SMALL COVER, INFRA-SOLVMANIFOLD AND CURVATURE

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ABSTRACT. It is shown that a small cover (resp. real moment-angle manifold) over a simple polytope is an infra-solvmanifold if and only if it is diffeomorphic to a real Bott manifold (resp. flat torus). Moreover, we obtain several equivalent conditions for a small cover being homeomorphic to a real Bott manifold. In addition, we study Riemannian metrics on small covers and real moment-angle manifolds with certain conditions on the Ricci or sectional curvature. We will see that these curvature conditions put very strong restrictions on the topology of the corresponding small covers and real moment-angle manifolds and the combinatorial structure of the underlying simple polytopes.

1. INTRODUCTION

The notion of small cover was first introduced by Davis and Januszkiewicz [11] as an analogue of smooth projective toric variety in the category of closed manifolds with $\mathbb{Z}_2$-torus actions. An n-dimensional *small cover* M^n is a closed n-manifold with a locally standard $(\mathbb{Z}_2)^n$ action whose orbit space can be identified with a simple convex polytope P^n in $\mathbb{R}^n$. The $(\mathbb{Z}_2)^n$-action on M^n determines a $(\mathbb{Z}_2)^n$-valued *characteristic function* λ_{M^n} on the facets of P^n, which encodes the information of isotropy subgroups of the non-free orbits. Conversely, we can recover M^n and the $(\mathbb{Z}_2)^n$-action, up to equivariant homeomorphism, by gluing 2^n copies of P^n according to the function λ_{M^n}. It is shown in [11] that many important topological invariants of M^n can be easily computed in terms of the combinatorial structure of P^n and the λ_{M^n}. For example, the fundamental group of M^n is a finite index subgroup of a right-angled Coxeter group W_{P^n}, where W_{P^n} is canonically determined by P^n. Note that not all simple convex polytopes admit small covers over them. But for any simple convex polytope P^n, we can

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canonically associate a closed manifold $\mathbb{R}\mathcal{Z}_{P^n}$ to P^n called *real moment-angle manifold* (see [11] and [3]).

In this paper, we will mainly use fundamental groups to study different kinds of geometric structures on small covers and real moment-angle manifolds. The following are some results proved in this paper.

Theorem 1.1. *If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of an n-dimensional simple convex polytope P^n is homeomorphic to an infra-solvmanifold, then P^n is an n-cube and hence $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to an n-dimensional torus. In particular, if a small cover is homeomorphic to an infra-solvmanifold, it must be homeomorphic to a real Bott manifold* (see Corollary 3.7)*.*

Theorem 1.2. *Let $\mathbb{R}\mathcal{Z}_{P^n}$ be the real moment-angle manifold of an n-dimensional simple convex polytope P^n.*

- (i) *$\mathbb{R}\mathcal{Z}_{P^n}$ admits a Riemannian metric with positive constant sectional curvature if and only if P^n is an n-simplex* (see Theorem 5.4)*.*
- (ii) *$\mathbb{R}\mathcal{Z}_{P^n}$ admits a flat Riemannian metric if and only if P^n is an n-cube* (see Corollary 3.7)*.*
- (iii) *If $\mathbb{R}\mathcal{Z}_{P^n}$ admits a Riemannian metric with negative (not necessarily constant) sectional curvature, then no 2-face of P^n can be a 3-gon or a 4-gon* (see Proposition 5.7)*.*

The paper is organized as follows. In section 2, we study when the fundamental group of a real moment-angle manifold (or a small cover) over a simple convex polytope is virtually nilpotent or finite (Corollary 2.5 and Corollary 2.6). Then we introduce a special class of small covers called *generalized real Bott manifolds* which are the main examples related to our study in this paper. In section 3, we will study when a small cover or real moment-angle manifold is an infra-solvmanifold. It turns out that such a small cover (or real moment-angle manifold) must be a real Bott manifold (or flat torus) (Corollary 3.7). The proof essentially uses a nice result in [12] that describes the asphericality of small covers in terms of flagness of the underlying simple polytope. In section 4, we obtain several equivalent conditions for a small cover being homeomorphic to a real Bott manifold (Theorem 4.3). In section 5, we study Riemannian metrics on small covers and real moment-angle manifolds with various conditions on the Ricci or sectional curvature. Most of the results obtained in this section follow from the study of fundamental groups in section 2. In addition, some problems are proposed for the future study.

2. Right-angled Coxeter group and fundamental group

Suppose P^n is an n-dimensional simple convex polytope in the Euclidean space $\mathbb{R}^n$. Here the word "simple" means that any vertex of P^n is the intersection of exactly n different facets of P^n. Let $\mathcal{F}(P^n)$ denote the set of all facets of P^n. Let W_{P^n} be a right-angled Coxeter group with one generator for each facet of P^n and relations $s^2 = 1$, $\forall\, s \in \mathcal{F}(P^n)$, and $(st)^2 = 1$ whenever s, t are adjacent facets of P^n.

Remark: Although we call W_{P^n} a right-angled Coxeter group, the dihedral angle between two adjacent facets of P^n may not be a right angle in reality. So generally speaking, W_{P^n} is not the group generated by the reflections of $\mathbb{R}^n$ about the hyperplanes passing the facets of P^n.

Suppose $F_1, \cdots, F_r$ are all the facets of P^n. Let $e_1, \cdots, e_r$ be a basis of $(\mathbb{Z}_2)^r$. Then we define a function $\lambda_0 : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^r$ by

$$\lambda_0(F_i) = e_i,\ 1 \leq i \leq r. \tag{1}$$

For any proper face f of P^n, let G_f denote the subgroup of $(\mathbb{Z}_2)^r$ generated by the set $\{\lambda_0(F_i) \,|\, f \subset F_i\}$. For any point $p \in P^n$, let $f(p)$ denote the unique face of P^n that contains p in its relative interior. In [11, Construction 4.1], the *real moment-angle manifold* $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n is defined to be the following quotient space

$$\mathbb{R}\mathcal{Z}_{P^n} := P^n \times (\mathbb{Z}_2)^r / \sim \tag{2}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_{f(p)}$. It is shown in [11] that the fundamental group of $\mathbb{R}\mathcal{Z}_{P^n}$ is the kernel of the abelianization $\mathrm{Ab}\colon W_{P^n} \to W^{ab}_{P^n} \cong (\mathbb{Z}_2)^r$, that is, there is an exact sequence

$$1 \longrightarrow \pi_1(\mathbb{R}\mathcal{Z}_{P^n}) \longrightarrow W_{P^n} \xrightarrow{\mathrm{Ab}} (\mathbb{Z}_2)^r \longrightarrow 1, \tag{3}$$

so we have

$$\pi_1(\mathbb{R}\mathcal{Z}_{P^n}) = \ker(\mathrm{Ab}) = [W_{P^n}, W_{P^n}] \text{ (the commutator subgroup of } W_{P^n}).$$

A *small cover* M^n over P^n is a closed n-manifold with a locally standard $(\mathbb{Z}_2)^n$-action so that its orbit space is homeomorphic to P^n. Here "locally standard" means that any point in M^n has an $(\mathbb{Z}_2)^n$-invariant open neighborhood which is equivariantly homeomorphic to an $(\mathbb{Z}_2)^n$-invariant open subset in an n-dimensional faithful linear representation space of $(\mathbb{Z}_2)^n$. Let $\pi : M^n \to P^n$ be the quotient map. For any facet F_i of P^n, the isotropy subgroup of $\pi^{-1}(F_i)$ in M^n under the $(\mathbb{Z}_2)^n$-action is a rank one subgroup of $(\mathbb{Z}_2)^n$ generated by a nonzero element, say $g_{F_i} \in (\mathbb{Z}_2)^n$. Then we obtain a map $\lambda_{M^n} : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^n$ where

$$\lambda_{M^n}(F_i) = g_{F_i},\ 1 \leq i \leq r.$$

We call λ_{M^n} the *characteristic function* associated to M^n. It is shown in [11] that up to equivariant homeomorphism, M^n can be recovered from (P^n, λ_{M^n}) in a similar way as the construction of $\mathbb{R}\mathcal{Z}_{P^n}$ in (2). Moreover, λ_{M^n} determines a group homomorphism $\overline{\lambda}_{M^n} : (\mathbb{Z}_2)^r \to (\mathbb{Z}_2)^n$ where

$$\overline{\lambda}_{M^n}(e_i) = \lambda_{M^n}(F_i) = g_{F_i},\ 1 \le i \le r.$$

It is shown in [11] that the fundamental group $\pi_1(M^n)$ of M^n is isomorphic to the kernel of the composition $\overline{\lambda}_{M^n} \circ \mathrm{Ab}$, that is, there is an exact sequence

$$1 \longrightarrow \pi_1(M^n) \longrightarrow W_{P^n} \xrightarrow{\overline{\lambda}_{M^n}\circ \mathrm{Ab}} (\mathbb{Z}_2)^n \longrightarrow 1. \tag{4}$$

Then it follows from (3) and (4) that we have an exact sequence

$$1 \longrightarrow \pi_1(\mathbb{R}\mathcal{Z}_{P^n}) \longrightarrow \pi_1(M^n) \longrightarrow (\mathbb{Z}_2)^{r-n} \longrightarrow 1. \tag{5}$$

In fact, it is easy to see that $\mathbb{R}\mathcal{Z}_{P^n}$ is a regular $(\mathbb{Z}_2)^{r-n}$-covering of any small cover over P^n.

Proposition 2.1. *For an n-dimensional simple convex polytope P^n, the following are equivalent:*

(i) *P^n is 2-neighborly (i.e. any two facets of P^n are adjacent).*

(ii) *The real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ over P^n is simply connected.*

Moreover, if there exists a small cover M^n over P^n, then the above statements are also equivalent to the following:

(iii) *The fundamental group of M^n is isomorphic to $(\mathbb{Z}_2)^{r-n}$, where r is the number of facets of P^n.*

Proof. By definition, P^n is 2-neighborly if and only if W_{P^n} is isomorphic to $(\mathbb{Z}_2)^r$ where r is the number of facets of P^n. Then the equivalence between (i) and (ii) follows from (3), and the equivalence between (ii) and (iii) follows from (5). $\square$

All the relations between the generators of W_{P^n} can be formally represented by a matrix $m = (m_{st})$, called *Coxeter matrix*.

$$m_{st} := \begin{cases} 1, & \text{if } s = t; \\ 2, & \text{if } s \text{ is adjacent to } t; \quad (s, t \text{ denote arbitrary facets of } P^n). \\ \infty, & \text{otherwise.} \end{cases}$$

The triple $(W_{P^n}, \mathcal{F}(P^n), m)$ is called a *Coxeter system* of W_{P^n}.

For a general Coxeter system (W, S, m), its *Coxeter graph* is a graph with a vertex set S, and with two vertices $s \neq t$ joined by an edge whenever $m_{st} \geq 3$. If $m_{st} \geq 4$, the corresponding edge is labeled by m_{st}. We say that (W, S, m) is *irreducible* if its Coxeter graph is connected.

A Coxeter group W is called *rigid* if, given any two systems (W, S, m) and (W, S', m') for W; there is an automorphism $\rho : W \to W$ such that $\rho(S, m) = (S', m')$, i.e. the Coxeter graphs of (W, S, m) and (W, S, m') are isomorphic.

Theorem 2.2 (Radcliffe [24])**.** *If (W, S, m) is a Coxeter system with $m_{st} \in \{2, \infty\}$ for all $s \neq t \in S$, then W is rigid. In other words, any right-angled Coxeter group is rigid.*

Associated to any Coxeter system (W, S, m), there is a symmetric bilinear form $(\ ,\ )$ on a real vector space V with a basis $\{\alpha_s \mid s \in S\}$ in one-to-one correspondence with the elements of S. The bilinear form $(\ ,\ )$ is defined by:

$$(\alpha_s, \alpha_t) := -\cos \frac{\pi}{m_{st}}, \tag{6}$$

where the value on the right-hand side is interpreted to be -1 when $m_{st} = \infty$.

It is well-known that a Coxeter group W is finite if and only if the bilinear form of a Coxeter system (W, S, m) is positive definite. All finite Coxeter groups have been classified by H. S. M. Coxeter in 1930s (see [10] and [16]). It is easy to see that if W is a finite right-angled Coxeter group, W must be isomorphic to $(\mathbb{Z}_2)^k$ for some $k \geq 0$. By Theorem 2.2, any Coxeter graph of $W \cong (\mathbb{Z}_2)^k$ should be k disjoint vertices.

An irreducible Coxeter group W is called *affine* if there is a Coxeter system (W, S, m) so that the bilinear form of the system (see (6)) is positive semi-definite but not positive definite. More generally, a Coxeter group is called *affine* if its irreducible components are either finite or affine, and at least one component is affine. Equivalently, a Coxeter group is affine if it is an infinite group and has a representation as a discrete, properly acting reflection group in $\mathbb{R}^n$. The reader is referred to [18] for more information on affine Coxeter groups. A Coxeter group is called *non-affine* if it is not affine.

Theorem 2.3 (Qi [23])**.** *The center of any finite index subgroup of an infinite, irreducible, non-affine Coxeter group is trivial.*

A group is called *virtually nilpotent* (*abelian, solvable*) if it has a nilpotent (abelian, solvable) subgroup of finite index.

Lemma 2.4. *If the right-angled Coxeter group W_P of a simple convex polytope P is virtually nilpotent, then $W_P \cong (\mathbb{Z}_2)^k \times (\widetilde{A}_1)^l$ for some $k, l \geq 0$, where $k + 2l$ equals the number of facets of P and $\widetilde{A}_1 = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle$.*

Proof. Suppose W_P is infinite and let N be a finite index nilpotent subgroup of W_P. If W_P is not affine, then W_P has at least one irreducible component which is neither finite nor affine, say W_1. Then $N_1 = N \cap W_1$ is a finite index nilpotent subgroup of W_1. Since W_1 is an infinite group, N_1 is also infinite hence nontrivial.

Now since W_1 is infinite, irreducible and non-affine, by Theorem 2.3, the center of N_1 must be trivial. But the center of any nontrivial nilpotent group is never trivial. This implies that the Coxeter group W_P must be either finite or affine.

Since W_P is right-angled, so is each of its irreducible components. Then by the classification of irreducible affine Coxeter group (see [18]), each connected component of the Coxeter graph of W_P is either a single vertex or a 1-simplex labeled by ∞. The Coxeter group corresponding to a single vertex is $\mathbb{Z}_2$, and the Coxeter group corresponding to a 1-simplex labeled by ∞ is $\widetilde{A}_1$. Suppose the Coxeter graph of W_P consists of k isolated vertices and l isolated 1-simplices labeled by ∞. Then $W_P \cong (\mathbb{Z}_2)^k \times (\widetilde{A}_1)^l$ and $k+2l$ equals the number of facets of P. □

Corollary 2.5. *If $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is virtually nilpotent, then $\pi_1(\mathbb{R}\mathcal{Z}_P) \cong \mathbb{Z}^l$ for some $l \leq r/2$ where r is the number of facets of P. In particular, if $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is finite, $\mathbb{R}\mathcal{Z}_P$ must be simply connected and so P is 2-neighborly.*

Proof. If $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is virtually nilpotent, so is the Coxeter group W_P. Then by Lemma 2.4, $W_P \cong (\mathbb{Z}_2)^k \times (\widetilde{A}_1)^l$ for some $k, l \geq 0$ with $k+2l = r$. So $\pi_1(\mathbb{R}\mathcal{Z}_P) = [W_P, W_P] \cong \mathbb{Z}^l$ since $[\widetilde{A}_1, \widetilde{A}_1] \cong \mathbb{Z}$. If $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is finite, then $l = 0$, i.e. $\pi_1(\mathbb{R}\mathcal{Z}_P)$ is trivial. And so P is 2-neighborly by Proposition 2.1. □

Corollary 2.6. *If a small cover M^n over an n-dimensional simple convex polytope P^n has finite fundamental group, then P^n is 2-neighborly and $\pi_1(M^n)$ is isomorphic to $(\mathbb{Z}_2)^{r-n}$, where r is the number of facets of P^n.*

Proof. Because of (5), $\pi_1(M^n)$ is finite if and only if $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$ is finite. So the claim follows from Corollary 2.5 and Proposition 2.1. □

Example 1. Let Δ^j denote a j-simplex. If P^n is a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$, then the number of facets of P^n is $n+m$ and P^n is 2-neighborly if and only if $n_i \geq 2$ for all $1 \leq i \leq m$. The real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ is a product of spheres $S^{n_1} \times \cdots \times S^{n_m}$, so $\pi_1(\mathbb{R}\mathcal{Z}_{P^n}) \cong \mathbb{Z}^l$ where l is the number of i's with $n_i = 1$.

Small covers over $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ naturally arise as follows. Recall that a *generalized real Bott manifold* is the total space B_m of an iterated fiber bundle:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{7}$$

where each B_i $(1 \leq i \leq m)$ is the projectivization of the Whitney sum of a finite collection (at least two) of real line bundles over B_{i-1}. We call the sequence in (7) a *generalized real Bott tower*. We consider B_m as a closed smooth manifold whose smooth structure is determined by the bundle structures of $\pi_i : B_i \to B_{i-1}$, $i = 1, \cdots, m$. Suppose the fiber of $\pi_i : B_i \to B_{i-1}$ is a real projective space of dimension n_i. Then it is easy to show that B_m is a small cover over a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$. Conversely, [8, Remark 6.5] tells us that any small cover over a product of simplices is homeomorphic to a generalized real Bott manifold. When $n_1 = \cdots = n_m = 1$, we call B_m a *real Bott manifold* (see [19]), which is a small cover over an m-cube.

Remark 2.7. The number of (weakly) equivariant homeomorphism types of real Bott manifolds was counted in [5] and [7]. Moreover, it was shown in [19] and [9] that the diffeomorphism types of real Bott manifolds are completely determined by their cohomology rings with $\mathbb{Z}_2$-coefficient. This is called *cohomological rigidity* of real Bott manifold. This property relates the diffeomorphism classification of real Bott manifolds with the classification of acyclic digraphs (directed graphs with no directed cycles) up to some equivalence (see [9]). But cohomological rigidity does not hold for generalized real Bott manifolds. Indeed, it is shown in [20] that there exist two generalized real Bott manifolds whose cohomology rings with $\mathbb{Z}_2$-coefficient are isomorphic, but they are not even homotopy equivalent.

Remark 2.8. There are many 2-neighborly simple convex polytopes which are not product of simplices. For example, the dual P^n of an n-dimensional cyclic polytope ($n \geq 4$) is always 2-neighborly. Recall that a *cyclic polytope* $C(k,n)$ ($k > n$) is the convex hull of k distinct points on the curve $\gamma(t) = (t, t^2, \cdots, t^n)$ in $\mathbb{R}^n$. Then the number of facets of its dual P^n has k facets. But when $k \geq 2^n$, P^n admits no small cover hence can not be a product of simplices (see p.428 of [11]). It is an interesting problem to find out all the 2-neighborly simple convex polytopes that admit small covers. Note that this problem is intimately related to the *Buchstaber invariant* of simple polytopes (see [3, section 7.5] and [13]).

3. Small cover, real moment-angle manifold and infra-solvmanifold

It is shown in [19] that any n-dimensional real Bott manifold admits a flat Riemannian metric which is invariant under the $(\mathbb{Z}_2)^n$-action. Conversely, any small cover of dimension n which admits a flat Riemannian metric invariant under the canonical $(\mathbb{Z}_2)^n$-action must be a real Bott manifold. In fact, this is proved for any real toric manifolds in [19, Theorem 1.2], but the same argument works for small covers. This suggests us to ask the following question.

Question: If a small cover M^n of dimension n admits a flat Riemannian metric (not necessarily invariant under the $(\mathbb{Z}_2)^n$-action), must M^n be diffeomorphic to a real Bott manifold? Or equivalently, must M^n be a small cover over an n-cube?

We will see that the answer to this question is yes (Corollary 3.7). In fact, we will obtain a much stronger result in Corollary 3.7. But first let us introduce a well-known notion in combinatorics.

Definition 3.1 (Flag Complex)**.** A simplicial complex K is called *flag* if a subset J of the vertex set of K spans a simplex in K whenever any two vertices in J are joined by a 1-simplex in K.

Let P be a simple convex polytope of dimension n. For simplicity, we say that P is *flag* if the boundary of P is dual to a flag complex. In other words, a collection of facets of P have a common intersection whenever any two of them intersect. Suppose $F_1, \cdots, F_r$ are all the facets of P. Then each F_k itself is an $(n-1)$-dimensional simple convex polytope whose facets are $\{F_{ki} := F_k \cap F_i \neq \emptyset, 1 \leq i \leq r\}$. Let $s(F_k)$ denote the generator of W_P corresponding to the facet F_k $(1 \leq k \leq r)$ of P. Similarly, let $s(F_{ki})$ denote the the generator of W_{F_k} corresponding to a facet F_{ki} of F_k.

Lemma 3.2. *Suppose a simple convex polytope P is flag and $F_1, \cdots, F_r$ are all the facets of P. If $F_k \cap F_i \neq \emptyset$ for $i = i_1, \ldots, i_q$, then $F_{ki_1} \cap \cdots \cap F_{ki_q} \neq \emptyset$ if and only if $F_{i_1} \cap \cdots \cap F_{i_q} \neq \emptyset$. So if P is flag, every facet of P must also be flag.*

Proof. The "only if part" is trivial. Suppose $F_{i_1} \cap \cdots \cap F_{i_q} \neq \emptyset$. Then any two of $F_k, F_{i_1}, \ldots, F_{i_q}$ intersect, so the whole intersection $F_k \cap F_{i_1} \cap \cdots \cap F_{i_q}$ is non-empty because P is flag, proving the "if part". □

Proposition 3.3. *Let P^n be a simple convex polytope of dimension n. If P^n is flag and the fundamental group of the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n is virtually solvable, then P^n must be an n-cube and so $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to the n-dimensional torus.*

Proof. When $n = 2$, P^n is a polygon. Then our assumption that P^n is flag and $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$ is virtually solvable will force P^n to be a 4-gon (see [3, Example 6.40]). In the rest, we assume $n \geq 3$. By (3), we have an exact sequence

$$1 \longrightarrow \pi_1(\mathbb{R}\mathcal{Z}_{P^n}) \longrightarrow W_{P^n} \longrightarrow (\mathbb{Z}_2)^r \longrightarrow 1.$$

Let F_k $(k = 1, \ldots, r)$ be the facets of P^n. Then similarly, we have an exact sequence for each F_k

$$1 \longrightarrow \pi_1(\mathbb{R}\mathcal{Z}_{F_k}) \longrightarrow W_{F_k} \longrightarrow (\mathbb{Z}_2)^{r_k} \longrightarrow 1$$

where r_k is the number of facets of F_k. Notice that W_{F_k} is generated by $s(F_{ki})$ with relations $s(F_{ki})^2 = 1$ and $(s(F_{ki})s(F_{kj}))^2 = 1$ whenever F_{ki} and F_{kj} intersect. But F_{ki} and F_{kj} intersect if and only if F_i and F_j (having non-empty intersection with F_k) intersect by Lemma 3.2. This implies that the group homomorphism $\varphi : W_{F_k} \to W_{P^n}$ sending each $s(F_{ki})$ to $s(F_i)$ is injective. Moreover, since

$$\pi_1(\mathbb{R}\mathcal{Z}_{P^n}) = [W_{P^n}, W_{P^n}], \quad \pi_1(\mathbb{R}\mathcal{Z}_{F_k}) = [W_{F_k}, W_{F_k}],$$

so φ maps $\pi_1(\mathbb{R}\mathcal{Z}_{F_k})$ injectively into $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$. Then $\pi_1(\mathbb{R}\mathcal{Z}_{F_k})$ is virtually solvable since so is $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$ by our assumption. In addition, F_k is also flag by Lemma 3.2.

By iterating the above arguments, we can show that for any 2-face f of P^n, f is flag and the fundamental group of the real moment-angle manifold $\mathbb{R}\mathcal{Z}_f$ is virtually solvable. We have shown that such an f must be a 4-gon. So any 2-face of P^n is a 4-gon, which implies that P^n is an n-cube (see [28, Problem 0.1]). □

It is shown in [12, Theorem 2.2.5] that a small cover over a simple convex polytope P is aspherical if and only if P is flag. Similarly, we can prove the following.

Proposition 3.4. *The real moment-angle manifold $\mathbb{R}\mathcal{Z}_P$ of a simple convex polytope P is aspherical if and only if P is flag.*

Proof. Define $\mathcal{M} = W_P \times P/\sim$, where the equivalence relation is defined by $(w, x) \sim (w', x') \iff x' = x$ and $w'w^{-1}$ belongs to the subgroup G_x of W_P generated by $\{s(F)\,;\, x \in F\}$. If x lies in the relative interior of a codimesion-k face of P, then the subgroup G_x of W_P is isomorphic to $(\mathbb{Z}_2)^k$.

It is shown in [11, Lemma 4.4] that $\mathcal{M}$ is simply connected. Let $\zeta : W_P \times P \to \mathcal{M}$ be the quotient map. There is a natural action of W_P on $\mathcal{M}$ defined by:

$$w' \cdot \zeta(w, x) = \zeta(w'w, x), \;\; w, w' \in W_P, \;\; x \in P. \tag{8}$$

The isotropy group of a point $\zeta(w, x) \in \mathcal{M}$ under this W_P-action is exactly G_x.

Claim-1: The commutator subgroup $[W_P, W_P]$ of W_P acts freely on $\mathcal{M}$.

It amounts to prove that $[W_P, W_P] \cap G_x = \{1\}$ for any point $x \in P$. In fact, it is easy to see that the abelianization $\mathrm{Ab} : W_P \to W_P^{ab}$ maps G_x injectively into W_P^{ab}. So $G_x \cap \ker(\mathrm{Ab}) = \{1\}$, proving Claim-1.

Claim-2: The quotient space $\mathcal{M}/[W_P, W_P]$ is homeomorphic to $\mathbb{R}\mathcal{Z}_P$.

Suppose $F_1, \cdots, F_r$ are all the facets of P. For each $1 \leq i \leq r$, let $\overline{s}(F_i)$ be the image of $s(F_i)$ under the abelianization $\mathrm{Ab} : W_P \to W_P^{ab}$. Then $\{\overline{s}(F_1), \ldots, \overline{s}(F_r)\}$ is a basis of $W_P^{ab} \cong (\mathbb{Z}_2)^r$. So the quotient $\mathcal{M}/[W_P, W_P]$ is homeomorphic to the space obtained by gluing 2^r copies of P according to the characteristic function μ on P where $\mu(F_i) = \overline{s}(F_i) \in (\mathbb{Z}_2)^r$, $1 \leq i \leq r$. This coincides with the definition of $\mathbb{R}\mathcal{Z}_P$ (see (2)). So the Claim-2 is proved.

Therefore, $\mathcal{M}$ is a universal covering of $\mathbb{R}\mathcal{Z}_P$. Then $\mathbb{R}\mathcal{Z}_P$ is aspherical if and only if $\mathcal{M}$ is contractible. But it is shown in [12, Theorem 2.2.5] that $\mathcal{M}$ is contractible if and only if P is flag. So the proposition is proved. □

The following corollary is an immediate consequence of Proposition 3.3 and Proposition 3.4.

Corollary 3.5. *Let P^n be a simple convex polytope of dimension n. If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n is aspherical with virtually solvable fundamental group, then P^n is an n-cube. Therefore, $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to the n-dimensional torus if and only if P^n is an n-cube.*

The typical examples of aspherical manifolds with virtually solvable fundamental group are infra-solvmanifolds. In fact, any compact aspherical manifold with virtually solvable fundamental group is homeomorphic to an infra-solvmanifold (see [14, Corollary 2.21]). But in general, we can not replace the "homeomorphic" by "diffeomorphic" in this statement.

Definition 3.6 (Infrahomogeneous Space)**.** Let G be a connected and simply connected Lie group, K be a maximal compact subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G, and Γ be a cocompact, discrete subgroup of $E(G) = G \rtimes K$. If the action of Γ on G is free and $[\Gamma : G \cap \Gamma] < \infty$, the orbit space $\Gamma\backslash G$ is called a (compact) *infrahomogeneous space modeled on G*. If G is solvable (nilpotent), $\Gamma\backslash G$ is called an *infra-solvmanifold* (*infra-nilmanifold*). When $G = \mathbb{R}^n$ and $K = O(n, \mathbb{R})$ (the orthogonal group), $\Gamma\backslash G$ is a compact *flat Riemannian manifold.*

In the above definition, the group law of $G \rtimes K < G \rtimes \mathrm{Aut}(G)$ is defined by:

$$(g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1 \cdot \tau_1(g_2), \tau_1 \circ \tau_2), \;\; g_1, g_2 \in G, \;\; \tau_1, \tau_2 \in \mathrm{Aut}(G).$$

The action of $G \rtimes \mathrm{Aut}(G)$ on G is defined by:

$$(g, \tau) \cdot g' = g \cdot \tau(g'), \;\; g, g' \in G, \;\; \tau \in \mathrm{Aut}(G).$$

By definition, we have the following hierarchy of notions:

Compact flat Riemannian manifolds $\subset$ Infra-nilmanifolds $\subset$ Infra-solvmanifolds
$\subset$ Compact aspherical manifolds with virtually solvable fundamental group

Infra-nilmanifolds and infra-solvmanifolds also have some Riemannian geometric interpretations as follows. By a theorem of Ruh [25] which is based on the work of Gromov [15], a compact connected manifold M is an infra-nilmanifold if and only if it is *almost flat*, which means that M admits a sequence of Riemannian metric $\{g_n\}$ with uniformly bounded sectional curvature so that the sectional curvatures of (M, g_n) converge uniformly to 0. Similarly for infra-solvmanifolds, it is shown in [27, Proposition 3.1] that a compact connected topological manifold M is homeomorphic to an infra-solvmanifolds if and only if M admits a sequence of Riemannian metric $\{g_n\}$ with uniformly bounded sectional curvature so that (M, g_n) collapses in the *Gromov-Hausdorff* sense to a *flat orbifold*.

In addition, it is shown in [2, Corollary 1.5] that compact infra-solvmanifolds are *smoothly rigid*, i.e. any two compact infra-solvmanifolds with isomorphic fundamental groups are diffeomorphic. From this fact and Corollary 3.5, we can easily prove the following.

Corollary 3.7. *Suppose P^n is an n-dimensional simple convex polytope.*

(i) *The real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n is (or homeomorphic to) an infra-solvmanifold if and only if $\mathbb{R}\mathcal{Z}_{P^n}$ is diffeomorphic (or homeomorphic) to the n-dimensional flat torus.*

(ii) *A small cover M^n over P^n is (or homeomorphic to) an infra-solvmanifold if and only if M^n is diffeomorphic (or homeomorphic) to a real Bott manifold. In particular, if M^n admits a flat Riemannian metric, then M^n is diffeomorphic to a real Bott manifold.*

Proof. The statement (i) follows easily from Corollary 3.5 and the smooth rigidity of infra-solvmanifolds. As for (ii), note that $\mathbb{R}\mathcal{Z}_{P^n}$ is a finite-sheeted covering space of any small cover (if exists) over P^n. So if a small cover over P^n is homeomorphic to an infra-solvmanifold, $\mathbb{R}\mathcal{Z}_{P^n}$ must be aspherical with virtually solvable fundamental group. Then (ii) also follows from Corollary 3.5 and the smooth rigidity of infra-solvmanifolds. □

Question: Let P and Q be two simple convex polytopes. If $\mathbb{R}\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_Q$ are homeomorphic, what can we conclude about the relationship between the combinatorial properties of P and Q?

By Proposition 3.4, if $\mathbb{R}\mathcal{Z}_P$ is homeomorphic to $\mathbb{R}\mathcal{Z}_Q$, then P and Q are either both flag or they are both non-flag. It is interesting to see more answers to this question.

4. Flag simple polytopes and Real Bott manifolds

In this section, we will get several different ways to describe a small cover that is homeomorphic to a real Bott manifolds. As we know from the definition, any real Bott manifold of dimension n is a small cover over an n-cube. It is clear that an n-cube is a flag simple polytope with $2n$ facets. Conversely, we can show the following.

Proposition 4.1. *Let P^n be a flag simple polytope of dimension n. Then P^n has at least $2n$ facets, and if P^n has exactly $2n$ facets, P must be an n-cube.*

Proof. This is probably known to many people. But since we do not know the literature, we shall give a proof here for the sake of completeness of the paper.

Since P^n is simple and of dimension n, there are n facets in P^n whose intersection is a vertex. We denote them by $F_1, \dots, F_n$ and the vertex $\bigcap_{i=1}^n F_i$ by v. For each $j \in [n] := \{1, \dots, n\}$, the intersection $\bigcap_{i \neq j} F_i$ is an edge of P^n which has v as an endpoint. Therefore, there is a unique facet of P^n, denoted G_j, such that $(\bigcap_{i\neq j} F_i) \cap G_j$ is the other endpoint of the edge $\bigcap_{i\neq j} F_i$ different from v.

We claim that G_j's must be mutually distinct. Indeed, if $G_p = G_q$ for some $p \neq q$, this implies that any two of the $n+1$ facets $F_1, \dots, F_n, G_p = G_q$ have non-empty intersection because $(\bigcap_{i\neq j} F_i) \cap G_j$ is non-empty for any j. Therefore the intersection of the $n+1$ facets must be non-empty since P^n is flag. However, this is impossible because P^n is simple and of dimension n. Therefore G_j's are mutually distinct and hence P^n has at least $2n$ facets, proving the former statement of the proposition.

Hereafter we assume that P^n has exactly $2n$ facets. Then the facets of P^n are exactly $F_1, \dots, F_n, G_1, \dots, G_n$. Since $\bigcap_{j=1}^n F_j$ and $(\bigcap_{i\neq j} F_i) \cap G_j$ are both non-empty, any two of the $n+1$ facets $F_1, \dots, F_n, G_j$ have non-empty intersection if $F_j \cap G_j \neq \emptyset$. However, this is impossible by the same reason as above. Therefore

$$F_j \cap G_j = \emptyset \text{ for any } j \in [n]. \tag{9}$$

We shall prove that

$$(\bigcap_{i\notin J} F_i) \cap (\bigcap_{j\in J} G_j) \neq \emptyset \quad \text{for any subset } J \text{ of } [n] \tag{10}$$

by induction on the cardinality $|J|$ of J. Since $(\bigcap_{i\neq j} F_i) \cap G_j$ is a vertex of P^n by the choice of G_j, (10) holds when $|J| = 1$. Suppose that (10) holds for J with $|J| = k-1$. Let J be a subset of $[n]$ with $|J| = k$. Without loss of generality, we may assume that $J = \{1, 2, \dots, k\}$. By the induction assumption we have $(\bigcap_{i=k}^n F_i) \cap (\bigcap_{j=1}^{k-1} G_j) \neq \emptyset$. Since P^n is simple, $(\bigcap_{i=k}^n F_i) \cap \bigcap_{j=1}^{k-1} G_j)$ is a vertex of P^n, denoted w, and $(\bigcap_{i=k+1}^n F_i) \cap (\bigcap_{j=1}^{k-1} G_j)$ is an edge of P^n which contains the vertex w. Therefore, there is a unique facet H of P^n such that

$$(\bigcap_{i=k+1}^{n} F_i) \cap (\bigcap_{j=1}^{k-1} G_j) \cap H \text{ is a vertex of } P^n \text{ different from } w. \tag{11}$$

We claim that $H = G_k$. In fact, since the intersection in (11) is a vertex, H must be either F_p for $1 \le p \le k$ or G_q for $k \le q \le n$. However, the intersection in (11) is empty unless $H = F_k$ or G_k by (9). Moreover, $H \neq F_k$ because the intersection in (11) is different from w. Therefore we can conclude $H = G_k$ and this shows that (10) holds for J with $|J| = k$, completing the induction step.

Let P^* be the simplicial polytope dual to P^n. Then the facts (9) and (10) show that the boundary complex ∂P^* is isomorphic to the boundary complex of a crosspolytope C of dimension n, which is isomorphic to the n-fold join of S^0. Therefore the simplicial polytopes P^* and C are isomorphic combinatorially and hence so are their duals P^n and C^*. Since C^* is an n-cube, this proves the latter statement of the proposition. □

Remark 4.2. The above argument shows that if the geometrical realization of a flag simplicial complex K is a pseudomanifold of dimension $n-1$, then the number of vertices of K is at least $2n$; and if it is exactly $2n$, then K is isomorphic to the boundary complex of the crosspolytope of dimension n.

Combining all our previous discussions, we get several descriptions of a small cover that is homeomorphic to a real Bott manifold as follows.

Theorem 4.3. *Suppose M^n is a small cover over a simple convex polytope P^n of dimension n. Let $b_1(M^n;\mathbb{Z}_2)$ denote the first Betti number of M with $\mathbb{Z}_2$-coefficient. Then the following statements are equivalent.*

- *M^n is homeomorphic to a real Bott manifold.*
- *M^n is aspherical and $b_1(M^n;\mathbb{Z}_2) \leq n$.*
- *P^n is flag and the number of facets of P^n is $\leq 2n$.*
- *P^n is an n-cube.*

Proof. Suppose P^n has r facets. It is known that $r = b_1(M^n;\mathbb{Z}_2) + n$ (see [11]). Then $b_1(M^n;\mathbb{Z}_2) \leq n \iff r \leq 2n$. In addition, M is aspherical $\iff P^n$ is flag by [12, Theorem 2.2.5]. Then this proposition follows from Proposition 4.1. □

5. Riemannian metrics on small covers and real moment-angle manifolds

Geometric structures on small covers were first discussed in [11, Example 1.21] for 3-dimensional cases. Later, Davis-Januszkiewcz-Scott [12] systematically studied the piecewise Euclidean structure on any small cover. A very nice result obtained in [12, Proposition 2.2.3] says that the natural piecewise Euclidean cubical metric on a small cover over a simple polytope P is nonpositively curved if and only if the boundary of P is dual to a flag complex (this is also equivalent to saying that the small cover is aspherical).

In this section, we will study Riemannian metrics on small covers and real moment-angle manifolds in any dimension with certain conditions on the Ricci or sectional curvature. By our discussion of the fundamental group of small covers and real moment-angle manifolds in section 2, we will see that these curvature conditions put very strong restrictions on the topology of the corresponding small covers and real moment-angle manifolds and the combinatorics of the underlying simple polytopes.

A Riemannian manifold is called *positively* (*nonnegatively*, *negatively*) *curved* if its *sectional curvature* is everywhere positive (nonnegative, negative). It is clear that a positively (nonnegatively, negatively) curved Riemannian manifold has positive (nonnegative, negative) *Ricci curvature*.

5.1. **Positive curvature.** By a classical theorem of Bonnet and Myers, any compact Riemannian manifold with positive Ricci curvature must have finite fundamental group (see Chapter 6 of [21]). Then by Corollary 2.5 and Corollary 2.6, we have the following corollary.

Corollary 5.1. *Let P^n be an n-dimensional simple convex polytope. If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ admits a Riemannian metric with positive Ricci curvature, then P^n must be 2-neighborly and $\mathbb{R}\mathcal{Z}_{P^n}$ is simply connected. Similarly, if a small cover M^n over P^n admits a Riemannian metric with positive Ricci curvature, then P^n must be 2-neighborly and the fundamental group of M^n is isomorphic to $(\mathbb{Z}_2)^{r-n}$ where r is the number of facets of P^n.*

The following is a well-known fact on positively curved Riemannian manifolds (see Chapter 6 of [21]).

Theorem 5.2 (Synge 1936)**.** *Let M be a compact Riemannian manifold with positive sectional curvature.*

(i) *If M is even-dimensional and orientable, then M is simply connected.*
(ii) *If M is odd-dimensional, then M is orientable.*

Notice that a small cover is never simply connected (this is an easy consequence of (4)). So by Synge's theorem, we can conclude the following.

Corollary 5.3. *If an even (odd) dimensional small cover admits a positively curved Riemannian metric, then it must be non-orientable (orientable).*

It is well known that a simply connected closed smooth n-manifold which admits a Riemannian metric with positive *constant* sectional curvature is diffeomorphic to the standard sphere S^n in $\mathbb{R}^{n+1}$.

Theorem 5.4. *Let P^n be an n-dimensional simple convex polytope. If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ (or a small cover M^n) of P^n admits a Riemannian metric with positive constant sectional curvature, then $\mathbb{R}\mathcal{Z}_{P^n}$ (or M^n) is diffeomorphic to the standard sphere S^n (or the real projective space $\mathbb{R}\mathbb{P}^n$).*

Proof. If $\mathbb{R}\mathcal{Z}_{P^n}$ admits a Riemannian metric with positive constant sectional curvature, then it is simply connected by Corollary 5.1. So it is diffeomorphic to S^n. If a small cover M^n over P^n admits a Riemannian metric with positive constant sectional curvature, so does $\mathbb{R}\mathcal{Z}_{P^n}$. Then $\mathbb{R}\mathcal{Z}_{P^n}$ is diffeomorphic to S^n and $\mathbb{R}\mathcal{Z}_{P^n}$ is a universal covering space of M^n. In addition, Corollary 5.1 tells us that the fundamental group of M^n is isomorphic to $(\mathbb{Z}_2)^{r-n}$ where r is the number of facets of P^n. So M^n is the quotient space of $\mathbb{R}\mathcal{Z}_{P^n} \cong S^n$ by a free $(\mathbb{Z}_2)^{r-n}$-action. But by the classical Smith's theory (see [26]), we must have $r - n \leq 1$. So P^n must be an n-dimensional simplex and then M^n is the n-dimensional real projective space $\mathbb{RP}^n$. □

The following geometric problem for small covers should be interesting to study.

Problem-1: find out all the small covers (or real moment-angle manifolds) which admit Riemannian metrics with positive sectional or Ricci curvature in each dimension.

It is well known that the only 2-neighborly simple polytopes in dimension 2 and 3 are the 2-simplex and 3-simplex. So by Corollary 5.1, the only small covers in dimension 2 and 3 that admit Riemannian metrics with positive Ricci curvature are $\mathbb{RP}^2$ and $\mathbb{RP}^3$. But in dimension ≥ 4, the answer to Problem-1 is not so clear. In particular, it is interesting to see if there exists any small cover which admits a positively curved Riemannian metric but not homeomorphic to a real projective space $\mathbb{RP}^n$.

5.2. **Nonnegative curvature.** By Cheeger-Gromoll splitting theorem (see [6]), the fundamental group of any compact Riemannian manifold with nonnegative Ricci curvature is virtually abelian. This fact leads to the following description of the fundamental group of the real moment-angle manifold or a small cover that admits a Riemannian metric with nonnegative Ricci curvature.

Proposition 5.5. *Let P^n be an n-dimensional simple convex polytope with r facets. If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n admits a Riemannian metric with nonnegative Ricci curvature, then $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$ is isomorphic to $\mathbb{Z}^l$ for some $l \leq r/2$. Similarly, if a small cover M^n over P^n admits a Riemannian metric with nonnegative Ricci curvature, then there is an exact sequence*

$$1 \longrightarrow \mathbb{Z}^l \longrightarrow \pi_1(M^n) \longrightarrow (\mathbb{Z}_2)^{r-n} \longrightarrow 1$$

where $l \leq r/2$.

Proof. If $\mathbb{R}\mathcal{Z}_{P^n}$ admits a Riemannian metric with nonnegative Ricci curvature, then $\pi_1(\mathbb{R}\mathcal{Z}_{P^n})$ is virtually abelian as remarked above. Therefore, the former statement in the proposition follows from Corollary 2.5. If a small cover M^n over P^n admits a Riemannian metric with nonnegative Ricci curvature, then so does

$\mathbb{R}\mathcal{Z}_{P^n}$ because $\mathbb{R}\mathcal{Z}_{P^n}$ is a finite cover of M^n. Therefore, the latter statement in the proposition follows from (5) and the former statement. □

Example 2. The real moment-angle manifold over a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ is a product of standard spheres $S^{n_1} \times \cdots \times S^{n_m} =: S$. In the product, let each sphere $S^{n_i} \subset \mathbb{R}^{n_i+1}$ be equipped with the standard Riemannian metric whose isometry group is the orthogonal group $O(n_i + 1, \mathbb{R})$. Then S is a nonnegatively curved Riemannian manifold with respect to the product metric.

A generalized real Bott manifold B_m discussed in Example 1 is a small cover over $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$. It is known that B_m is the quotient of S by a free $(\mathbb{Z}_2)^m$-action on it (see [8, Proposition 6.2]) and one can easily see that the $(\mathbb{Z}_2)^m$-action preserves the product metric on S. So the quotient space of this free $(\mathbb{Z}_2)^m$-action, that is B_m, inherits a nonnegatively curved Riemannian metric from S.

Remark 5.6. It is possible that B_m is the total space of several different generalized real Bott towers. Therefore, we will get several different nonnegatively curved Riemannian metrics on B_m which may not be isometric.

Problem-2: find out all the small covers which admit Riemannian metrics with nonnegative sectional or Ricci curvature in each dimension.

In dimension 2 and dimension 3, the small covers which admit Riemannian metric with nonnegative Ricci curvature are exactly the generalized real Bott manifolds (this follows from the classification of 3-dimensional compact Riemannian manifolds with nonnegative Ricci curvature in [17]). So by Example 2, we can conclude that all the nonnegatively curved small covers in dimension 2 and dimension 3 are exactly generalized real Bott manifolds. In dimension ≥ 4, the answer to Problem-2 is not so easy. In particular, it is interesting to see if there exists any small cover which admits a nonnegatively curved Riemannian metric but not homeomorphic to any generalized real Bott manifold.

5.3. **Negative curvature.** A theorem due to Preissmann [22] says that any abelian subgroup of the fundamental group of a negatively curved compact Riemannian manifold is infinite cyclic (also see [4, section 9.3]). Using this fact, we can easily show the following.

Proposition 5.7. *If the real moment-angle manifold $\mathbb{R}\mathcal{Z}_P$ (or a small cover) of a simple polytope P admits a negatively curved Riemannian metric, then no 2-face of P can be a 3-gon or a 4-gon.*

Proof. If $\mathbb{R}\mathcal{Z}_P$ admits a negatively curved Riemannian metric, then it follows from Cartan-Hadamard theorem that $\mathbb{R}\mathcal{Z}_P$ is aspherical. So the simple polytope P is flag by Proposition 3.4. Then by Lemma 3.2, any 2-face f of P must be flag. So f

can not be a 3-gon. In addition, since P is flag, the proof of Proposition 3.3 implies that there is an injective group homomorphism from $\pi_1(\mathbb{R}\mathcal{Z}_f)$ into $\pi_1(\mathbb{R}\mathcal{Z}_P)$. This implies that f can not be a 4-gon, because otherwise $\pi_1(\mathbb{R}\mathcal{Z}_f) \cong \mathbb{Z} \oplus \mathbb{Z}$ and so $\pi_1(\mathbb{R}\mathcal{Z}_P)$ contains an abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ which contradicts the Preissmann's theorem mentioned above. Finally, if a small cover over P admits a negatively curved Riemannian metric, then so does $\mathbb{R}\mathcal{Z}_P$. □

Remark 5.8. It was shown in [11, Example 1.21] that a small cover over a 3-dimensional simple convex polytope P admits a *hyperbolic structure* (i.e. Riemannian metric with negative constant sectional curvature) if and only if P has no 3-gon or 4-gon facets. Its proof uses the Andreev's theorem [1] on when a 3-dimensional simple polytope can be realized as a right-angled polytope in the hyperbolic 3-space. But since there is no analogue of Andreev's theorem in higher dimensions, it is not clear how to judge the existence of hyperbolic structures on small covers (or real moment-angle manifolds) in general.

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