TOPOLOGY OF ITERATED S^1-BUNDLES

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Abstract. In this paper we investigate what kind of manifolds arise as the total spaces of iterated S^1-bundles. A real Bott tower studied in [2], [13] and [14] is an example of an iterated S^1-bundle. We show that the total space of an iterated S^1-bundle is homeomorphic to an infra-nilmanifold. A real Bott manifold, which is the total space of a real Bott tower, provides an example of a closed flat Riemannian manifold. We also show that real Bott manifolds are the only closed flat Riemannian manifolds obtained from iterated $\mathbb{R}P^1$-bundles. Finally we classify the homeomorphism types of the total spaces of iterated S^1-bundles in dimension 3.

1. Introduction

In this paper, an S^1-bundle is a fiber bundle with the circle S^1 as a fiber and an *iterated S^1-bundle* of height n is a sequence of smooth S^1-bundles starting with a point:

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \{\text{a point}\}. \tag{1.1}$$

The total space X_n of an iterated S^1-bundle is a closed aspherical manifold of dimension n. Our concern is what kind of aspherical manifolds arise as the total space X_n. If all the S^1-bundles $X_i \to X_{i-1}$ in (1.1) are principal, then one sees that the fundamental group of X_n is nilpotent and hence X_n is homeomorphic to a nilmanifold, and conversely any closed nilmanifold arises as the total space of an iterated *principal* S^1-bundle (see [5, Proposition 3.1]). Our first main result is the following.

Theorem 1.1. *The total space X_n of an iterated S^1-bundle of height n is homeomorphic to an infra-nilmanifold. In fact, some 2^{n-1}-cover of X_n is homeomorphic to a nilmanifold.*

The projectivization of a plane bundle, called an $\mathbb{R}P^1$-bundle, is an S^1-bundle, so an iterated $\mathbb{R}P^1$-bundle is an iterated S^1-bundle. The total spaces

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of iterated $\mathbb{R}P^1$-bundles are somewhat special. For instance, the first Betti number $b_1(X_n;\mathbb{Z}_2)$ of the total space X_n in (1.1) with $\mathbb{Z}_2$-coefficient, where $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$, is at most n and it turns out that X_n is the total space of an iterated $\mathbb{R}P^1$-bundle if and only if $b_1(X_n;\mathbb{Z}_2)=n$.

When every plane bundle used to projectivise in the iterated $\mathbb{R}P^1$-bundle is a Whitney sum of two line bundles, the iterated $\mathbb{R}P^1$-bundle is called a *real Bott tower* and the total space X_n is called a *real Bott manifold.* A real Bott manifold provides an example of a flat Riemannian manifold and the diffeomorphism classification of real Bott manifolds has been completed in [2], see also [13] and [14]. Unless every plane bundle used to projectivise in the iterated $\mathbb{R}P^1$-bundle is a Whitney sum of two line bundles, the total space X_n is not necessarily flat Riemannian. However, we may ask whether more flat Riemannian manifolds than real Bott manifolds can be produced from iterated $\mathbb{R}P^1$-bundles. The following theorem says that the answer is no.

Theorem 1.2. *If the total space of an iterated $\mathbb{R}P^1$-bundle is homeomorphic to a closed flat Riemannian manifold, then it is homeomorphic to a real Bott manifold.*

The total space X_2 of an iterated S^1-bundle of height 2 is either the torus $(S^1)^2$ or the Klein bottle. However, the total spaces X_3 of iterated S^1-bundles of height 3 are abundant and we classify them up to homeomorphism. It turns out that there are six flat Riemannian manifolds, an infinite family of nilmanifolds and an infinite family of infra-nilmanifolds, see Theorem 5.9 for details. It is known that there are ten homeomorphism classes of closed flat Riemannian manifolds in dimension 3 and our result shows that six of them arise from iterated S^1-bundles while four of them arise from iterated $\mathbb{R}P^1$-bundles. In a forthcoming paper, we will classify 4-dimensional closed flat Riemannian manifolds arising from iterated S^1-bundles. It is known in [1] that there are 74 homeomorphism classes of closed flat Riemannian manifolds in dimension 4 and it turns out that 35 of them arise from iterated S^1-bundles while 12 of them arise from iterated $\mathbb{R}P^1$-bundles.

This paper is organized as follows. We study fundamental groups of S^1-bundles in Section 2 and of iterated S^1-bundles in Section 3. In Section 4 we prove that the total space of an iterated S^1-bundle of height n contains a nilpotent normal subgroup of index 2^{n-1} in its fundamental group, which implies Theorem 1.1. In Section 5 we classify isomorphism classes of possible fundamental groups of the total spaces of iterated S^1-bundles of height 3 and then show that those isomorphism classes can be realized by iterated S^1-bundles of height 3. Section 6 is devoted to the proof of Theorem 1.2.

2. S^1-BUNDLES

When ξ is a plane bundle with an Euclidean fiber metric, the unit circle bundle $S(\xi)$ of ξ is an S^1-bundle. Conversely, if the base space B is a closed smooth manifold, then any S^1-bundle over B can be regarded as the unit circle bundle of some plane bundle with an Euclidean fiber metric because the inclusion map $\mathrm{O}(2) \to \mathrm{Diff}(S^1)$ is known to be homotopy equivalent so that the structure group of the circle bundle, that is $\mathrm{Diff}(S^1)$, reduces to $\mathrm{O}(2)$. This also shows that a smooth S^1-bundle over a closed smooth manifold is isomorphic to a principal S^1-bundle if and only if the S^1-bundle is orientable (see [16, Proposition 6.15] for a direct proof).

The projectivization $P(\eta)$ of a plane bundle η, called the $\mathbb{R}P^1$*-bundle*, is also an S^1-bundle and fiber-wisely double covered by $S(\eta)$. If η is orientable, then η admits a complex structure so that one can form its 2-fold tensor product $\eta \otimes_{\mathbb{C}} \eta$ over the complex numbers $\mathbb{C}$ and then $P(\eta) = S(\eta \otimes_{\mathbb{C}} \eta)$.

Lemma 2.1. *Let $S^1 \xrightarrow{i} X \xrightarrow{\pi} B$ be an S^1-bundle over an arcwise connected space B and let $\pi_1(B)$ be finitely presented as follows*

$$\langle x_1, \cdots, x_p \mid f_j(x_1, \cdots, x_p) = 1 \ (1 \le j \le q)\rangle$$

and let $i_* : \pi_1(S^1) \to \pi_1(X)$ be injective. Then $\pi_1(X)$ has a presentation of the form

$$\langle x_1, \cdots, x_p, \sigma \mid x_i \sigma x_i^{-1} = \sigma^{\pm 1}, f_j(x_1, \cdots, x_p) = \sigma^{a_j} \ (1 \le i \le p, 1 \le j \le q)\rangle$$

for some integers a_j.

Moreover, the following are equivalent:

1. *the S^1-bundle $X \to B$ is fiber-wisely double covered by another S^1-bundle,*
2. *every integer a_j above is even,*
3. *$b_1(X, \mathbb{Z}_2) = b_1(B, \mathbb{Z}_2)$ where $b_1(\ , \mathbb{Z}_2)$ denotes the first Betti number with $\mathbb{Z}_2$-coefficient.*

Proof. The S^1-bundle $X \to B$ induces a short exact sequence $1 \to \pi_1(S^1) \to \pi_1(X) \to \pi_1(B) \to 1$. Taking σ as a generator of $\pi_1(S^1)$, it can be seen easily that the first part of the Lemma holds.

(1) $\Longrightarrow$ (2). Assume that the S^1-bundle $X \to B$ is fiber-wisely double covered by another S^1-bundle $Y \to B$. Then there is a fiber preserving map ϕ between them

$$\begin{tikzcd} S^1 \arrow[r, "\bar{\phi}"] \arrow[d, "i'"] & S^1 \arrow[d, "i"] \\ Y \arrow[r, "\phi"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

where $\phi : Y \to X$ and the restriction on the fiber $\bar{\phi} : S^1 \to S^1$ are double covering projections. Therefore ϕ induces the following commuting diagram between exact sequences of groups

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(S^1) \arrow[r, "i'_*"] \arrow[d, "\bar\phi_*"] & \pi_1(Y) \arrow[r, "\pi'_*"] \arrow[d, "\phi_*"] & \pi_1(B) \arrow[r] \arrow[d, "="] & 1 \\
\cdots \arrow[r] & \pi_1(S^1) \arrow[r, "i_*"] & \pi_1(X) \arrow[r, "\pi_*"] & \pi_1(B) \arrow[r] & 1
\end{tikzcd}$$

Since i_* and $\bar\phi_*$ are injective, $i_*\bar\phi_* = \phi_* i'_*$ yields that i'_* is injective. Let τ be a generator of $\pi_1(S^1)$ of the fiber S^1 of the bundle $Y \to B$ so that $\bar\phi_*(\tau) = \sigma^2$. Recalling that

$$\begin{aligned}
\pi_1(Y) &= \langle x_1, \cdots, x_p, \tau \mid x_i\tau x_i^{-1} = \tau^{\pm 1}, f_j(x_1, \cdots, x_p) = \tau^{b_j}\rangle, \\
\pi_1(X) &= \langle x_1, \cdots, x_p, \sigma \mid x_i\sigma x_i^{-1} = \sigma^{\pm 1}, f_j(x_1, \cdots, x_p) = \sigma^{a_j}\rangle
\end{aligned}$$

for some integers a_i and b_j, we must have that $\phi_*(f_j(x_1, \cdots, x_p)) = \phi_*(\tau^{b_j})$ and so $f_j(x_1, \cdots, x_p) = \sigma^{2b_j}$. Hence $a_j = 2b_j$ for all j.

(2) $\Longrightarrow$ (1). Conversely suppose that the fundamental group of the total space X of the S^1-bundle $X \to B$ has a presentation of the form as above with all the integers a_j even. Consider the subgroup H of $\pi_1(X)$ generated by $x_1, \cdots, x_p$ and σ^2. Then H has index 2 in $\pi_1(X)$. Let Y be the double covering space of X corresponding to H with covering projection $\phi : Y \to X$. Then $\pi' = \pi\phi : Y \to B$ is a bundle with fiber $F = \phi^{-1}(S^1)$ and we have the commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\bar\phi"] \arrow[d, "i'"] & S^1 \arrow[d, "i"] \\
Y \arrow[r, "\phi"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\
B \arrow[r, "="] & B
\end{tikzcd}$$

where $\bar\phi$ is the restriction of ϕ.

Now we will show that $F = S^1$ and $\bar\phi$ is a double covering projection. Notice that $\phi_* : \pi_1(Y) = H \to \pi_1(X)$ is the inclusion $H \hookrightarrow \pi_1(X)$ and hence the composition $\pi'_* = \pi_*\phi_* : \pi_1(Y) \to \pi_1(B)$ is surjective by the choice of H. It follows that $\pi_0(F) = 1$, i.e., F is arcwise connected. Hence $\bar\phi : F \to S^1$ is a (double) covering projection [15, Lemma 2.1, p. 150] and so $F = S^1$.

(2) $\Leftrightarrow$ (3). We note that $H_1(X;\mathbb{Z}_2) = H_1(X;\mathbb{Z}) \otimes \mathbb{Z}_2$ which follows from the universal coefficient theorem for homology groups because $H_0(X;\mathbb{Z})$ is torsion free. Therefore, $H_1(X;\mathbb{Z}_2)$ agrees with the abelianization of $\pi_1(X)$ over $\mathbb{Z}_2$. Looking at the description of $\pi_1(X)$ and $\pi_1(B)$, one easily sees that every a_j is even if and only if the abelianization of $\pi_1(X)$ over $\mathbb{Z}_2$ agrees with that of B. These show the equivalence between (2) and (3). $\square$

The last part of the proof above essentially proves that $b_1(X;\mathbb{Z}_2) = b_1(B;\mathbb{Z}_2)$ or $b_1(B;\mathbb{Z}_2)+1$. This fact can also be seen in terms of 2nd Stiefel-Whitney class as follows.

Lemma 2.2. *Let $p\colon X \to B$ be the unit circle bundle of a plane bundle ξ over a connected space B. Then $b_1(X;\mathbb{Z}_2) = b_1(B;\mathbb{Z}_2)$ or $b_1(B;\mathbb{Z}_2)+1$ and the former occurs when $w_2(\xi) \neq 0$ and the latter occurs when $w_2(\xi) = 0$.*

Proof. Consider the Thom-Gysin sequence associated with the S^1-bundle $p\colon X \to B$:

$$0 = H^{-1}(B) \xrightarrow{\cup w_2(\xi)} H^1(B) \xrightarrow{p^*} H^1(X) \longrightarrow H^0(B) \xrightarrow{\cup w_2(\xi)} H^2(B)$$

where the coefficients are taken with $\mathbb{Z}_2$. Here the last map above is injective if $w_2(\xi) \neq 0$ and zero if $w_2(\xi) = 0$. This implies the lemma since $H^0(B) \cong \mathbb{Z}_2$. $\square$

3. Iterated S^1-bundles

An *iterated S^1-bundle* of height n is a sequence of smooth S^1-bundles starting with a point:

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \{\text{a point}\}. \tag{3.1}$$

Each X_i is a closed connected aspherical manifold of dimension i for $i = 1, 2, \ldots, n$ and the S^1-fibration $X_i \to X_{i-1}$ induces a short exact sequence:

$$1 \to \pi_1(S^1) \to \pi_1(X_i) \to \pi_1(X_{i-1}) \to 1.$$

The total space X_n is an n-dimensional torus if every S^1-bundle $X_i \to X_{i-1}$ in (3.1) is trivial. However, the topology of X_n is complicated in general.

Lemma 3.1. *$\pi_1(X_n)$ has a presentation of the form*

$$\left\langle s_1, \cdots, s_n \;\middle|\; s_i s_j s_i^{-1} = s_n^{a_{i,j}^n} \cdots s_{j+1}^{a_{i,j}^{j+1}} s_j^{\epsilon_{ij}} \ (1 \le i < j \le n) \right\rangle$$

where $\epsilon_{ij} = \pm 1$ and the $a_{i,j}^k$ are some integers. Moreover, the S^1-bundle $X_j \to X_{j-1}$ in (3.1) *is orientable* (*equivalently, principal*) *if and only if $\epsilon_{ij} = 1$ for all $i = 1, \cdots, j-1$.*

Proof. The former statement follows by applying Lemma 2.1 inductively.

The proof of the latter is as follows. Note that the S^1-bundle $X_j \to X_{j-1}$ is orientable if and only if any loop in the base space X_{j-1} induces the identity map on the first cohomology group of the fiber over the base point of the loop. Equivalently, $\pi_1(X_{j-1}, b)$ acts on $H^1(F_b)$ trivially for all $b \in B$. This exactly means that $s_i s_j s_i^{-1} = s_j$ for all the generators s_i of $\pi_1(X_{j-1})$. $\square$

Since the projectivization of a plane bundle, that is an $\mathbb{R}P^1$-bundle, is an S^1-bundle, an iterated $\mathbb{R}P^1$-bundle is an iterated S^1-bundle.

Proposition 3.2. *Let X_n be the total space of an iterated S^1-bundle* (3.1)*. Then $b_1(X_n;\mathbb{Z}_2) \le n$. Moreover, the following are equivalent.*

(1) *X_n is the total space of an iterated $\mathbb{R}P^1$-bundle.*
(2) *All the exponents $a^k_{i,j}$ in Lemma 3.1 are even.*
(3) *$b_1(X_n;\mathbb{Z}_2) = n$.*

Proof. The former statement follows from Lemma 2.2. The latter follows by applying Lemma 2.1 repeatedly. □

Finally, we give an example of iterated $\mathbb{R}P^1$-bundle which motivated the study of this paper.

Example 3.3 (Real Bott tower)**.** An iterated $\mathbb{R}P^1$-bundle of height n:

$$B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 = \{\text{a point}\}, \tag{3.2}$$

where each fibration $B_i \to B_{i-1}$ for $i = 1, 2, \ldots, n$ is the projectivization of a Whitney sum of two line bundles over B_{i-1} is called a *real Bott tower* of height n, and the total space B_n is called a *real Bott manifold*. At each stage, one of the two line bundles may be assumed to be trivial without loss of generality because projectivization remain unchanged under tensor product with a line bundle. The same construction works in the complex category and in this case the tower is called a Bott tower and the total space B_n is called a Bott manifold. A two stage Bott manifold is nothing but a Hirzebruch surface. A Bott manifold provides an example of a closed smooth toric variety and a real Bott manifold provides an example of a closed smooth real toric variety.

A real Bott manifold B_n also provides an example of a flat Riemannian manifold. In fact, it can be described as the quotient of $\mathbb{R}^n$ by a group π_n generated by Euclidean motions s_i's ($i = 1, \ldots, n$) on $\mathbb{R}^n$ defined by

$$s_i(x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, x_i + \frac{1}{2}, \epsilon^i_{i+1}x_{i+1}, \ldots, \epsilon^i_n x_n),$$

where $\epsilon^i_j = \pm 1$ for $1 \le i < j \le n$ and ϵ^i_j's are determined by the line bundles used to define the real Bott tower (3.2). The action of π_n on $\mathbb{R}^n$ is free so that π_n is the fundamental group of the real Bott manifold B_n. It is generated by s_i's ($i = 1, \ldots, n$) with relations

$$s_i s_j s_i^{-1} = s_j^{\epsilon^i_j} \quad \text{for } 1 \le i < j \le n.$$

The subgroup of π_n generated by s_i^2's ($i = 1, \ldots, n$) is the translations $\mathbb{Z}^n$ and the quotient $\pi_n/\mathbb{Z}^n$ is an elementary 2-group of rank n. Note that the natural projections $\mathbb{R}^n \to \mathbb{R}^{n-1} \to \cdots \to \mathbb{R}^1 \to \mathbb{R}^0 = \{\text{a point}\}$ induce a real Bott tower.

The diffeomorphism classification of real Bott manifolds has been completed in [2]. The paper [2] also relates the diffeomorphism classification of real Bott manifolds with the classification of acyclic digraphs (directed graphs with no direct cycles) up to some equivalence.

4. Infra-nilmanifolds

The purpose of this section is to prove Theorem 1.1 in the Introduction. We continue to use notations in Section 3. A group G is called *supersolvable* if there exists a finite normal series

$$G = \bar{G}_1 \supset \bar{G}_2 \supset \cdots \supset \bar{G}_c \supset \bar{G}_{c+1} = 1$$

such that each quotient group $\bar{G}_i/\bar{G}_{i+1}$ is cyclic and each $\bar{G}_i$ is normal in G.

Lemma 4.1. *$\pi_1(X_n)$ is a supersolvable group.*

Proof. We consider the subgroups $\bar{\pi}_j$ of $\pi_1(X_n)$ generated by $s_j, \cdots, s_n$. Then we have a finite normal series

$$\pi_1(X_n) = \bar{\pi}_1 \supset \bar{\pi}_2 \supset \cdots \supset \bar{\pi}_n \supset \bar{\pi}_{n+1} = 1$$

such that $\bar{\pi}_i/\bar{\pi}_{i+1} \cong \langle s_i \rangle$. By Lemma 3.1, it follows easily that each $\bar{\pi}_i$ is a normal subgroup of $\pi_1(X_n)$. □

However, the normal series in the above proof is not always a central series. This implies that $\pi_1(X_n)$ is not always a nilpotent group. We will show in Theorem 4.3 that $\pi_1(X_n)$ is always virtually nilpotent. Note further that the subgroup $\bar{\pi}_i$ of $\pi_1(X_n)$ is isomorphic to $\pi_1(X_{n-i+1})$.

The projection $\pi_1(X_i) \to \pi_1(X_{i-1})$ sends s_j to s_j for $j = 1, \cdots, i-1$ with kernel $\pi_1(S^1) = \langle s_i \rangle$. For the simplicity, we will write $\pi_i = \pi_1(X_i)$ and $A_i = \pi_1(S^1)$ with generator s_i. Thus we have a short exact sequence

$$1 \longrightarrow A_i \longrightarrow \pi_i \longrightarrow \pi_{i-1} \longrightarrow 1. \tag{4.1}$$

Let Γ_i be the subgroup of π_i generated by $s_1^2, \cdots, s_i^2$. Then Γ_i is mapped onto Γ_{i-1} under the projection $\pi_i \to \pi_{i-1}$ with kernel $\langle s_i^2 \rangle$, which induces a short exact sequence

$$1 \longrightarrow \langle s_i^2 \rangle \longrightarrow \Gamma_i \longrightarrow \Gamma_{i-1} \longrightarrow 1. \tag{4.2}$$

Lemma 4.2. *Γ_n is a normal subgroup of π_n with index 2^n.*

Proof. For each s_i, we denote by $c(s_i)$ the conjugation by s_i. Since $c(s_i)(s_n) = s_n^{\epsilon_{in}}$, the conjugate automorphism $c(s_i)$ on π_n induces the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & A_n \arrow[r] \arrow[d, "c(s_i)"] & \pi_n \arrow[r] \arrow[d, "c(s_i)"] & \pi_{n-1} \arrow[r] \arrow[d, "c(\bar{s}_i)"] & 1 \\
1 \arrow[r] & A_n \arrow[r] & \pi_n \arrow[r] & \pi_{n-1} \arrow[r] & 1
\end{tikzcd}$$

where $\bar{s}_i$ is the image of s_i under $\pi_n \to \pi_{n-1}$. This diagram gives rise to the following commutative diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \langle s_n^2 \rangle \arrow[r] \arrow[d, "c(s_i)"] & \Gamma_n \arrow[r] \arrow[d, "c(s_i)"] & \Gamma_{n-1} \arrow[r] \arrow[d, "c(\bar{s}_i)"] & 1 \\
1 \arrow[r] & \langle s_n^2 \rangle \arrow[r] & \Gamma'_n \arrow[r] & \Gamma'_{n-1} \arrow[r] & 1
\end{tikzcd}$$

where Γ'_n and Γ'_{n-1} are the images of Γ_n and Γ_{n-1} under $c(s_i)$ and $c(\bar{s}_i)$ respectively. In order to show that Γ_n is a normal subgroup of π_n, it suffices to show that $\Gamma'_n = \Gamma_n$. For this purpose we will use induction on n. It is clear that $\Gamma'_1 = \Gamma_1$. Assume that $\Gamma'_{n-1} = \Gamma_{n-1}$. Consider an element $s_i s_j^2 s_i^{-1}$ of Γ'_n. It is mapped to the element $\bar{s}_i \bar{s}_j^2 \bar{s}_i^{-1}$ of $\Gamma'_{n-1} = \Gamma_{n-1}$. Hence $\bar{s}_i \bar{s}_j^2 \bar{s}_i^{-1}$ is a word of $\bar{s}_1^2, \cdots, \bar{s}_{n-1}^2$. This therefore implies that $s_i s_j^2 s_i^{-1}$ is a word of $s_1^2, \cdots, s_{n-1}^2, s_n^2$, which means that $s_i s_j^2 s_i^{-1} \in \Gamma_n$. Consequently $\Gamma'_n = \Gamma_n$.

Furthermore, we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \pi_n/\Gamma_n & \longrightarrow & \pi_{n-1}/\Gamma_{n-1} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & A_n & \longrightarrow & \pi_n & \longrightarrow & \pi_{n-1} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \langle s_n^2 \rangle & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma_{n-1} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

This, in particular, shows that the order of π_n/Γ_n equals 2^n by induction. $\square$

Lemma 4.3. *Γ_n is a nilpotent group of rank n. Therefore, π_n is a torsion-free virtually nilpotent group of rank n.*

Proof. It suffices to show that Γ_n has a finite central series

$$\Gamma_n = \Gamma^1 \supset \Gamma^2 \supset \cdots \supset \Gamma^c \supset \Gamma^{c+1} = 1$$

such that the quotient groups Γ^i/Γ^{i+1} are isomorphic to some $\mathbb{Z}^{k_i}$. We will use induction on n to show that the series

$$\Gamma_n = \langle s_1^2, \cdots, s_n^2 \rangle \supset \langle s_2^2, \cdots, s_n^2 \rangle \supset \cdots \supset \langle s_{n-1}^2, s_n^2 \rangle \supset \langle s_n^2 \rangle \supset \{1\} \tag{4.3}$$

is a required central series with successive quotient groups isomorphic to $\mathbb{Z}$.

The case where $n = 1$ is obvious and hence we assume the following: Γ_{n-1} has such a central series. To avoid confusion let us use $\bar{s}_1, \cdots, \bar{s}_{n-1}$ in the presentation of π_{n-1} given in Lemma 3.1 so that $s_i \in \pi_n$ is mapped to $\bar{s}_i \in \pi_{n-1}$ for $i = 1, \cdots, n-1$. Then $\Gamma_{n-1} = \langle \bar{s}_1^2, \cdots, \bar{s}_{n-1}^2 \rangle$ with index 2^{n-1} in π_{n-1}, and by induction hypothesis, Γ_{n-1} has a central series

$$\Gamma_{n-1} = \langle \bar{s}_1^2, \cdots, \bar{s}_{n-1}^2 \rangle \supset \langle \bar{s}_2^2, \cdots, \bar{s}_{n-1}^2 \rangle \supset \cdots \supset \langle \bar{s}_{n-1}^2 \rangle \supset \{1\} \tag{4.4}$$

with successive quotient groups isomorphic to $\mathbb{Z}$. Using the short exact sequence $1 \to \langle s_n^2 \rangle \to \Gamma_n \xrightarrow{p} \Gamma_{n-1} \to 1$, we take the pullback of the series (4.4). Namely, for each subgroup $\langle \bar{s}_i^2, \cdots, \bar{s}_{n-1}^2 \rangle$ of Γ_{n-1}, we consider the subgroup $p^{-1}(\langle \bar{s}_i^2, \cdots, \bar{s}_{n-1}^2\rangle)$ of Γ_n. This group fits in a short exact sequence $1 \to \langle s_n^2\rangle \to p^{-1}(\langle \bar{s}_i^2, \cdots, \bar{s}_{n-1}^2\rangle) \to \langle \bar{s}_i^2, \cdots, \bar{s}_{n-1}^2\rangle \to 1$, which induces that $p^{-1}(\langle \bar{s}_i^2, \cdots, \bar{s}_{n-1}^2\rangle) = \langle s_i^2, \cdots, s_n^2\rangle$. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \langle s_n^2\rangle \arrow[r] & \Gamma_n \arrow[r] & \Gamma_{n-1} \arrow[r] & 1 \\
1 \arrow[r] & \langle s_n^2\rangle \arrow[r] \arrow[u, "="'] & \langle s_2^2, \cdots, s_n^2\rangle \arrow[r] \arrow[u] & \langle \bar{s}_2^2, \cdots, \bar{s}_{n-1}^2\rangle \arrow[r] \arrow[u] & 1 \\
 & \vdots \arrow[u, "="'] & \vdots \arrow[u] & \vdots \arrow[u] & \\
1 \arrow[r] & \langle s_n^2\rangle \arrow[r] \arrow[u, "="'] & \langle s_{n-2}^2, s_n^2\rangle \arrow[r] \arrow[u] & \langle \bar{s}_{n-1}^2\rangle \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \langle s_n^2\rangle \arrow[r] \arrow[u, "="'] & \langle s_n^2\rangle \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Finally we note that since the most right vertical is a central series, so is the induced middle vertical. Clearly the rank of Γ_n is n. $\square$

In fact, π_n contains another nilpotent normal subgroup which is slightly larger than Γ_n as is shown in the following lemma.

Lemma 4.4. *Let Λ_n be the subgroup of π_n generated by $s_1^2, \cdots, s_{n-1}^2, s_n$. Then Λ_n is a nilpotent normal subgroup of π_n which has rank n and index 2^{n-1}.*

Proof. Under the short exact sequence $1 \to \langle s_n\rangle \to \pi_n \to \pi_{n-1} \to 1$, we take the pullback of the subgroup Γ_{n-1} of π_{n-1}. Then we obtain the short exact sequence $1 \to \langle s_n\rangle \to \Lambda_n \to \Gamma_{n-1} \to 1$. Since Γ_{n-1} is normal in π_{n-1}, it follows that Λ_n is a normal subgroup of π_n.

On the other hand, Λ_n fits in the following short exact sequence $1 \to \Gamma_n \to \Lambda_n \to \mathbb{Z}_2 \to 1$. Since $s_i s_n s_i^{-1} = s_n^{\epsilon_{in}}$, we have $s_i^2 s_n s_i^{-2} = s_n$ and so the extension is central. Hence since Γ_n is nilpotent, we see that Λ_n is nilpotent. $\square$

Now we are in a position to prove our first main theorem stated in the Introduction.

Theorem 4.5. *The total space X_n of an iterated S^1-bundle of height n is homeomorphic to an infra-nilmanifold. In fact, some 2^{n-1}-cover of X_n is homeomorphic to a nilmanifold.*

Proof. Let X_n be the total space of an iterated S^1-bundle and let π_n be its fundamental group as before. By [12, Corollary 3.2.1], there is an infra-nilmanifold X whose fundamental group is isomorphic to π_n. Therefore, two aspherical manifolds X_n and X are homotopic. By [6, Theorem 6.3], X_n and X are homeomorphic except possibly for $n = 3, 4$.

Since X_4 is aspherical and π_4 is virtually nilpotent, X_4 has an infra-nil structure by [7, Corollary 2.21]. (In fact, this is true for all $n \neq 3$. See also F. Quinn's Math Review of the paper [6].) Namely, X_4 is homeomorphic to an infra-nilmanifold.

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds. It is evident that the Seifert manifolds X_3 and X are sufficiently large, see [10, Proposition 2]. By works of Waldhausen [19] and Heil [9, Theorem A], X_3 is homeomorphic to X.

By Lemmas 4.2 and 4.4, π_n has a normal nilpotent subgroup Λ_n of index 2^{n-1}. The covering space associated with the nilpotent group Λ_n is a 2^{n-1}-cover of X_n and it is homeomorphic to a nilmanifold. □

Remark 4.6. The closed nilmanifolds are precisely the total spaces of iterated *principal* S^1-bundles up to homeomorphism as remarked in the Introduction.

We conclude this section with the following lemma.

Lemma 4.7. *π_n is isomorphic to a Bieberbach group* (*in other words, X_n is homeomorphic to a flat Riemannian manifold*) *if and only if Γ_n is isomorphic to $\mathbb{Z}^n$.*

Proof. The if part is clear. Suppose that π_n is a Bieberbach group. Then $\mathbb{R}^n/\pi_n$ is a flat Riemannian manifold, so is its finite cover $\mathbb{R}^n/\Gamma_n$. On the other hand, it is known by Gromoll-Wolf [8] and Yau [21] that if the fundamental group of a compact nonpositively curved manifold is nilpotent, then it is abelian. Therefore, Γ_n is isomorphic to $\mathbb{Z}^n$. □

5. Iterated S^1-bundles of height 3

In this section we classify the 3-dimensional total spaces obtained as iterated S^1-bundles of height 3 up to homeomorphism (equivalently up to diffeomorphism because diffeomorphism classification is the same as homeomorphism classification in dimension 3). This classification reduces to the classification of isomorphism classes of their fundamental groups by Theorem 4.5.

5.1. **Isomorphism classes of π_3.** In the 3-dimensional case, by Lemma 3.1, the fundamental group π_3 of the total space of an iterated S^1-bundle of height 3 is generated by s_1, s_2, s_3 with relations

$$s_1 s_2 s_1^{-1} = s_3^a s_2^\epsilon, \quad s_1 s_3 s_1^{-1} = s_3^{\epsilon_1}, \quad s_2 s_3 s_2^{-1} = s_3^{\epsilon_2} \tag{5.1}$$

where $a \in \mathbb{Z}$ and $\epsilon, \epsilon_1, \epsilon_2 \in \{\pm 1\}$. We shall denote the group π_3 with the relation (5.1) by $\Pi(a, \epsilon, \epsilon_1, \epsilon_2)$.

Lemma 5.1. *$\Pi(a, \epsilon, \epsilon_1, \epsilon_2)$ is a Bieberbach group if and only if $(\epsilon + \epsilon_1)(\epsilon_2 + 1)a = 0$.*

Proof. By Lemma 4.7, $\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$ is a Bieberbach group if and only if $s_i^2s_j^2=s_j^2s_i^2$ for $1\le i<j\le 3$. The latter two identities in (5.1) imply that s_3 commutes with s_1^2 and s_2^2. Therefore it suffices to show that $s_1^2s_2^2=s_2^2s_1^2$ if and only if $(\epsilon+\epsilon_1)(\epsilon_2+1)a=0$. We note that the latter two identities in (5.1) imply

$$s_is_3^b=s_3^{\epsilon_i b}s_i \quad \text{for } i=1,2 \text{ and } b\in\mathbb{Z}. \tag{5.2}$$

We distinguish two cases according to the value of ϵ.

The case where $\epsilon=1$. In this case $s_1s_2=s_3^as_2s_1$ by the first identity in (5.1). Using this together with (5.2), we have

$$\begin{aligned} s_1^2s_2^2&=s_1(s_1s_2)s_2=s_1(s_3^as_2s_1)s_2=s_3^{\epsilon_1a}(s_1s_2)(s_1s_2)\\ &=s_3^{\epsilon_1a}(s_3^as_2s_1)(s_3^as_2s_1)=s_3^{\epsilon_1a+a+\epsilon_1\epsilon_2a}s_2(s_1s_2)s_1\\ &=s_3^{\epsilon_1a+a+\epsilon_1\epsilon_2a}s_2(s_3^as_2s_1)s_1=s_3^{\epsilon_1a+a+\epsilon_1\epsilon_2a+\epsilon_2a}s_2^2s_1^2. \end{aligned}$$

Therefore $s_1^2s_2^2=s_2^2s_1^2$ if and only if the exponent of s_3 in the last term above is zero. This is equivalent to the assertion in the lemma because $\epsilon=1$.

The case where $\epsilon=-1$. In this case $s_1s_2=s_3^as_2^{-1}s_1$ by the first identity in (5.1). Moreover, by taking inverse at the both sides of the first identity in (5.1) and using (5.2), we obtain $s_1s_2^{-1}=s_3^{-\epsilon_2a}s_2s_1$. Using these two identities together with (5.2), we have

$$\begin{aligned} s_1^2s_2^2&=s_1(s_1s_2)s_1=s_1(s_3^as_2^{-1}s_1)s_2=s_3^{\epsilon_1a}(s_1s_2^{-1})(s_1s_2)\\ &=s_3^{\epsilon_1a}(s_3^{-\epsilon_2a}s_2s_1)(s_3^as_2^{-1}s_1)=s_3^{\epsilon_1a-\epsilon_2a+\epsilon_1\epsilon_2a}s_2(s_1s_2^{-1})s_1\\ &=s_3^{\epsilon_1a-\epsilon_2a+\epsilon_1\epsilon_2a}s_2(s_3^{-\epsilon_2a}s_2s_1)s_1=s_3^{\epsilon_1a-\epsilon_2a+\epsilon_1\epsilon_2a-a}s_2^2s_1^2. \end{aligned}$$

Therefore $s_1^2s_2^2=s_2^2s_1^2$ if and only if the exponent of s_3 in the last term above is zero. This is equivalent to the assertion in the lemma because $\epsilon=-1$. $\square$

Lemma 5.1 implies that when $(\epsilon,\epsilon_1,\epsilon_2)=(1,1,1)$ or $(-1,-1,1)$, $\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$ is a Bieberbach group if and only if $a=0$. This condition that $(\epsilon,\epsilon_1,\epsilon_2)=(1,1,1)$ or $(-1,-1,1)$ appears from another viewpoint as is seen in (the proof of) the following lemma.

Lemma 5.2. *Unless* $(\epsilon,\epsilon_1,\epsilon_2)=(1,1,1)$ *or* $(-1,-1,1)$,

$$\Pi(a,\epsilon,\epsilon_1,\epsilon_2)\cong\begin{cases}\Pi(0,\epsilon,\epsilon_1,\epsilon_2) & \textit{if } a \textit{ is even},\\ \Pi(1,\epsilon,\epsilon_1,\epsilon_2) & \textit{if } a \textit{ is odd}.\end{cases}$$

Proof. Changing the lift of s_1 and s_2, we may replace

$$s_1\mapsto s_3^{-b}s_1,\ s_2\mapsto s_3^{-c}s_2,\ s_3\mapsto s_3$$

where b and c can be any integers. Setting

$$t_1=s_3^{-b}s_1,\ t_2=s_3^{-c}s_2,\ t_3=s_3,$$

the second and third identities of (5.1) remain unchanged with s replaced by t but the first one turns into

$$(t_3^bt_1)(t_3^ct_2)(t_3^bt_1)^{-1}=t_3^a(t_3^ct_2)^\epsilon. \tag{5.3}$$

The left hand side of (5.3) reduces to

$$t_3^{b+\epsilon_1 c-\epsilon_2 b} t_1 t_2 t_1^{-1}$$

while the right hand side of (5.3) reduces to

$$\begin{cases} t_3^{a+c} t_2 & \text{when } \epsilon = 1, \\ t_3^{a-\epsilon_2 c} t_2^{-1} & \text{when } \epsilon = -1. \end{cases}$$

Therefore the first identity in (5.1) turns into

$$t_1 t_2 t_1^{-1} = \begin{cases} t_3^{a+(\epsilon_2-1)b-(\epsilon_1-1)c} t_2 & \text{when } \epsilon = 1, \\ t_3^{a+(\epsilon_2-1)b-(\epsilon_1+\epsilon_2)c} t_2^{-1} & \text{when } \epsilon = -1. \end{cases}$$

This implies the lemma. $\square$

There are more isomorphisms among groups $\Pi(a, \epsilon, \epsilon_1, \epsilon_2)$.

Lemma 5.3. *The following isomorphisms hold:*

(1) $\Pi(a, \epsilon, \epsilon_1, \epsilon_2) \cong \Pi(-a, \epsilon, \epsilon_1, \epsilon_2)$.
(2) $\Pi(a, \epsilon, \epsilon_1, \epsilon_2) \cong \Pi(a, \epsilon, \epsilon_1\epsilon_2, \epsilon_2)$.
(3) $\Pi(a, 1, \epsilon_1, \epsilon_2) \cong \Pi(a, 1, \epsilon_2, \epsilon_1)$.
(4) $\Pi(a, \epsilon, -\epsilon, 1) \cong \Pi(a, -\epsilon, \epsilon, 1)$.

Proof. The following isomorphisms are desired ones for the first three cases:
(1) $s_1 \to s_1,\ s_2 \to s_2,\ s_3 \to s_3^{-1}$.
(2) $s_1 \to s_1 s_2,\ s_2 \to s_2,\ s_3 \to s_3$.
(3) $s_1 \to s_2,\ s_2 \to s_1,\ s_3 \to s_3^{-1}$.
It would be obvious that the first two above are the desired isomorphisms. We shall check it for (3). We set $t_1 = s_2$, $t_2 = s_1$ and $t_3 = s_3^{-1}$. Then

$$\begin{aligned} t_1 t_2 t_1^{-1} &= s_2 s_1 s_2^{-1} = s_3^{-a} s_1 = t_3^a t_2, \\ t_1 t_3 t_1^{-1} &= s_2 s_3^{-1} s_2^{-1} = s_3^{-\epsilon_2} = t_3^{\epsilon_2}, \\ t_2 t_3 t_2^{-1} &= s_1 s_3^{-1} s_1^{-1} = s_3^{-\epsilon_1} = t_3^{\epsilon_1}, \end{aligned}$$

and this proves the isomorphism (3) in the lemma.

The proof of (4) is as follows. By Lemma 5.2 we may assume that $a = 0$ or 1. Then
$s_1 \to s_1,\ s_2 \to s_3,\ s_3 \to s_2$ when $a = 0$,
$s_1 \to s_1,\ s_2 \to s_2,\ s_3 \to s_3 s_2^{2\epsilon}$ when $a = 1$.
are the desired isomorphisms. The check is left to the reader. $\square$

There are ten diffeomorphism classes of closed flat 3-dimensional Riemannian manifolds; six orientable ones $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4, \mathfrak{G}_5, \mathfrak{G}_6$ and four non-orientable ones $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$, see [20, Theorems 3.5.5 and 3.5.9]. It is known that $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{B}_1, \mathfrak{B}_3$ appear as real Bott manifolds ([13], [14]).

Proposition 5.4. *The isomorphism classes of* $\pi_3 = \Pi(a, \epsilon, \epsilon_1, \epsilon_2)$ *are classified into the following three types:*

(1) *Six Bieberbach groups:*

$\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$	a	$(\epsilon,\epsilon_1,\epsilon_2)$
$\mathfrak{G}_1$	0	$(1,1,1)$
$\mathfrak{G}_2$	0	$(-1,-1,1)$
$\mathfrak{B}_1$	even	$(1,1,-1),(1,-1,1),(1,-1,-1),(-1,1,1)$
$\mathfrak{B}_2$	odd	$(1,1,-1),(1,-1,1),(1,-1,-1),(-1,1,1)$
$\mathfrak{B}_3$	even	$(-1,1,-1),(-1,-1,-1)$
$\mathfrak{B}_4$	odd	$(-1,1,-1),(-1,-1,-1)$

(2) *An infinite family of nilpotent groups*

$$\Pi(a,1,1,1)\cong\Pi(-a,1,1,1)\ \textit{with } a\neq 0.$$

(3) *An infinite family of virtually nilpotent groups*

$$\Pi(a,-1,-1,1)\cong\Pi(-a,-1,-1,1)\ \textit{with } a\neq 0.$$

Proof. (1) First we note that groups $\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$'s for values of $(a,\epsilon,\epsilon_1,\epsilon_2)$ in a same row in the table above are isomorphic to each other by Lemmas 5.2 and 5.3. Three dimensional Bieberbach groups are classified and presented in [20, Theorems 3.5.5 and 3.5.9] with generators and relations, and we shall identify our groups $\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$ with them.

($\mathfrak{G}_1$). Clearly $\Pi(0,0,0,0)$ is $\mathbb{Z}^3$ and isomorphic to $\mathfrak{G}_1$.

($\mathfrak{G}_2$). Taking $\alpha=s_1, t_2=s_2$ and $t_3=s_3$, we see that $\Pi(0,-1,-1,1)$ is isomorphic to $\mathfrak{G}_2$.

($\mathfrak{B}_1$). We take $e=s_1,\ t_1=s_1^2,\ t_2=s_3,\ t_3=s_2^{-1}$. Then $\{e,t_1,t_2,t_3\}$ generates $\Pi(0,-1,1,1)$ and satisfies

$$e^2=t_1,\ et_2e^{-1}=t_2,\ et_3e^{-1}=t_3^{-1}.$$

This shows that $\Pi(0,-1,1,1)$ is isomorphic to $\mathfrak{B}_1$.

($\mathfrak{B}_2$). We take $e=s_1,\ t_1=s_1^2,\ t_2=s_1^{-2}s_3,\ t_3=s_2$. Then $\{e,t_1,t_2,t_3\}$ generates $\Pi(1,-1,1,1)$ and satisfies

$$e^2=t_1,\ et_2e^{-1}=t_2,\ et_3e^{-1}=t_1t_2t_3^{-1}.$$

This shows that $\Pi(1,-1,1,1)$ is isomorphic to $\mathfrak{B}_2$.

($\mathfrak{B}_3$). We take $\alpha=s_1,\ e=s_2,\ t_1=s_1^2,\ t_2=s_2^2,\ t_3=s_3$. Then $\{\alpha,e,t_1,t_2,t_3\}$ generates $\Pi(0,-1,-1,-1)$ and satisfies

$$\alpha^2=t_1,\ \alpha t_2\alpha^{-1}=t_2^{-1},\ \alpha t_3\alpha^{-1}=t_3^{-1},$$
$$e^2=t_2,\ et_1e^{-1}=t_1,\ et_3e^{-1}=t_3^{-1},\ e\alpha e^{-1}=t_2\alpha.$$

This shows that $\Pi(0,-1,-1,-1)$ is isomorphic to $\mathfrak{B}_3$.

($\mathfrak{B}_4$). We take $\alpha=s_1,\ e=s_2,\ t_1=s_1^2,\ t_2=s_2^2,\ t_3=s_3^{-1}$. Then $\{\alpha,e,t_1,t_2,t_3\}$ generates $\Pi(1,-1,-1,-1)$ and satisfies

$$\alpha^2=t_1,\ \alpha t_2\alpha^{-1}=t_2^{-1},\ \alpha t_3\alpha^{-1}=t_3^{-1},$$
$$e^2=t_2,\ et_1e^{-1}=t_1,\ et_3e^{-1}=t_3^{-1},\ e\alpha e^{-1}=t_2t_3\alpha.$$

This shows that $\Pi(1,-1,-1,-1)$ is isomorphic to $\mathfrak{B}_4$.

(2) The isomorphism in (2) of the proposition follows from Lemma 5.3 (1). Since the first homology group of $\Pi(a,0,0,0)$ is a cyclic group of order

$|a|$, $\Pi(a,0,0,0)$ is isomorphic to $\Pi(b,0,0,0)$ if and only if $|a|=|b|$. For $\Pi(a,0,0,0)$ with $a\neq 0$,

$$[s_1,s_2]=s_3^a,\quad [s_1,s_3]=[s_2,s_3]=1,$$

so the group has a central series

$$\Pi(a,0,0,0)=\langle s_1,s_2,s_3\rangle\supset\langle s_3\rangle$$

and hence is nilpotent.

(3) The isomorphism in (3) of the proposition also follows from Lemma 5.3 (1). For $\Pi(a,-1,-1,1)$ with $a\neq 0$, we have

$$[s_1^2,s_2]=s_3^{-2a},\quad [s_1^2,s_3]=[s_2,s_3]=1.$$

So the subgroup $H_a=\langle s_1^2,s_2,s_3\rangle$ of $\Pi(a,-1,-1,1)$ with $a\neq 0$ is isomorphic to the nilpotent group $\Pi(2a,0,0,0)$ in (2). Moreover, since the subgroup H_a is of index 2, it is normal and the quotient group $\Pi(2a,0,0,0)/H_a$ is an order two cyclic group. Therefore H_a is the unique maximal nilpotent normal subgroup of $\Pi(2a,0,0,0)$ and $\Pi(2a,0,0,0)$ is virtually nilpotent. Finally, if $\Pi(a,-1,-1,1)$ is isomorphic to $\Pi(b,-1,-1,1)$, then their maximal normal nilpotent subgroups H_a and H_b are isomorphic; so $|a|=|b|$ by (2) above. $\square$

Remark 5.5. One can see that $\Pi(a,-1,-1,-1)$ with $a\neq 0$ is isomorphic to an almost Bieberbach group (in short, an AB-group) of Seifert bundle type 3 in [4, Proposition 6.1], or π_3 (the subscript 3 also stands for Seifert bundle type 3) in the list of [3, p. 799]. Since the unique maximal normal nilpotent subgroup H_a of $\Pi(a,-1,-1,-1)$ is isomorphic to $\Pi(2a,0,0,0)$, our class $\Pi(a,-1,-1,-1)$ consists of all of the infinitely many non-isomorphic AB-groups of type 3.

5.2. **Realization.** We shall observe that all the isomorphism classes of the groups $\Pi(a,\epsilon,\epsilon_1,\epsilon_2)$ in Proposition 5.4 can be realized by iterated S^1-bundles of height 3.

By Theorem 4.5, the total space of an iterated S^1-bundle of height 3 is a 3-dimensional infra-nilmanifold. The 3-dimensional infra-nilmanifolds are well understood. In fact, these are ten flat Riemannian manifolds mentioned before or infra-nilmanifolds covered by the simply connected 3-dimensional nilpotent Lie group Nil, called the Heisenberg group,

$$\mathrm{Nil}=\left\{\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix}\;\middle|\; x,y,z\in\mathbb{R}\right\}. \tag{5.4}$$

As mentioned before, $\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{B}_1,\mathfrak{B}_3$ appear as real Bott manifolds ([13], [14]), and in addition to them, $\mathfrak{B}_2$ and $\mathfrak{B}_4$ appear as iterated S^1-bundles as is shown in the following example.

Example 5.6 (Flat Riemannian manifolds of types $\mathfrak{B}_2$ and $\mathfrak{B}_4$)**.**

$$\begin{aligned}
s_1(x,y,z) &= (x+\frac{1}{2}, \epsilon y, -z+\frac{1}{4}),\\
s_2(x,y,z) &= (x, y+\frac{1}{2}, -z),\\
s_3(x,y,z) &= (x, y, z+\frac{1}{2}).
\end{aligned}$$

The group generated by s_1, s_2, s_3 acts freely on $\mathbb{R}^3$ and has relations

$$s_1 s_2 s_1^{-1} = s_3 s_2^{\epsilon}, \quad s_1 s_3 s_1^{-1} = s_3^{-1}, \quad s_2 s_3 s_2^{-1} = s_3^{-1}.$$

The subgroup generated by s_1^2, s_2^2, s_3^2 is the group $\mathbb{Z}^3$ of translations. The natural projections $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$ induce an iterated S^1-bundle of height 3.

The group $\Pi(a,0,0,0)$ in Proposition 5.4 (2) can be realized by an iterated S^1-bundle as follows.

Example 5.7 (Nilmanifolds)**.** It is well known that a lattice (i.e., a torsion free discrete cocompact subgroup) of Nil is isomorphic to

$$\Pi_a := \Pi(a,0,0,0) = \langle s_1, s_2, s_3 \mid [s_1,s_2] = s_3^a,\ [s_1,s_3] = [s_2,s_3] = 1\rangle \tag{5.5}$$

for some $a \neq 0$. This group is realized as a lattice of Nil if one takes

$$s_1 = \begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \ s_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}, \ s_3 = \begin{bmatrix} 1 & 0 & \frac{1}{a}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Then the orbit space Nil/Π_a is a nilmanifold with Π_a as the fundamental group.

The product of the matrix in (5.4) with s_i $(i = 1,2,3)$ from the left is respectively given by

$$\begin{bmatrix} 1 & x+1 & z+y\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix} 1 & x & z\\ 0 & 1 & y+1\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix} 1 & x & z+\frac{1}{a}\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}.$$

Therefore, if we identify the matrix in (5.4) with the point (x,y,z) in $\mathbb{R}^3$, then the left multiplication by s_i on Nil for $i = 1,2,3$ can respectively be identified with the following diffeomorphism of $\mathbb{R}^3$:

$$\begin{aligned}
s_1(x,y,z) &= (x+1, y, z+y),\\
s_2(x,y,z) &= (x, y+1, z),\\
s_3(x,y,z) &= (x, y, z+\frac{1}{a}).
\end{aligned}$$

So, the natural projections $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$ induce an iterated S^1-bundle of height 3:

$$\mathrm{Nil}/\Pi_a \to \mathbb{R}^2/\mathbb{Z}^2 = (S^1)^2 \to \mathbb{R}/\mathbb{Z} = S^1.$$

Note that $\mathrm{Nil}/\Pi_a \to (S^1)^2$ above is the unit sphere bundle of an oriented plane bundle over $(S^1)^2$ whose Euler class is a times a generator of $H^2((S^1)^2;\mathbb{Z})$.

It is well known that all 3-dimensional infra-nilmanifolds M covered by Nil are Seifert manifolds (see [18]); namely, M is a circle bundle over a 2-dimensional orbifold with singularities. It is known [4, Proposition 6.1] that there are fifteen classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

It is known (cf. [3, 4]) that the group $\mathrm{Aut}(\mathrm{Nil})$ of automorphisms of Nil is isomorphic to $\mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$. In fact, an element

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \in \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$$

acts on Nil as follows:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & ax+by & z' \\ 0 & 1 & cx+dy \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$z' = (ad-bc)z + \frac{1}{2}(acx^2 + 2bcxy + bdy^2) - (au+cv)x - (bu+dv)y.$$

An infra-nilmanifold of dimension 3 is an orbit space of Nil by a cocompact discrete subgroup of the affine group $\mathrm{Aff}(\mathrm{Nil}) = \mathrm{Nil} \rtimes \mathrm{Aut}(\mathrm{Nil})$ of Nil acting on Nil freely.

Example 5.8 (Infra-nilmanifolds). Let $a \neq 0$ as before. We consider affine diffeomorphisms s_1, s_2, s_3 in $\mathrm{Aff}(\mathrm{Nil})$ given as follows:

$$s_1 = \left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right),$$
$$s_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right),$$
$$s_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{2a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right).$$

In other words, if we identify Nil with $\mathbb{R}^3$ as before, then the diffeomorphisms s_1, s_2, s_3 are described as follows:

$$\begin{aligned} s_1(x,y,z) &= (x+\frac{1}{2}, -y, -z-\frac{y}{2}), \\ s_2(x,y,z) &= (x, y+1, z), \\ s_3(x,y,z) &= (x, y, z-\frac{1}{2a}). \end{aligned}$$

One can check that the group Δ_a generated by s_1, s_2, s_3 has relations

$$s_1s_2s_1^{-1} = s_3^a s_2^{-1}, \quad s_1s_3s_1^{-1} = s_3^{-1}, \quad s_2s_3s_2^{-1} = s_3 \tag{5.6}$$

and the action of Δ_a on Nil is free. The subgroup of Δ_a generated by s_1^2, s_2, s_3 agrees with Π_{-2a} in Example 5.7 and $\mathrm{Nil}/\Pi_{-2a} \to \mathrm{Nil}/\Delta_a$ is a double covering. Note that the natural projections $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$ induce an iterated S^1-bundle of height 3 with Nil/Δ_a as the total space.

We summarize what we have observed as follows.

Theorem 5.9. *The total spaces of iterated S^1-bundles of height 3 are classified into the following three types up to homeomorphism:*

(1) *Closed flat Riemannian manifolds of types $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$.*
(2) *Nilmanifolds Nil/Π_a in Example 5.7 parametrized by positive integers a.*
(3) *Infra-nilmanifolds Nil/Δ_a in Example 5.8 parametrized by positive integers a.*

6. Flat Riemannian iterated $\mathbb{R}P^1$-bundles

A real Bott manifold (see Example 3.3) is flat Riemannian although the total space of an iterated $\mathbb{R}P^1$-bundle is not necessarily flat Riemannian. The purpose of this section is to show that real Bott manifolds are precisely flat Riemannian manifolds among the total spaces of iterated $\mathbb{R}P^1$-bundles. In fact, we prove the following, which is essentially same as Theorem 1.2 in the Introduction.

Theorem 6.1. *Let X_n be the total space of an iterated $\mathbb{R}P^1$-bundle of height n. If the fundamental group of X_n is a Bieberbach group, then it is isomorphic to the fundamental group of a real Bott manifold.* (*This means that if X_n is homeomorphic to a flat Riemannian manifold, then it is homeomorphic to a real Bott manifold.*)

We consider the following setting:

$$\begin{aligned} &s_is_js_i^{-1} = s_n^{a_{ij}} s_j^{\epsilon_{ij}} \quad \text{with } a_{ij} \in \mathbb{Z},\ \epsilon_{ij} = \pm 1 \text{ for } 1 \le i < j < n, \\ &s_is_ns_i^{-1} = s_n^{\epsilon_i} \quad \text{with } \epsilon_i = \pm 1 \text{ for } 1 \le i < n. \end{aligned} \tag{6.1}$$

Lemma 6.2. $s_i^2 s_j^2 = s_j^2 s_i^2$ *for $i < j$ if and only if* $(\epsilon_i + \epsilon_{ij})(\epsilon_j + 1)a_{ij} = 0$.

Proof. The proof is essentially same as that in Lemma 5.1, so we omit it. $\square$

Lemma 6.3. *Fix $1 \le k < n$ and suppose*

$$a_{ij} = 0 \text{ for all } i > k. \tag{6.2}$$

Then, for $k < i < j < n$, we have

$$\begin{aligned} (\epsilon_j - 1)a_{ki} &= (\epsilon_i - 1)a_{kj} \quad \text{if } \epsilon_{ij} = 1, \\ (\epsilon_j - 1)a_{ki} &= (\epsilon_i + \epsilon_j)a_{kj} \quad \text{if } \epsilon_{ij} = -1. \end{aligned}$$

Proof. We conjugate the both sides of the former identity in (6.1) by s_k. Then the left hand side turns into

$$
\begin{aligned}
s_k(s_is_js_i^{-1})s_k^{-1} &= (s_ks_is_k^{-1})(s_ks_js_k^{-1})(s_ks_i^{-1}s_k^{-1}) \\
&= (s_n^{a_{ki}}s_i^{\epsilon_{ki}})(s_n^{a_{kj}}s_j^{\epsilon_{kj}})(s_n^{a_{ki}}s_i^{\epsilon_{ki}})^{-1} \\
&= s_n^{a_{ki}+\epsilon_i a_{kj}-\epsilon_j a_{ki}}s_i^{\epsilon_{ki}}s_j^{\epsilon_{kj}}s_i^{-\epsilon_{ki}}
\end{aligned} \tag{6.3}
$$

while since $a_{ij}=0$ for $i>k$ by assumption, the right hand side of (6.1) conjugated by s_k turns into

$$
s_ks_j^{\epsilon_{ij}}s_k^{-1} = \begin{cases} s_n^{a_{kj}}s_j^{\epsilon_{kj}} & \text{when } \epsilon_{ij}=1, \\ s_n^{-\epsilon_j a_{kj}}s_j^{-\epsilon_{kj}} & \text{when } \epsilon_{ij}=-1. \end{cases} \tag{6.4}
$$

When $\epsilon_{ij}=1$, $s_is_js_i^{-1}=s_j$ and hence $s_i^{\epsilon_{ki}}s_j^{\epsilon_{kj}}s_i^{-\epsilon_{ki}}=s_j^{\epsilon_{kj}}$. Therefore, comparing exponents of s_n at (6.3) and (6.4), we obtain the former identity in the lemma. When $\epsilon_{ij}=-1$, a similar argument yields the latter identity in the lemma. □

Lemma 6.4. *Let* (6.2) *be satisfied and $k<i<j<n$ as in Lemma 6.3. Then the following hold.*

(1) *If $\epsilon_i=\epsilon_j=-1$, then $a_{ki}=a_{kj}$.*
(2) *If $\epsilon_i=1$ and $a_{ki}\neq 0$, then $\epsilon_j=1$ and $\epsilon_{\ell i}=\epsilon_\ell$ for $k<\ell<i$.*

Proof. (1) This is obvious from Lemma 6.3.

(2) The first assertion $\epsilon_j=1$ is obvious from Lemma 6.3. To prove the latter assertion, we apply Lemma 6.3 with $i=\ell$ and $j=i$. When $\epsilon_{\ell i}=1$, we have $a_{k\ell}(\epsilon_i-1)=a_{ki}(\epsilon_\ell-1)$. This implies $\epsilon_\ell=1$ because $\epsilon_i=1$ and $a_{ki}\neq 0$ by assumption. When $\epsilon_{\ell i}=-1$, we have $a_{k\ell}(\epsilon_i-1)=a_{ki}(\epsilon_\ell+\epsilon_i)$. This implies $\epsilon_\ell=-1$ because $\epsilon_i=1$ and $a_{ki}\neq 0$. In any case, $\epsilon_{\ell i}=\epsilon_\ell$. □

Proof of Theorem 6.1. It suffices to prove that $\pi_1(X_n)$ is generated by $s_1,\dots,s_n$ with relations of the form (6.1) with $a_{ij}=0$ because the fundamental group of a real Bott manifold has such a presentation, see Example 3.3. We prove this assertion by induction on n. The assertion is clearly true when $n=2$. When $n=3$, $\pi_1(X_3)$ is of the form (5.1), that is

$$
s_1s_2s_1^{-1}=s_3^as_2^\epsilon, \quad s_1s_3s_1^{-1}=s_3^{\epsilon_1}, \quad s_2s_3s_2^{-1}=s_3^{\epsilon_2}.
$$

Here $a=0$ when $(\epsilon,\epsilon_1,\epsilon_2)=(1,1,1)$ or $(-1,-1,1)$ by Lemma 5.1 and a is even otherwise by Proposition 3.2 (2). Therefore one can assume $a=0$ by Lemma 5.2, so the assertion is true when $n=3$.

Now we assume that the assertion is true for $\pi_1(X_{n-1})$ with some $n\geq 4$. Then $\pi_1(X_n)$ is generated by $s_1,\dots,s_n$ with relations of the form (6.1). We shall show that we can achieve $a_{ij}=0$ by replacing s_i $(1\leq i<n)$ by $s_n^{b_i}s_i$ with suitable $b_i\in\mathbb{Z}$.

First we look at the following (last) three relations among s_{n-2},s_{n-1},s_n:

$$
s_{n-2}s_{n-1}s_{n-2}^{-1}=s_n^as_{n-1}^\epsilon, \quad s_{n-2}s_ns_{n-2}^{-1}=s_n^{\epsilon_{n-2}}, \quad s_{n-1}s_ns_{n-1}^{-1}=s_n^{\epsilon_{n-1}}
$$

where $a = a_{n-2\ n-1}$ and $\epsilon = \epsilon_{n-2\ n-1}$. Since X_n is an iterated $\mathbb{R}P^1$-bundle, one can assume $a = 0$ by the same reason as the case $n = 3$.

Now suppose that for some $k < n-2$, we have achieved $a_{ij} = 0$ for all $i > k$; so we are under the situation of Lemmas 6.3 and 6.4. What we shall prove is that we can achieve $a_{ij} = 0$ for all $i \geq k$. Let $p > k$. If $a_{kp} = 0$, we have nothing to do; so we assume $a_{kp} \neq 0$. We distinguish two cases according to the value of ϵ_p.

The case where $\epsilon_p = -1$. We replace s_k by $s_n^b s_k$. This replacement does not affect relations for s_i and s_j with $k < i < j$, so it keeps $a_{ij} = 0$ for $k < i < j$. But the relation $s_k s_p s_k^{-1} = s_n^{a_{kp}} s_p^{\epsilon_{kp}}$ turns into

$$s_k s_p s_k^{-1} = s_n^{a_{kp}+2b} s_p^{\epsilon_{kp}}$$

because $\epsilon_p = -1$. Since a_{kp} is even by Proposition 3.2 (2), one can take $b = -a_{kp}/2$ so that the exponent of s_n above becomes zero. For other $q > k$ with $\epsilon_q = -1$, $a_{kq} = a_{kp}$ by Lemma 6.4 (1). Therefore, b is independent of p with $\epsilon_p = -1$.

The case where $\epsilon_p = 1$. We note $\epsilon_j = 1$ for any $j > k$ by Lemma 6.4 (2). We replace s_p by $s_n^c s_p$. In this case, it is not obvious that this replacement keeps $a_{ij} = 0$ for $k < i < j$ but it does. In fact, its suffices to check that the relations $s_p s_j s_p^{-1} = s_j^{\epsilon_{pj}}$ for $p < j$ and $s_\ell s_p s_\ell^{-1} = s_p^{\epsilon_{\ell p}}$ for $\ell < p$ remain unchanged and one can easily check that the former identity remains unchanged because $\epsilon_j = 1$ and the latter one remains unchanged because $\epsilon_p = 1$ by assumption and $\epsilon_{\ell p} = \epsilon_\ell$ by Lemma 6.4 (2). However, the relation $s_k s_p s_k^{-1} = s_n^{a_{kp}} s_p^{\epsilon_{kp}}$ turns into

$$s_k s_p s_k^{-1} = s_n^{a_{kp}+(\epsilon_k-\epsilon_{kp})c} s_p^{\epsilon_{kp}}. \tag{6.5}$$

On the other hand, since $s_k^2 s_p^2 = s_p^2 s_k^2$, we have $(\epsilon_k + \epsilon_{kp})(\epsilon_p + 1)a_{kp} = 0$ by Lemma 6.2. Since $a_{kp} \neq 0$ and $\epsilon_p = 1$, this implies $\epsilon_k = -\epsilon_{kp}$ and hence $\epsilon_k - \epsilon_{kp} = \pm 2$. Since a_{kp} is even by Proposition 3.2 (2), one can take $c = -a_{kp}/(\epsilon_k - \epsilon_{kp})$ so that the exponent of s_n becomes zero in (6.5).

In any case, we can achieve $a_{kp} = 0$ for any $p > k$ keeping $a_{ij} = 0$ for $k < i < j$. This completes the induction step and proves the theorem. □

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