A criterion of sampling theorems on Banach function spaces

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Dedicated to Professor Masami Okada for his 60th birthday

Abstract

In the present paper, we consider sampling theorems on Banach function spaces. Here we obtain a necessary and sufficient condition. For the latter half of the paper, we consider sampling theorems in terms of wavelets.

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1 Introduction

Recently to unify the results in harmonic analysis, we consider Banach function spaces. Following [3], let us recall the definition of Banach function spaces. By Banach function norms on $\mathbb{R}^n$, we mean the mapping $\|\cdot\|_X : L^1_{\rm loc}(\mathbb{R}^n) \to [0,\infty]$ satisfying the following conditions below.

1. For all $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $\|f\|_X \ge 0$ and equality holds if and only if $f = 0$.
2. For all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $a \in \mathbb{C}$, $\|a \cdot f\|_X = |a| \cdot \|f\|_X$.
3. For all $f, g \in L^1_{\rm loc}(\mathbb{R}^n)$, $\|f+g\|_X \le \|f\|_X + \|g\|_X$.
4. For all $f, g \in L^1_{\rm loc}(\mathbb{R}^n)$, if $|f| \le |g|$, then $\|f\|_X \le \|g\|_X$.
5. We have $\left\|\sup\limits_{j\in\mathbb{N}} f_j\right\|_X = \sup\limits_j \|f_j\|_X$, whenever $\{f_j\}_{j=1}^\infty \subset L^1_{\rm loc}(\mathbb{R}^n)$ satisfies $0 \le f_1 \le f_2 \le \cdots$.
6. If F is of finite measure, then χ_F, the indicator function of F, satisfies $\|\chi_F\|_X < \infty$ and there exists a constant $c_F > 0$ such that $\displaystyle\int_F |f(x)|\,dx \le c_F\|f\|_X$.

We denote by X the set of all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ such that $\|f\|_X < \infty$. Then X is a normed space. We assume in the present paper that X is a Banach space. A Banach function space on $\mathbb{R}^n$ is a Banach space that can be realized in this way.

The sampling theorem is fundamental not only in harmonic analysis but also in other fields of engineering. We review the classical results. For definiteness, define $\mathcal{F}f(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$, $\mathcal{F}^{-1}f(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\,d\xi$.

Theorem 1.1. *If $f \in L^2(\mathbb{R})$ has frequency support in $[-2\pi W, 2\pi W]$ for some $W > 0$, then the reproducing formula $f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2W}\right) \frac{\sin(\pi(2Wt-k))}{\pi(2Wt-k)}$ holds, where the right-hand side converges in $L^2(\mathbb{R})$ and uniformly over $\mathbb{R}$. Additionally we have the norm equality $\|f\|^2_{L^2(\mathbb{R})} = \frac{1}{2W} \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{2W}\right)\right|^2$.*

This result has a long history. The main people involved are Ogura, Shannon, Someya and Whitaker. There are a vast amount of literatures and we refer especially to the original papers [18, 20] and an account [4, 17] for more details. It is known that the classical theorem above is generalized to the case $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ as the following form.

Theorem 1.2. *Let $1 < p < \infty$. If $f \in L^p(\mathbb{R}^n)$ has frequency support on $[-\pi r, \pi r]^n$ for some $r > 0$, then the norm equivalence*

$$C_p^{-n} r^{n/p} \|f\|_{L^p(\mathbb{R}^n)} \le \left(\sum_{k\in\mathbb{Z}^n} \left| f\left(\frac{k}{r}\right)\right|^p \right)^{1/p} \le C_p{}^n r^{n/p} \|f\|_{L^p(\mathbb{R}^n)}$$

holds, where $C_p \ge 1$ is a constant independent of n, r and f. Moreover the reproducing formula $f(x) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k}{r}\right) \prod_{\nu=1}^{n} \frac{\sin(\pi(rx_\nu - k_\nu))}{\pi(rx_\nu - k_\nu)}$ holds, where the right-hand side converges unconditionally in $L^p(\mathbb{R}^n)$.

Gensun has initially proved Theorem 1.2 in [10] where the statement on unconditionally convergence is unclear. Later an alternative proof containing unconditional convergence has been given by Ashino–Mandai [1].

To formulate our results, we introduce notations. Write $Q(r) := [-r, r]^n$ for $r > 0$. For a closed subset Z of $\mathbb{R}^n$, let us denote by $\mathcal{S}'(\mathbb{R}^n)_Z$ the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform is supported on Z. For $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, we write $Q_{j,m} := 2^{-j}m + [0, 2^{-j})^n$. We employ sampling theorems to learn the size of $f \in \mathcal{S}'(\mathbb{R}^n)_Z$ in terms of a certain norm from the data $\{f(m)\}_{m\in\mathbb{Z}^n}$. Let $A, B \ge 0$ be a real numbers. The notation $A \lesssim B$ stands for $A \le CB$ for some constant $C > 0$ independent of the main parameters. Meanwhile, $A \gtrsim B$ means $B \lesssim A$. Finally $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. With this notation, we prove the following theorem first.

Theorem 1.3. *Assume that*

$$f_N(x) := (1 + |x_1|)^{-2N}(1 + |x_2|)^{-2N} \cdots (1 + |x_n|)^{-2N} \in X \tag{1.1}$$

for some $N \in \mathbb{N}$.

(1) *We have the following norm equivalence.*

$$\left\| \sup_{y \in \mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^{2N}} \right\|_X \sim \left\| \sup_{y \in \mathbb{R}^n} \frac{1}{(1+|y|)^{2N}} \sum_{m \in \mathbb{Z}^n} |f(m)| \chi_{Q_{0,m}}(\cdot+y) \right\|_X$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$.

(2) *We denote by* $\mathcal{K}$ *the set of all non-negative sequences* $\lambda = \{\lambda_m\}_{m \in \mathbb{Z}^n}$ *such that* $\lambda_m = 0$ *with finite number of exception. Then the following are equivalent.*

(a) *For all* $y \in \mathbb{R}^n$ *and* $\lambda \in \mathcal{K}$,

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}}(\cdot - y) \right\|_X \lesssim (1+|y|)^{2N} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}} \right\|_X . \tag{1.2}$$

(b) *For all* $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$, *we have*

$$\|f\|_X \lesssim \left\| \sum_{m \in \mathbb{Z}^n} f(m) \chi_{Q_{0,m}} \right\|_X . \tag{1.3}$$

(3) *The following are equivalent.*

(c) *For all* $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$ *and* $y \in \mathbb{R}^n$, *we have*

$$\|f(\cdot - y)\|_X \lesssim (1+|y|)^{2N} \|f\|_X. \tag{1.4}$$

(d) *For all* $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$, *we have*

$$\left\| \sum_{m \in \mathbb{Z}^n} f(m) \chi_{Q_{0,m}} \right\|_X \sim \|f\|_X. \tag{1.5}$$

(4) *If we assume* (1.4) *for all* $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$, *then*

$$\begin{aligned} \|f\|_X &\sim \left\| \sup_{y \in \mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^{2N+n+1}} \right\|_X \\ &\sim \left\| \sup_{y \in \mathbb{R}^n} \frac{1}{(1+|y|)^{2N+n+1}} \sum_{m \in \mathbb{Z}^n} |f(m)| \chi_{Q_{0,m}}(\cdot+y) \right\|_X \end{aligned}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$.

Motivated by [19], we are led to consider the translation in the theorem above.

We denote by $M^{(\eta)}$ the Hardy–Littlewood maximal operator. That is,

$$M^{(\eta)} f(x) := \sup_{Q \in \mathcal{Q}(x)} \left(\frac{1}{|Q|} \int_Q |f(y)|^\eta \, dy \right)^{1/\eta}$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, where $\mathcal{Q}(x)$ denotes the set of all cubes whose sides are parallel to the coordinate axis and which contain x. Also, we write $M := M^{(1)}$. As a corollary, when the maximal operator is bounded, then our assumptions are automatically fulfilled.

Corollary 1.4. *If there exists $\eta > 0$ such that $\|M^{(\eta)}f\|_X \lesssim \|f\|_X$ for all measurable functions f, then*

$$\|f\|_X \sim \left\| \sum_{m\in\mathbb{Z}^n} f(m)\chi_{Q_{0,m}} \right\|_X \tag{1.6}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$.

Here are some examples of X we envisage.

Example 1.5.

1. The simplest case is the one when $X = L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. Namely

   $$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \sum_{m\in\mathbb{Z}^n} f(m)\chi_{Q_{0,m}} \right\|_{L^p(\mathbb{R}^n)}$$

   holds for all $f \in L^p(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$. Comparing with Theorem 1.2, we note that this norm equivalence is true even if $p = 1$ or $p = \infty$.

2. Our theory is readily applicable to the weighted Lebesgue space $X = L^p(\mathbb{R}^n, (1+|x|)^a\,dx)$ for some $a \in \mathbb{R}$.

   The same can be said for the space $X = L^p(\mathbb{R}^n, w(x)\,dx)$ for some $w \in A_p$ with $1 \le p < \infty$, where A_p denotes the set of all weights w for which the quantity

   $$A_p(w) = \sup_{Q\,:\,\text{cubes}} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty$$

   when $1 < p < \infty$ and

   $$A_1(w) = \lim_{\kappa\downarrow 1} A_\kappa(w) < \infty$$

   when $p = 1$.

3. Orlicz spaces fall under the scope of Theorem 1.3. The definition of Orlicz spaces are given as follows: First, by a Young function, we mean a continuous bijection $\Phi : [0,\infty) \to [0,\infty)$. For such a function Φ, we define

   $$\|f\|_{L^\Phi} := \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

   It is easy to see that $\|\cdot\|_{L^\Phi}$ is a Banach function norm. For example, Orlicz spaces can be used to describe the intersection spaces as the example $\Phi(t) = t + t^2$ shows.

4. Let $p(\,\cdot\,) : \mathbb{R}^n \to [1, \infty]$ be a measurable function and consider the variable Lebesgue space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ (cf. [15]). According to [5, 7], if we assume

   (a) For all $|x-y| \le 1/2$,

   $$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \lesssim \frac{1}{-\log(|x-y|)},$$

   (b) There exists a constant $\tilde{p}$ such that for all $x \in \mathbb{R}^n$,

   $$\left|\frac{1}{p(x)} - \tilde{p}\right| \lesssim \frac{1}{\log(e+|x|)},$$

   then M is bounded on $L^{p(\,\cdot\,)/\eta}(\mathbb{R}^n)$ for every number η such that $0 < \eta < \operatorname{ess\,inf}\{p(x) \,:\, x \in \mathbb{R}^n\}$. Hence $\|M^{(\eta)} f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}$ holds for all $f \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$.

5. If we modify the argument, then we see that we can take $\mathrm{BMO}(\mathbb{R}^n)$ which consists of all functions f satisfying

   $$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q\,:\,\text{cubes}} \frac{1}{|Q|}\int_Q \left| f(x) - \left(\frac{1}{|Q|}\int_Q f(y)\,dy\right)\right| dx < \infty.$$

   The quasi-norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ becomes a norm when we identify functions which differ by a constant. The maximal operator M is shown to be bounded in [2].

Below we describe how we organized the present paper. Theorem 1.3 is proved in Section 2. Section 3 deals with sampling theorems from the viewpoint of wavelet characterization.

2 Proof of Theorem 1.3

The following lemma, which dates back to [8], was the starting point of the sampling theorem and the (modern) theory of function spaces. This lemma shall be used in the proof of Theorem 1.3.

Lemma 2.1. *Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$ be an auxiliary bump function satisfying*

$$\chi_{Q(3)} \le \kappa \le \chi_{Q(3+1/100)}.$$

Then any $f \in \mathcal{S}'(\mathbb{R}^n)_{Q(3)}$ has the following expansion:

$$f = (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} f(m)\mathcal{F}^{-1}\kappa(\cdot - m), \tag{2.1}$$

where the convergence takes place in $\mathcal{S}'(\mathbb{R}^n)$.

To prove Theorem 1.3 (2), we need the following lemma as well.

Lemma 2.2. *Let $N \in \mathbb{N}$ be arbitrary and f_N given by* (1.1)*. Then there exists a function $\varphi_N \in \mathcal{S}'(\mathbb{R}^n)_{Q(1)}$ such that $\varphi_N(x) \sim f_N(x)$.*

Proof. First observe that the function τ of a variable $t \in \mathbb{R}$, which is given by

$$\tau(t) := \mathcal{F}^{-1}[\chi_{[-1,1]}](t) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin t}{t}, \tag{2.2}$$

belongs to $\mathcal{S}'(\mathbb{R})_{[-1,1]}$ and vanishes at $2\pi\mathbb{Z} \setminus \{0\}$. Therefore, the function given by

$$\psi(x) := \prod_{j=1}^{2N} \tau(x_j)^{2N} \tag{2.3}$$

belongs to $\mathcal{S}'(\mathbb{R}^n)_{Q(2N)}$. Observe that it is non-negative and that it vanishes at $(2\pi\mathbb{Z})^n \setminus \{(0,0,\cdots,0)\}$. If we define

$$\varphi_N(x) := \sum_{l_1=-10N}^{10N} \sum_{l_2=-10N}^{10N} \cdots \sum_{l_n=-10N}^{10N} \psi\left(\frac{x-(l_1,l_2,\cdots,l_n)}{2N}\right), \tag{2.4}$$

then we see that φ_N satisfies the desired property. $\square$

Proof of Theorem 1.3.

(1) Let us first establish that

$$\sup_{z \in \mathbb{R}^n} \frac{|f(x+z)|}{(1+|z|)^{2N}} \gtrsim \frac{1}{(1+|y|)^{2N}} \sum_{m \in \mathbb{Z}^n} |f(m)| \chi_{Q_{0,m}}(x+y) \tag{2.5}$$

for all $x, y \in \mathbb{R}^n$. To this end, we freeze x and y arbitrarily and let us estimate the right-hand side. Denote by $m_{x,y}$ the unique integer such that $x+y \in Q_{0,m_{x,y}}$. Then we have

$$\frac{1}{(1+|y|)^{2N}} \sum_{m \in \mathbb{Z}^n} |f(m)| \chi_{Q_{0,m}}(x+y) = \frac{1}{(1+|y|)^{2N}} |f(m_{x,y})|. \tag{2.6}$$

Note that

$$(1+|x|)(1+|y|) \geq 1+|x+y|. \tag{2.7}$$

If we use (2.7) and the fact that $|x+y-m_{x,y}| \leq n$, then we obtain

$$1+|x-m_{x,y}| \leq (1+|y|)(1+|x+y-m_{x,y}|) \leq (1+n)(1+|y|). \tag{2.8}$$

If we insert (2.8) to (2.6), then we have

$$\begin{aligned}\frac{1}{(1+|y|)^{2N}} |f(m_{x,y})| &\lesssim \frac{1}{(1+|x-m_{x,y}|)^{2N}} |f(m_{x,y})| \\ &\lesssim \sup_{w \in \mathbb{R}^n} \frac{1}{(1+|x-w|)^{2N}} |f(w)|,\end{aligned}$$

which proves (2.5). Next, let us establish that

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x+y)|}{(1+|y|)^{2N}} \lesssim \frac{1}{(1+|y|)^{2N}} \sum_{m\in\mathbb{Z}^n} |f(m)|\chi_{Q_{0,m}}(x+y). \tag{2.9}$$

To this end, we fix $y \in \mathbb{R}^n$. Then, by virtue of Lemma 2.1, we have

$$\frac{|f(x+y)|}{(1+|y|)^{2N}} \lesssim \sum_{m\in\mathbb{R}^n} |f(m)| \frac{|\mathcal{F}^{-1}\kappa(x+y-m)|}{(1+|y|)^{2N}}.$$

If we use the fact that $\kappa \in \mathcal{S}(\mathbb{R}^n)$ and (2.7), then we obtain

$$\begin{aligned}\frac{|f(x+y)|}{(1+|y|)^{2N}} &\lesssim \sum_{m\in\mathbb{R}^n} \frac{|f(m)|}{(1+|x+y-m|)^{2N}(1+|y|)^{2N}} \\ &\lesssim \sum_{m\in\mathbb{R}^n} \frac{|f(m)|}{(1+|x-m|)^{2N}} \\ &\lesssim \frac{1}{(1+|y|)^{2N}} \sum_{m\in\mathbb{Z}^n} |f(m)|\chi_{Q_{0,m}}(x+y).\end{aligned}$$

This proves (2.9).

From (2.5) and (2.9), we obtain

$$\left\| \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^{2N}} \right\|_X \sim \left\| \sup_{y\in\mathbb{R}^n} \frac{1}{(1+|y|)^{2N}} \sum_{m\in\mathbb{Z}^n} |f(m)|\chi_{Q_{0,m}}(\cdot+y) \right\|_X$$

and (1) is therefore proved.

(2) Assume (a) and that f belongs to $\mathcal{S}'(\mathbb{R}^n)_{Q(1)}$. Then we have by (2.1) and the fact that $\kappa \in \mathcal{S}(\mathbb{R}^n)$,

$$|f(x)| \lesssim \sum_{m\in\mathbb{Z}^n} |f(m)|(1+|x-m|)^{-2N-n-1}.$$

If we use (2.7), then we obtain

$$\begin{aligned}|f(x)| &\lesssim \sum_{l\in\mathbb{Z}^n} |f(l)|(1+|x-l|)^{-2N-n-1} \\ &\lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left(\sum_{l\in\mathbb{Z}^n} |f(l)|\chi_{Q_{0,l}}(x-m) \right).\end{aligned} \tag{2.10}$$

Consequently, we have

$$
\begin{aligned}
\|f\|_X &\lesssim \left\| \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left(\sum_{l\in\mathbb{Z}^n} |f(l)|\chi_{Q_{0,l}}(\cdot - m) \right) \right\|_X \\
&\le \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}}(\cdot - m) \right\|_X \\
&= \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left\| \sum_{l\in\mathbb{Z}^n} f(l-m)\chi_{Q_{0,l}} \right\|_X \\
&= \lim_{N\to\infty} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left\| \sum_{l\in\mathbb{Z}^n,\, |l-m|\le N} f(l-m)\chi_{Q_{0,l}} \right\|_X .
\end{aligned}
$$

If we use (a) with $\lambda = \{\lambda_l\}_{l\in\mathbb{Z}^n} = \{|f(l)\chi_{\{|k|\le N\}}(l)|\}_{l\in\mathbb{Z}^n}$, then we have

$$
\|f\|_X \lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{n+1}} \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}} \right\|_X \sim \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}} \right\|_X . \tag{2.11}
$$

Now assume (b). Since $\chi_{[0,1]^n} \lesssim f_N$, we obtain

$$
\left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}}(\cdot - y) \right\|_X \lesssim \left\| \sum_{m\in\mathbb{Z}^n} \lambda_m f_N(\cdot - y - m) \right\|_X .
$$

By the definition of f_N (1.1), we have

$$
\begin{aligned}
\left\| \sum_{m\in\mathbb{Z}^n} \lambda_m f_N(\cdot - y - m) \right\|_X &\lesssim (1+|y|)^{2N} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_m f_N(\cdot - m) \right\|_X \\
&\lesssim (1+|y|)^{2N} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \varphi_N(\cdot - m) \right\|_X .
\end{aligned}
$$

Assuming (b), then we have

$$
\left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \varphi_N(\cdot - m) \right\|_X \lesssim \left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}} \right\|_X . \tag{2.12}
$$

If we use (2.12), then we obtain

$$
\left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}}(\cdot - y) \right\|_X \lesssim (1+|y|)^{2N} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_m \chi_{Q_{0,m}} \right\|_X ,
$$

which proves (a).

(3) Assume first (c). We take a function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}\Phi$ equals 1 on $Q(1)$. Note that $f = c_n \Phi * f$ for some constant c_n depending only on n. By using (2.7) we have

$$\begin{aligned}\frac{|f(x+y)|}{(1+|y|)^{2N+n+1}} &\lesssim \int_{\mathbb{R}^n} \frac{|\Phi(x+y-z)f(z)|}{(1+|y|)^{2N+n+1}}\, dz \\ &\lesssim \int_{\mathbb{R}^n} \frac{|f(z)|}{(1+|x+y-z|)^{2N+n+1}(1+|y|)^{2N+n+1}}\, dz \\ &\lesssim \int_{\mathbb{R}^n} \frac{|f(z)|}{(1+|x-z|)^{2N+n+1}}\, dz.\end{aligned}$$

By replacing x with $x+y$, we obtain

$$\frac{|f(x+y)|}{(1+|y|)^{2N+n+1}} \lesssim \int_{\mathbb{R}^n} \frac{|f(z)|}{(1+|x-z|)^{2N+n+1}}\, dz = \int_{\mathbb{R}^n} \frac{|f(x-z)|}{(1+|z|)^{2N+n+1}}\, dz.$$

Consequently, since X is a Banach function space, we have

$$\begin{aligned}\left\| \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^{2N+n+1}} \right\|_X &\lesssim \left\| \int_{\mathbb{R}^n} \frac{|f(\cdot-z)|}{(1+|z|)^{2N+n+1}}\, dz \right\|_X \\ &\lesssim \int_{\mathbb{R}^n} \left\| \frac{|f(\cdot-z)|}{(1+|z|)^{2N+n+1}} \right\|_X dz.\end{aligned}$$

We obtain by using (c)

$$\left\| \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^{2N+n+1}} \right\|_X \lesssim \int_{\mathbb{R}^n} \frac{\|f\|_X}{(1+|z|)^{n+1}}\, dz \lesssim \|f\|_X. \tag{2.13}$$

If we combine (a), (b), (2.11) and (2.13), we obtain (d).

Assume (d). By (2.10) we obtain

$$|f(x-y)| \lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left(\sum_{l\in\mathbb{Z}^n} |f(l)|\chi_{Q_{0,l}}(x-y-m) \right).$$

Since (d) implies (a), we have

$$\begin{aligned}\|f(\cdot-y)\|_X &\lesssim \left\| \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left(\sum_{l\in\mathbb{Z}^n} |f(l)|\chi_{Q_{0,l}}(\cdot-y-m) \right) \right\|_X \\ &\lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{2N+n+1}} \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}}(\cdot-y-m) \right\|_X \\ &\lesssim \sum_{m\in\mathbb{Z}^n} \frac{(1+|y|+|m|)^{2N}}{(1+|m|)^{2N+n+1}} \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}} \right\|_X \\ &\sim (1+|y|)^{2N} \left\| \sum_{l\in\mathbb{Z}^n} f(l)\chi_{Q_{0,l}} \right\|_X.\end{aligned}$$

Assuming (d), we obtain

$$\|f(\cdot - y)\|_X \lesssim (1+|y|)^{2N}\|f\|_X.$$

Hence (c) was obtained.

(4) If we mimic the proof of (1) and recall Lemma 2.1, then we have

$$\begin{aligned}\|f\|_X &\lesssim \left\|\sup_{y\in\mathbb{R}^n}\frac{|f(\cdot+y)|}{(1+|y|)^{2N+n+1}}\right\|_X\\ &\sim \left\|\sup_{y\in\mathbb{R}^n}\frac{1}{(1+|y|)^{2N+n+1}}\sum_{m\in\mathbb{Z}^n}|f(m)|\chi_{Q_{0,m}}(\cdot+y)\right\|_X\end{aligned}$$

for all $f\in\mathcal{S}'(\mathbb{R}^n)_{Q(1)}$. If we combine this inequality with (2.13), then we obtain (4).

□

Proof of Corollary 1.4. To prove Corollary 1.4, we need the following lemma.

Lemma 2.3 (Planchrel–Polya–Nikols'kij [21, p.16])**.** *Let $0<\eta<\infty$. Then for $f\in\mathcal{S}'(\mathbb{R}^n)_{Q(1)}$, we have*

$$\sup_{y\in\mathbb{R}^n}\frac{|f(x-y)|}{(1+|y|)^{n/\eta}} \lesssim M^{(\eta)}f(x).$$

Once we admit Lemma 2.3, we obtain

$$\left\|\sum_{m\in\mathbb{Z}^n}f(m)\chi_{Q_{0,m}}\right\|_X \lesssim \|M^{(\eta)}f\|_X \lesssim \|f\|_X.$$

Observe also that

$$\left|\sum_{m\in\mathbb{Z}^n}\lambda_m\chi_{Q_{0,m}}(x-y)\right| \lesssim (1+|y|)^{n/\eta}M^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_m\chi_{Q_{0,m}}\right](x) \qquad (2.14)$$

for all $x,y\in\mathbb{R}^n$. To see this, first we observe that the left-hand side is made up of at most 1 term. Let m be such that $\lambda_m\chi_{Q_{0,m}}(x-y)\neq 0$. Note that

$$M^{(\eta)}\left[\lambda_m\chi_{Q_{0,m}}\right](x) \sim |\lambda_m|(1+|x-m|)^{-n/\eta}.$$

Consequently, we obtain

$$\begin{aligned}|\lambda_m|\chi_{Q_{0,m}}(x-y) &= (1+|y|)^{n/\eta}(1+|y|)^{-n/\eta}|\lambda_m|\chi_{Q_{0,m}}(x-y)\\ &\lesssim (1+|y|)^{n/\eta}(1+|x-m|)^{-n/\eta}(1+|x-m-y|)^{n/\eta}|\lambda_m|\\ &\lesssim (1+|y|)^{n/\eta}(1+|x-m|)^{-n/\eta}|\lambda_m|\\ &\lesssim (1+|y|)^{n/\eta}M^{(\eta)}\left[\lambda_m\chi_{Q_{0,m}}\right](x).\end{aligned}$$

Thus, (2.14) is proved.

If we use the boundedness of $M^{(\eta)}$, we see that (a) holds. Therefore, the reverse inequality being obtained in Theorem 1.3, we obtain the desired result.

□

3 The sampling theorems in terms of wavelets

Let $\varphi^0 \in \mathcal{S}(\mathbb{R})$ be a real-valued function satisfying

(a) $\mathcal{F}\varphi^0(\xi) \equiv \dfrac{1}{\sqrt{2\pi}} \quad \left(\xi \in \left(-\dfrac{2}{3}\pi, \dfrac{2}{3}\pi\right)\right)$,

(b) $\operatorname{supp} \mathcal{F}\varphi^0 \subset \left[-\dfrac{4}{3}\pi, \dfrac{4}{3}\pi\right]$,

(c) $|\mathcal{F}\varphi^0(\xi)|^2 + |\mathcal{F}\varphi^0(2\pi - \xi)|^2 \equiv \dfrac{1}{2\pi} \quad (\xi \in (0, 2\pi))$,

(d) $0 \le \mathcal{F}\varphi^0(\xi) = \mathcal{F}\varphi^0(-\xi) \le \dfrac{1}{\sqrt{2\pi}} \quad (\xi \in \mathbb{R})$.

Define φ^1 by

$$\mathcal{F}\varphi^1(\xi) := -\sqrt{2\pi} e^{i\xi/2} \mathcal{F}\varphi^0(\xi/2) \left\{\mathcal{F}\varphi^0(\xi + 2\pi) + \mathcal{F}\varphi^0(\xi - 2\pi)\right\}.$$

Note that $\varphi^1 \in \mathcal{S}(\mathbb{R})$ is real-valued and its frequency support is contained in the closed set $\left[-\dfrac{8}{3}\pi, -\dfrac{2}{3}\pi\right] \cup \left[\dfrac{2}{3}\pi, \dfrac{8}{3}\pi\right]$. Let $E := \{0,1\}^n \setminus \{(0, 0, \dots, 0)\}$. For each $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in E$, we define

$$\varphi(x) := \prod_{\nu=1}^n \varphi^0(x_\nu), \;\; \psi^\epsilon(x) := \prod_{\nu=1}^n \varphi^{\epsilon_\nu}(x_\nu) \quad (x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n). \tag{3.1}$$

The construction of such functions is due to Meyer [16]. Actually, Meyer established that the system

$$\{\varphi_{0,k}\}_{k \in \mathbb{Z}^n} \cup \{\psi^\epsilon_{j,k}\}_{j \in \mathbb{N} \cup \{0\},\, k \in \mathbb{Z}^n,\, \varepsilon \in E}$$

can be arranged so that it is an orthonormal basis in $L^2(\mathbb{R}^n)$. We also define the following square functions:

$$\begin{aligned} If &:= \left(\sum_{k \in \mathbb{Z}^n} \left|\langle f, \varphi_{0,k}\rangle \chi_{Q_{0,k}}\right|^2\right)^{1/2}, \\ Jf &:= \left(\sum_{\epsilon \in E} \sum_{j=0}^\infty \sum_{k \in \mathbb{Z}^n} \left|\langle f, \psi^\epsilon_{j,k}\rangle 2^{jn/2} \chi_{Q_{j,k}}\right|^2\right)^{1/2}. \end{aligned}$$

To formulate our results, we recall the following definitions.

Definition 3.1. Let X be a Banach function space on $\mathbb{R}^n$.

1. The space X is said to have absolutely continous norm, if $f, g \in X$ and $\{f_j\}_{j=1}^{\infty} \subset X$ satisfies

$$\sup_{j\in\mathbb{N}} |f_j(x)| \le g(x), \ \lim_{j\to\infty} f_j(x) = f(x) \quad (\text{a.e.} x \in \mathbb{R}^n),$$

then

$$\lim_{j\to\infty} \|f_j - f\|_X = 0.$$

2. A series $\sum_{j=1}^{\infty} f_j$ of X is said to converge unconditionally, if, for all bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{j=1}^{\infty} f_{\sigma(j)}$ converges.

Recall that X' denotes the associate space X [3, Chapter 1].

Lemma 3.2. *Let X be a Banach function space on $\mathbb{R}^n$. Suppose the following:*

$(A1)$ *The operotors M and J are bounded both on X and on X'.*

$(A2)$ *The space $X \cap L^2(\mathbb{R}^n)$ is dense in X and the space $X' \cap L^2(\mathbb{R}^n)$ is dense in X'.*

Then we have

$$\|If\|_X + \|Jf\|_X \sim \|f\|_X \tag{3.2}$$

for all $f \in X$. If X has absolutely continuous norm, then

$$f = \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \varphi_{0,k} + \sum_{\epsilon\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^{\epsilon}\rangle \psi_{j,k}^{\epsilon}$$

the convergence takes place unconditionally.

Proof. This theorem seems somehow known. However, it has never explicitly appeared in any literature at least for Banach function spaces in general. So we outline the proof. Since M is assumed bounded in X, I is bounded on X as well. Since J is assumed bounded too, it follows that

$$\|If\|_X + \|Jf\|_X \lesssim \|f\|_X.$$

So the heart of the matter is to prove the reverse inequality, that is,

$$\|If\|_X + \|Jf\|_X \gtrsim \|f\|_X \tag{3.3}$$

for all $f \in X$. Since $X \cap L^2(\mathbb{R}^n)$ is dense in X, it suffices to prove (3.3) for $f \in X \cap L^2(\mathbb{R}^n)$. In this case, for all $g \in X' \cap L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx\right| \\
&\leq \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |\langle f,\varphi_{0,k}\rangle\chi_{Q_{0,k}}(x)\overline{\langle g,\varphi_{0,k}\rangle}\chi_{Q_{0,k}}(x)|\,dx \\
&\quad+\sum_{\epsilon\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |\langle f,\psi^{\epsilon}_{j,k}\rangle 2^{jn/2}\chi_{Q_{j,k}}(x)\overline{\langle g,\psi^{\epsilon}_{j,k}\rangle}2^{jn/2}\chi_{Q_{j,k}}(x)|\,dx \\
&\leq \|If\|_X\|Ig\|_{X'}+\|Jf\|_X\|Jg\|_{X'}.
\end{aligned}$$

In view of the density assumption $(A2)$ and the fact that $(X')' = X$ [3, Chapter 1], we obtain the result.

Once we obtain (3.2), the unconditional convergence follows immediately. □

Remark 3.3. The idea of using duality dates back to [11].

Here we envisage the following setting as examples.

Examples 3.4.

1. Let $1 < p < \infty$ and w be a Muckenhoupt's A_p weight. Then J is bounded on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ (cf. [9]). Thus $(A1)$ and $(A2)$ hold with $X = L^p_w(\mathbb{R}^n)$.
2. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a measurable function and consider the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. The associate space of $L^{p(\cdot)}(\mathbb{R}^n)$ coincides with $L^{p'(\cdot)}(\mathbb{R}^n)$, where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$. If $p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\} < \infty$, then the set of all infinitely differentiable functions is dense in $L^{p(\cdot)}(\mathbb{R}^n)$. According to [5, 6, 7], if we additionally assume

   (a) $1 < p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}$,

   (b) For all $|x - y| \leq 1/2$,
   $$|p(x) - p(y)| \lesssim \frac{1}{-\log(|x - y|)},$$

   (c) There exists a constant $p(\infty)$ such that for all $x \in \mathbb{R}^n$,
   $$|p(x) - p(\infty)| \lesssim \frac{1}{\log(e + |x|)},$$

   then M and J are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Hence $(A1)$ and $(A2)$ are true with $X = L^{p(\cdot)}(\mathbb{R}^n)$, provided the conditions above are satisfied.

If we apply Lemma 3.2, then we obtain the following results. Theorem 3.6 is recorded in [13].

Theorem 3.5. *Let $1 < p < \infty$, $r \geq \frac{2}{3}$, $a = \frac{3}{2}r$ and let w be a Muckenhoupt's A_p weight. Then for all $f \in L^p_w(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)_{Q(\pi r)}$ we have the norm equivalence*

$$\left\| \left(\sum_{k \in \mathbb{Z}^n} \left| f\left(\frac{k}{a}\right) \chi_{Q_{0,k}}(a\,\cdot\,) \right|^2 \right)^{1/2} \right\|_{L^p_w(\mathbb{R}^n)} \sim \|f\|_{L^p_w(\mathbb{R}^n)}, \tag{3.4}$$

where the implicit constants are independent of r and f. Moreover we obtain the reproducing formula

$$f(x) = \sum_{k \in \mathbb{Z}^n} f\left(\frac{k}{a}\right) \varphi(ax - k), \tag{3.5}$$

where the right-hand side converges unconditionally on $L^p_w(\mathbb{R}^n)$.

Theorem 3.6. *Let $p(\,\cdot\,) : \mathbb{R}^n \to [1, \infty]$ be a measurable function satisfying the following:*

1. $1 < p_- \leq p_+ < \infty$,
2. *For all* $|x - y| \leq 1/2$,
$$|p(x) - p(y)| \lesssim \frac{1}{-\log(|x - y|)},$$
3. *There exists a constant $p(\infty)$ such that for all $x \in \mathbb{R}^n$,*
$$|p(x) - p(\infty)| \lesssim \frac{1}{\log(e + |x|)}.$$

Let $r \geq 2/3$, $R := r^{n(1/p_- - 1/p_+)}$, $a := 3r/2$. Then we have the norm equivalence

$$R^{-1}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{k \in \mathbb{Z}^n} \left| f\left(\frac{k}{a}\right) \chi_{Q_{0,k}}(a\,\cdot\,) \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim R\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)_{Q(\pi r)}$, where the implicit constants are independent of r and f. Moreover we obtain the reproducing formula

$$f(x) = \sum_{k \in \mathbb{Z}^n} f\left(\frac{k}{a}\right) \varphi(ax - k), \tag{3.6}$$

where the right-hand side converges unconditionally on $L^{p(\cdot)}(\mathbb{R}^n)$.

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