PROJECTIVE EMBEDDINGS OF THE TEICHMÜLLER SPACES OF BORDERED RIEMANN SURFACES

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Abstract. We will show that except few cases, by using the hyperbolic length functions of simple closed geodesics, we can embed the Teichmüller space of a bordered Riemann surface into the real projective space of the same dimension. The key idea is to study the hyperbolic structure on a subsurface conformally isomorphic to a torus with a hole (named as a "cook-hat"), or a thrice-punctured sphere with a hole (named as a "crown").

1. Introduction

Let M be a hyperbolic Riemann surface of genus g with n punctures and r holes. In this paper we assume that M has at least one boundary geodesic, i.e. $r \geq 1$. Then the Teichmüller space $\mathcal{T}_{g,n,r}$ is the space of isotopy classes of hyperbolic metrics on M which has a metric space structure homeomorphic to the real affine space $\mathbb{R}^{6g+2n+3r-6}$.

By using hyperbolic lengths of simple closed geodesics we can embed $\mathcal{T}_{g,n,r}$ into the real affine space. In practice we can embed $\mathcal{T}_{g,n,r}$ into $\mathbb{R}^{9g-9+3n+4r}$: Fix a pants decomposition $\mathcal{P}$ on M, i.e. a multicurve such that $M \setminus \mathcal{P}$ is homeomorphic to the disjoint union of thrice punctured spheres. $\mathcal{P}$ consists of $3g-3+n+r$ numbers of disjoint simple close curves. The Fenchel-Nielsen coordinates associate to each $m \in \mathcal{T}_{g,n,r}$ the length of each components of $\mathcal{P}$ and boundary geodesics, and the twist of each components of $\mathcal{P}$, which is a diffeomorphism from $\mathcal{T}_{g,n,r}$ onto $\mathbb{R}_+^{3g-3+n+2r} \times \mathbb{R}^{3g-3+n+r}$ (see [IT]). On the other hand the twist of each components of $\mathcal{P}$ can be determined by the lengths of two more curves for each components so that $\mathcal{T}_{g,n,r}$ can be embedded into $\mathbb{R}^{9g-9+3n+4r}$ by length functions of $9g-9+3n+4r$ number of simple closed geodesics. In his paper [S1], Schmutz showed that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}_{g,n,r}$ is equal to $dim_{\mathbb{R}}\mathcal{T}_{g,n,r}$, so that the image of $\mathcal{T}_{g,n,r}$ in $\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}_{g,n,r}}$ should be an unbounded domain.

Now we have the following natural question:

> Can we find $dim_{\mathbb{R}}\mathcal{T}_{g,n,r}+1$-number of simple closed geodesics whose hyperbolic lengths embed $\mathcal{T}_{g,n,r}$ into the finite dimensional real projective space $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}_{g,n,r}+1})$?

Because of the PL-Structure of the Thurston boundary, we might expect that the image of $\mathcal{T}_{g,n,r}$ should be the interior of some convex polyhedron in $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}_{g,n,r}+1})$.

In this paper we answer this question affirmatively except for the cases when $g=0$ and $r=0,1,2$. The key idea is to look for a subsurface homeomorphic to a thrice-punctured sphere with a hole or a torus with a hole, which is a tubular

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neighborhood of two geodesics contained in the members of geodesics parametrizing $\mathcal{T}_{g,n,r}$ in $P(\mathbb{R}^{dim_{\mathbb{R}}\mathcal{T}_{g,n,r}+1})$.

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2. Review the results of Schmutz

2.1. **Surfaces with no handles.** Let M be a Riemann surface of type $(0, n, r)$. From our assumption, n and r satisfy $n + r \geq 3$ and $r \geq 1$. We denote the boundary geodesics $x, a_1, a_2, \cdots, a_{n+r-1}$ and dividing geodesics $b_1, b_2, \cdots, b_{n+r-3}$ which decompose M into disjoint union of (degenerate) pair of pants (see Figure 1).

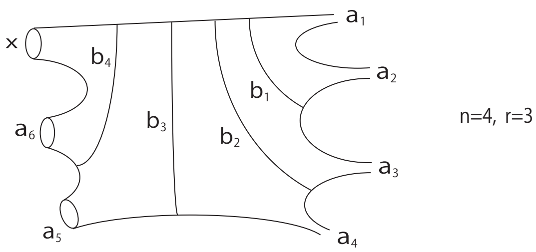

Figure 1

For each $i = 1, 2, \cdots, n+r-3$, let X_i be the subsurface of type $(0, n_i, r_i)$ where $n_i + r_i = 4$ with boundary geodesics $a_{i+1}, a_{i+2}, b_{i-1}, b_{i+1}$. Choose geodesics c_i and d_i in X_i so that the triple $\{b_i, c_i, d_i\}$ mutually intersect exactly twice. Then Schmutz proved that

Proposition 2.1. (cf. Proposition2 [S1])
The hyperbolic lengths of $2n + 3r - 6$ *geodesics*

$$a_1, a_2, \cdots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3}, , d_1, d_2, d_{n+r-3}$$

embeds $T_{0,n,r}$ *into* $\mathbb{R}^{2n+3r-6}$. *Here we remark that the length of* a_k *is equal to* 0 *when* a_k *corresponds to a puncture.*

2.2. **Surfaces with at least one handle.** Next we consider a Riemann surface M of type (g, n, r) where $g \geq 1$.

First we consider the case $(g, 0, 1)$. We denote the boundary geodesic by x. Choose non-dividing geodesics $a_1, a_2, \cdots, a_g, b_2, b_3, \cdots, b_g, c_2, c_3, \cdots, c_g$ which decompose M into disjoint union of pair of pants (see Figure 2).

For each $i = 2, \cdots, g-1$, let X_i be the subsurface of type $(0, 0, 4)$ with boundary geodesics $b_i, c_i, b_{i+1}, c_{i+1}$, Choose geodesics d_{i+1} and e_{i+1} in X_i so that the triple $\{a_{i+1}, d_{i+1}, e_{i+1}\}$ mutually intersect exactly twice. Let X_1 be the subsurface of M of type $(0, 0, 4)$ with boundary geodesics a_1, a_1, b_2, c_2, and choose d_2 and e_2 on X_1 so that the triple $\{a_2, d_2, e_2\}$ mutually intersect exactly twice. Moreover let f be a geodesic intersecting with $a_1, b_2, b_3, \cdots, b_g, c_2, c_3, \cdots, c_g$ exactly once. Then for $i = 2, \cdots, g$, we can find geodesics $r_1, s_2, s_3, \cdots, s_g, t_2, t_3, \cdots .t_g$ so that $\{a_1, r_1, f\}, \{b_i, s_i, f\}$ and $\{c_i, t_i, f\}$ mutually intersect exactly once. In this case, Schmutz proved that

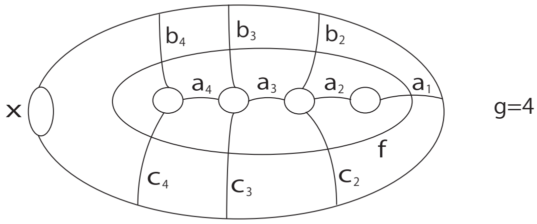

FIGURE 2

Proposition 2.2. (cf. Proposition3 [S1])
The hyperbolic lengths of $6g-3$ *geodesics*

$$a_1, a_2, \cdots, a_g, b_2, \cdots, b_g, d_2, \cdots, d_g, e_2, \cdots, e_g, f, r_1, s_2, \cdots, s_g, t_2, \cdots, t_g$$

embeds $T_{g,0,1}$ *into* $\mathbb{R}^{6g-3}$.

Finally we consider a Riemann surface M of type (g, n, r) where $g \geq 1$ in general. First we choose a dividing geodesic x to decompose M into subsurfaces M' of type $(g, 0, 1)$ and N' of type $(0, n, r+1)$ (see Figure 3).

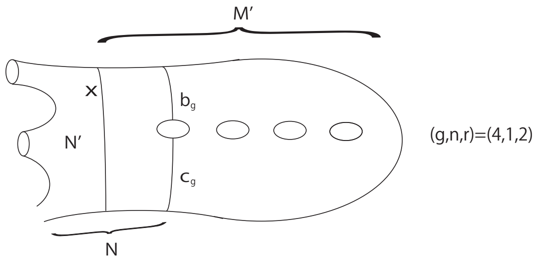

FIGURE 3

Let N be the subsurface of M consisting of N' and the pair of pants whose boundary curves are x, b_g and c_g. Then from the above argument we can choose $6g-3$ curves from M' and $2n+3(r+2)-6$ curves from N which determines M' and N in $T_{g,0,1}$ and $T_{0,n,r+2}$ respectively. On the other hand the lengths of curves x, b_g and c_g are counted twice in M' and N so that we can find $6g-3+2n+3(r+2)-6-3 = 6g+2n+3r-6$ geodesics whose hyperbolic lengths embed $T_{g,n,r}$ into $\mathbb{R}^{6g+2n+3r-6}$.

3. MAIN RESULT

First let M be a Riemann surface of type $(0, n, r)$. We assume that $n \geq 3$ and a_1, a_2, a_3 are punctures. Then the subsurface X_1 bounded by a_1, a_2, a_3 and b_2 is a thrice-punctured sphere with a hole, on which the triple $\{b_1, c_1, d_1\}$ mutually intersect exactly twice (see Figure 1). Therefore by means of Corollary 5.6, the hyperbolic lengths of $2n+3r-5$ geodesics

$$a_1, a_2, \cdots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3}, , d_1, d_2, d_{n+r-3}, b_2$$

embeds $T_{0,n,r}$ into $P(\mathbb{R}^{2n+3r-5})$.

Next we suppose M is a Riemann surface of type (g, n, r) where $g \geq 1$. Then there is a subsurface X of M with a geodesic boundary, which is a tubular neighborhood of the union of geodesics a_1 and f. X is homeomorphic to a torus with a hole on which the triple $\{a_1, r_1, f\}$ mutually intersect exactly once (see Figure 2). Then by means of Theorem 4.4, the proportion of the hyperbolic lengths of $6g + 2n + 3r - 5$ geodesics embeds $T_{g,n,r}$ into $P(\mathbb{R}^{6g+2n+3r-5})$.

Summarizing the above arguments,

Theorem 3.1. *Assume that $g \geq 1$ or $n \geq 3$. Then the Teichmüller space $T_{g,n,r}$ of a bordered Riemann surface can be embedded into the real projective space of $dim_{\mathbb{R}}\mathcal{T}_{g,n,r}$ by the hyperbolic length functions of $dim_{\mathbb{R}}\mathcal{T}_{g,n,r}+1$ simple closed geodesics.*

For a sphere (i.e., $g = 0$) with holes (i.e., $r \geq 1$), this question is still open for the cases $n = 0, 1, 2$.

4. Cook-hats

In this section we will consider complete hyperbolic structures on a torus with a hole. We call a hyperbolic torus with a hole a **cook-hat**.

Definition 4.1. Three simple closed geodesics (α, β, γ) on a cook-hat is called a **canonical triple** if each pair of them has the intersection number equal to one.

We remark that the hyperbolic lengths of a canonical triple (α, β, γ) satisfy triangle inequalities.

For the hyperbolic lengths of a canonical triple (α, β, γ) and the boundary geodesic δ on a cook-hat, we have the following equality and inequality.

Proposition 4.2. *For any cook-hat with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following equality and inequality:*

$$\cosh^2 \frac{l(\delta)}{4} = (\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2}). \tag{4.1}$$

$$l(\alpha) + l(\beta) + l(\gamma) > l(\delta). \tag{4.2}$$

Proof. We uniformize a cook-hat by a Fuchsian group $\Gamma \subset SL(2, \mathbb{R})$, and denote the traces of elements representing α, β, γ and δ by $t(\alpha), t(\beta), t(\gamma)$ and $t(\delta)$. Then they satisfy

$$t(\delta) - 2 = t(\alpha)t(\beta)t(\gamma) - (t(\alpha)^2 + t(\beta)^2 + t(\gamma)^2). \tag{4.3}$$

By means of the relation between trace functions and length functions

$$|t(\alpha)| = 2 \cosh \frac{l(\alpha)}{2} \tag{4.4}$$

and the equality

$$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y),$$

we can rewrite (4.3) in terms of length functions

$$\begin{aligned} & 2\cosh\frac{l(\delta)}{2} - 2 = t(\delta) - 2 \\ = \ & t(\alpha)t(\beta)t(\gamma) - (t(\alpha)^2 + t(\beta)^2 + t(\gamma)^2) \\ = \ & 4(2\cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)}{2}\cosh\frac{l(\gamma)}{2} - \cosh^2\frac{l(\alpha)}{2} - \cosh^2\frac{l(\beta)}{2} - \cosh^2\frac{l(\gamma)}{2}) \\ = \ & 4(\cosh\frac{l(\beta)+l(\gamma)}{2} - \cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2} - \cosh\frac{l(\beta)-l(\gamma)}{2}) - 4. \end{aligned}$$

Therefore

$$\begin{aligned} \cosh^2\frac{l(\delta)}{4} &= \frac{1}{2}(\cosh\frac{l(\delta)}{2} + 1) \\ &= (\cosh\frac{l(\beta)+l(\gamma)}{2} - \cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2} - \cosh\frac{l(\beta)-l(\gamma)}{2}) \end{aligned}$$

which is the equality (4.1).

Since $\cosh x$, hence $\cosh^2 x$ is monotonely increasing function of x, the equality (4.1) implies that it is enough to show that

$$(\cosh\frac{l(\beta)+l(\gamma)}{2} - \cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2} - \cosh\frac{l(\beta)-l(\gamma)}{2}) < \cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4}$$

for the proof of the inequality (4.2). In practice

$$\begin{aligned} & \cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4} \\ & -(\cosh\frac{l(\beta)+l(\gamma)}{2} - \cosh\frac{l(\alpha)}{2})(\cosh\frac{l(\alpha)}{2} - \cosh\frac{l(\beta)-l(\gamma)}{2}) \\ = \ & \cosh^2\frac{l(\alpha)+l(\beta)+l(\gamma)}{4} + \cosh^2\frac{l(\alpha)}{2} + \cosh\frac{l(\beta)+l(\gamma)}{2}\cosh\frac{l(\beta)-l(\gamma)}{2} \\ & - \cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)+l(\gamma)}{2} - \cosh\frac{l(\alpha)}{2}\cosh\frac{l(\beta)-l(\gamma)}{2} \\ = \ & \frac{1}{4}\{(e^{l(\alpha)} - e^{\frac{l(\alpha)+l(\beta)-l(\gamma)}{2}}) + (e^{l(\beta)} - e^{\frac{l(\beta)+l(\gamma)-l(\alpha)}{2}}) + (e^{l(\gamma)} - e^{\frac{l(\gamma)+l(\alpha)-l(\beta)}{2}}) \\ & +(1 - e^{\frac{l(\alpha)-l(\beta)-l(\gamma)}{2}}) + (1 - e^{\frac{l(\beta)-l(\gamma)-l(\alpha)}{2}}) + (1 - e^{\frac{l(\gamma)-l(\alpha)-l(\beta)}{2}}) \\ & +e^{-l(\alpha)} + e^{-l(\beta)} + e^{-l(\gamma)} + 1\} > 0. \end{aligned}$$

□

Remark 4.3. (1) The equality (4.1) also follows from the plane hyperbolic geometry of the right angled hexagon which is the symmetric half of the pair of pants $T \setminus \alpha$.
(2) The inequality (4.2) also comes from the fact that the curve $\alpha \cup \beta \cup \gamma$ is freely homotopic to the geodesic δ.

By means of the equality (4.1) in Proposition 4.2, we can embed the Teichmüller space $\mathcal{T}(T)$ of a torus with a hole into the 3-dimensional real projective space $P(\mathbb{R}^4)$.

Theorem 4.4. *For a cook hat with a canonical triple* (α, β, γ) *and the boundary geodesic* δ*, their hyperbolic lengths* $l(\alpha), l(\beta), l(\gamma)$ *and* $l(\delta)$ *satisfy*

$$\cosh^2\frac{sl(\delta)}{4} < (\cosh\frac{sl(\beta)+sl(\gamma)}{2} - \cosh\frac{sl(\alpha)}{2})(\cosh\frac{sl(\alpha)}{2} - \cosh\frac{sl(\beta)-sl(\gamma)}{2})$$

for any $s > 1$. In particular the system of length functions $L := (l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $\mathcal{T}(T)$ of a torus with a hole into $P(\mathbb{R}^4)$.

Proof. For simplicity we will write

$$a = l(\alpha), b = l(\beta), c = l(\gamma), d = l(\delta).$$

Then our claim is rewritten as

$$\frac{d}{4}s < \cosh^{-1}\sqrt{f(s)}, \ \ \forall s > 1$$

where

$$f(s) := (\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s)(\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s),$$

for which it is enough to show that

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} > \frac{d}{4}, \ \ \forall s > 1.$$

By the inequality (4.2), it is enough to show that

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} > \frac{a+b+c}{4}, \ \ \forall s > 1.$$

By the following simple estimation

$$\frac{d}{ds}\cosh^{-1}\sqrt{f(s)} = \frac{f'(s)}{2\sqrt{f(s)}\sqrt{f(s)-1}} > \frac{f'(s)}{2f(s)}$$

we will show that

$$\frac{f'(s)}{f(s)} > \frac{a+b+c}{2}, \ \ \forall s > 1.$$

In practice

$$\begin{aligned}\frac{f'(s)}{f(s)} &= \frac{\frac{d}{ds}(\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s)}{\cosh\frac{b+c}{2}s - \cosh\frac{a}{2}s} + \frac{\frac{d}{ds}(\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s)}{\cosh\frac{a}{2}s - \cosh\frac{b-c}{2}s} \\ &> \frac{b+c}{2} + \frac{a}{2} = \frac{a+b+c}{2}.\end{aligned}$$

Here we use the following lemma:

Lemma 4.5. *For $0 < p < q$,*

$$g(s) := \frac{\frac{d}{ds}(\cosh qs - \cosh ps)}{\cosh qs - \cosh ps} = \frac{q\sinh qs - p\sinh ps}{\cosh qs - \cosh ps} > q, \ \ \forall s > 1.$$

Proof. It is enough to show that the derivative of $g(s)$ is negative for $\forall s > 1$, since

$$\lim_{s\to\infty} g(s) = \lim_{s\to\infty}\frac{q\sinh qs - p\sinh ps}{\cosh qs - \cosh ps} = q.$$

Hence we will show the negativity of the numerator of $g'(s)$:

$$g'(s) = \frac{(q^2\cosh qs - p^2\cosh ps)(\cosh qs - \cosh ps) - (q\sinh qs - p\sinh ps)^2}{(\cosh qs - \cosh ps)^2}.$$

In practice

$$\begin{aligned}
& (q^2\cosh qs - p^2\cosh ps)(\cosh qs - \cosh ps) - (q\sinh qs - p\sinh ps)^2 \\
= \ & q^2\cosh^2 qs + p^2\cosh^2 ps - (q^2+p^2)\cosh qs\cosh ps \\
& -q^2\sinh^2 qs - p^2\sinh^2 ps + 2pq\sinh qs\sinh ps \\
= \ & q^2+p^2 - \frac{1}{2}(q+p)^2\cosh(q-p)s - \frac{1}{2}(q-p)^2\cosh(q+p)s \\
< \ & q^2+p^2 - \frac{1}{2}(q+p)^2 - \frac{1}{2}(q-p)^2 = 0.
\end{aligned}$$

□

□

By means of the triangle inequalities of $l(\alpha), l(\beta), l(\gamma)$ and the inequality (4.2) in Proposition 4.2, we can determine the image of $\mathcal{T}(T)$ in $\mathcal{P}(\mathbb{R}^4)$ as follows.

Theorem 4.6. *The image of $\mathcal{T}(T)$ the Teichmüller space of a cook-hat under the map $L := (l(\alpha) : l(\beta) : l(\gamma) : l(\delta))$ is the convex polyhedron Δ in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\begin{aligned}
\Delta \ := \ & \{(a:b:c:d) \in \mathcal{P}(\mathbb{R}^4) \mid a>0,\ b>0,\ c>0,\ d>0, \\
& a<b+c,\ b<c+a,\ c<a+b,\ d<a+b+c\}.
\end{aligned}$$

Proof. By means of the inequality (4.2) in Proposition 4.2, we have $L(T) \subset \Delta$. Hence we will prove that $\Delta \subset L(T)$. Take any point $p \in \Delta$ and four positive real numbers $(a,b,c,d) \in \mathbb{R}^4_+$ satisfying $p = (a:b:c:d)$. Then there exist $s>0$ and a hyperbolic structure $m \in \mathcal{T}(T)$ such that

$$(l(\alpha), l(\beta), l(\gamma), l(\delta)) = (as, bs, cs, d_s)$$

where $l(\alpha) = l(m,\alpha)$ and $d_s > 0$ is defined by

$$d_s := 4\cosh^{-1}\sqrt{(\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2})}.$$

To conclude that $L(m) = p$, It is enough to show that there is $s>0$ such that $d_s = sd$. We will show that d_s/s takes any value between 0 and $a+b+c$ when s varies. In practice d_s/s is a continuous function on s and

$$(\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2}) \to 1$$

when s decreases, hence $d_s/s \to 0$. On the other hand,

$$\begin{aligned}
& (\cosh\frac{sb+sc}{2} - \cosh\frac{sa}{2})(\cosh\frac{sa}{2} - \cosh\frac{sb-sc}{2}) \\
= \ & e^{\frac{(a+b+c)s}{2}} O(1), \quad s \to \infty
\end{aligned}$$

and

$$\cosh\frac{d_s}{4} = e^{\frac{d_s}{4}} O(1), \quad s \to \infty$$

imply that $\lim_{s\to\infty} d_s/s = a+b+c$. Hence d_s/s takes any value between 0 and $a+b+c$. □

5. Crowns

In this section we will consider complete hyperbolic structures on a thrice-punctured sphere with a hole. We call a hyperbolic thrice-punctured sphere with a hole a **crown**.

Definition 5.1. Three simple closed geodesics (α, β, γ) on a crown is called a **canonical triple** if each pair of them has the intersection number equal to two.

We will show that similar results in section 2 also hold for $\mathcal{T}(S)$ the Teichmüller space of a thrice-punctured sphere with a hole with the help of the geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$ explained below. For this purpose we realize $\mathcal{T}(T)$ and $\mathcal{T}(S)$ as hypersurfaces in $\mathbb{R}^4$ in terms of trace functions:

Theorem 5.2. (Theorem 2 of [L] and Proposition 3.1 of [NN])

(1) *We uniformize a cook-hat $m \in \mathcal{T}(T)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_T : \mathcal{T}(T) \to \mathbb{R}^4$ defined by $\varphi_T(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_T(\mathcal{T}(T))$ is described as follows:*

$$\{(a,b,c,d) \in \mathbb{R}^4 \quad | \quad a > 2, b > 2, c > 2, d > 2, \\ abc - a^2 - b^2 - c^2 + 2 = d\}.$$

(2) *We uniformize a crown $m \in \mathcal{T}(S)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_S : \mathcal{T}(S) \to \mathbb{R}^4$ defined by $\varphi_S(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_S(\mathcal{T}(S))$ is described as follows:*

$$\{(p,q,r,s) \in \mathbb{R}^4 \quad | \quad p > 2, q > 2, r > 2, s > 2, s^2 + 2(p+q+r+4)s \\ +4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 = 0\}.$$

Than by means of trace functions, we have the following geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$:

Theorem 5.3. *There is a bijection from $\mathcal{T}(T)$ to $\mathcal{T}(S)$ which sends a cook-hat T with the lengths of a canonical triple and the boundary geodesic equal to (l_1, l_2, l_3, l_4) to a crown S with the lengths of a canonical triple and the boundary geodesic equal to $(2l_1, 2l_2, 2l_3, l_4)$.*

Proof. When we substitute (a^2-2, b^2-2, c^2-2, d) for (p,q,r,s), the equation $s^2 + 2(p+q+r+4)s + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8$ factorizes as

$$\begin{aligned} & d^2 + 2(p+q+r+4)d + 4(p+q+r) + p^2 + q^2 + r^2 - pqr + 8 \\ = \quad & (d - (abc - a^2 - b^2 - c^2 + 2))(d - (-abc - a^2 - b^2 - c^2 + 2)). \end{aligned}$$

Hence the map $\Psi : \varphi_T(\mathcal{T}(T)) \to \varphi_S(\mathcal{T}(S))$ defined by $\Psi(a,b,c,d) := (a^2-2, b^2-2, c^2-2, d)$ is bijective. Also the relation between trace functions and length functions

$$|t(\alpha)| = 2\cosh\frac{l(\alpha)}{2}$$

tells us the length relations between $m \in \mathcal{T}(T)$ and $\varphi_S^{-1} \circ \Psi \circ \varphi_T(m) \in \mathcal{T}(S)$. $\square$

Remark 5.4. For the limiting case $l(\delta) = 0$, this bijection reduces to the well-known correspondence between punctured tori and forth-punctured spheres, which follows from the commensurability of uniformizing Fuchsian groups (see [ASWY]).

This bijection induces the next corollaries: The following inequality is the counterpart of the inequality (4.2) in Proposition 4.2 for crowns.

Corollary 5.5. *For any crown with the boundary geodesic δ and a canonical triple (α, β, γ), their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following inequality:*

$$l(\alpha) + l(\beta) + l(\gamma) > 2l(\delta).$$

Next result is the counterpart of Theorem 4.4 and 4.6 for crowns.

Corollary 5.6. *For a crown with a canonical triple (α, β, γ) and the boundary geodesic δ, the system of length functions $(l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $\mathcal{T}(S)$ into $P(\mathbb{R}^4)$. The image of $\mathcal{T}(S)$ is the convex polyhedron in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\{(a : b : c : d) \in P(\mathbb{R}^4) \mid a > 0,\ b > 0,\ c > 0,\ d > 0,$$
$$a < b + c,\ b < c + a,\ c < a + b,\ 2d < a + b + c\}.$$

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