 0\}$ and $y_0 \in \Omega_0$, we also have $-\Delta \overline{G}_{p-1}((-x_1, x'), y_0) = \overline{G}_{p-1}((-x_1, x'), y_0)$, thus $-\Delta \overline{G}_{p-1}((-x_1, x'), y_0) = \overline{G}_{p-1}(x, y_0)$ for $x \in \Omega$, since $\overline{G}_{p-1}(x, y_0)$ is even with respect to x_1.

Multiplying ϕ to these equations, we get

$$\int_{\Omega} \left\{ \nabla \overline{G}_{p-2}((x_1, x'), y_0) - \nabla \overline{G}_{p-2}((-x_1, x'), y_0) \right\} \cdot \nabla \phi(x) dx = 0$$

for any $\phi \in C_0^{\infty}(\Omega)$. This implies that $\overline{G}_{p-2}(\cdot, y_0)$ is even with respect to x_1. By induction, we obtain the result. $\square$

Lemma 4.3 *Assume Ω is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and let g_k $(k = 0, 1, 2, \cdots, p-1)$ be odd functions with respect to x_1. Then, the unique solution u of the problem*

$$\begin{cases} (-\Delta)^p u = 0 \quad \text{in } \Omega, \\ (-\Delta)^k u = g_k \quad \text{on } \partial\Omega \ (k = 0, 1, \cdots p-1), \end{cases} \tag{4.1}$$

is also an odd function with respect to x_1.

Proof. For $x = (x_1, x') \in \overline{\Omega}$, let us denote $x^* = (-x_1, x')$. By the symmetry, we see x^* is also a point in $\overline{\Omega}$. Define $v(x) = -u(x^*)$ for $x \in \overline{\Omega}$. Then, by the oddness of g_k, we obtain

$$\begin{aligned} &(-\Delta)^p v(x) = -\left((-\Delta)^p u\right)(x^*) = 0, \quad x \in \Omega, \\ &(-\Delta)^k v(x) = -\left((-\Delta)^k u\right)(x^*) = -g_k(x^*) = g_k(x) = (-\Delta)^k u(x), \quad x \in \partial\Omega, \end{aligned}$$

for $k = 0, 1, \cdots p-1$. That is, v is also a solution of (4.1). Therefore by the uniqueness of the solution, we obtain that $v = u$, which proves the lemma. $\square$

Now, we prove Theorem 4.1.

Proof of Theorem 4.1.

By Lemma 4.2, we see that $\left(\frac{\partial \overline{G}_k}{\partial x_1}\right)(\cdot, y_0)$ is an odd function with respect to x_1 for $k = 0, 1, \cdots, p-1$. Now, let u be the unique solution of the problem

$$\begin{cases} (-\Delta)^p u = 0 \quad \text{in } \Omega, \\ (-\Delta)^k u = -\left(\frac{\partial \overline{G}_k}{\partial x_1}\right)(\cdot, y_0) \quad \text{on } \partial\Omega \ (k = 0, 1, \cdots p-1). \end{cases} \tag{4.2}$$

By Lemma 4.3, we confirm that u is also odd in x_1. Therefore, we have $u \equiv 0$ on the hyperplane $\{x_1 = 0\}$, which implies

$$\left(\frac{\partial u}{\partial x_j}\right)(y_0) = 0 \quad \text{for } j = 2, 3, \cdots, N. \tag{4.3}$$

Note that the same oddness holds for $\overline{u}_k(x) = (-\Delta)^k u(x)$ for $k = 0, 1, \cdots, p-1$. We see $\overline{u}_k$ satisfies

$$\begin{cases} -\Delta \overline{u}_k = \overline{u}_{k+1} \quad \text{in } \Omega, \ (k = 0, 1, \cdots, p-2), \\ -\Delta \overline{u}_{p-1} = 0 \quad \text{in } \Omega, \\ \overline{u}_k = -\left(\frac{\partial \overline{G}_k}{\partial x_1}\right)(\cdot, y_0) \quad \text{on } \partial\Omega, \quad (k = 0, 1, \cdots, p-1). \end{cases} \tag{4.4}$$

Recall $\overline{G}_k(x, y_0) > 0$ for $x \in \Omega$ and $\overline{G}_k(x, y_0) = 0$ for $x \in \partial\Omega$ for any $k \in \{0, 1, \cdots, p-1\}$. By the assumption (H2), we have $\left(\frac{\partial \overline{G}_k}{\partial x_1}\right)(\cdot, y_0) \geq 0$ on $\{x_1 < 0\} \cap \partial\Omega$. Also $\overline{u}_k \equiv 0$ on $\Omega \cap \{x_1 = 0\}$ by the oddness of $\overline{u}_k$ in x_1. Then the maximum principle applied to the cooperative system (4.4) on the domain $\Omega \cap \{x_1 < 0\}$ implies that $\overline{u}_k(x) < 0$ for $x \in \Omega \cap \{x_1 < 0\}$ for any $k = 0, 1, \cdots, p-1$. By applying Hopf lemma in the domain $\Omega \cap \{x_1 < 0\}$, we also have $\left(\frac{\partial \overline{u}_k}{\partial x_1}\right)(y_0) > 0$ for all $k = 0, 1, \cdots, p-1$. In particular, we have $u(x) < 0$ in $\Omega \cap \{x_1 < 0\}$ and

$$\left(\frac{\partial u}{\partial x_1}\right)(y_0) > 0. \tag{4.5}$$

On the other hand, by Green's representation formula (2.3), we see the unique solution of (4.2) can be written as

$$u(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \overline{G}_{k-1}(x, y)}{\partial \nu_x}\right) \left(\frac{\partial \overline{G}_{p-k}(x, y_0)}{\partial x_1}\right) ds_x.$$

Differentiating both sides with respect to y_j leads to

$$\frac{\partial u}{\partial y_j}(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial}{\partial y_j} \left(\frac{\partial \overline{G}_{k-1}(x, y)}{\partial \nu_x}\right) \left(\frac{\partial \overline{G}_{p-k}(x, y_0)}{\partial x_1}\right) ds_x.$$

Now, compared this to the formula (3.4) in Theorem 3.1:

$$\frac{1}{2} \frac{\partial^2 R}{\partial y_1 \partial y_j}(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial}{\partial y_j} \left(\frac{\partial \overline{G}_{k-1}(x, y)}{\partial \nu_x}\right) \left(\frac{\partial \overline{G}_{p-k}(x, y)}{\partial x_1}\right) ds_x,$$

and using (4.3), (4.5), we confirm that

$$\begin{aligned} &\frac{1}{2} \left(\frac{\partial^2 R}{\partial y_1 \partial y_j}\right)(y_0) = \left(\frac{\partial u}{\partial y_j}\right)(y_0) = 0, \ (j = 2, \cdots, N), \\ &\frac{1}{2} \left(\frac{\partial^2 R}{\partial y_1^2}\right)(y_0) = \left(\frac{\partial u}{\partial y_1}\right)(y_0) > 0. \end{aligned}$$

By changing x_1-axis to another one, we obtain the desired result. □

Acknowledgement.

Part of this work was done while the author visited S.S. Chern Institute of Mathematics (CIM) at Nankai Universiy, Tianjin. The author would like to thank the hospitality of CIM. Part of this work was supported by JSPS Grant-in-Aid for Scientic Research (Kakenhi) (B), No. 23340038.

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