On the number of maximum points of least energy solutions to a two-dimensional Hénon equation with large exponent

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Abstract. In this note, we prove that least energy solutions of the two-dimensional Hénon equation

$$-\Delta u = |x|^{2\alpha} u^p \quad x \in \Omega, \quad u > 0 \quad x \in \Omega, \quad u = 0 \quad x \in \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with $0 \in \Omega$, $\alpha \geq 0$ is a constant and $p > 1$, have only one global maximum point when $\alpha > e - 1$ and the nonlinear exponent p is sufficiently large. This answers positively to a recent conjecture by C. Zhao (preprint, 2011).

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1. Introduction.

In this note we consider the problem

$$\begin{cases} -\Delta u = |x|^{2\alpha} u^p & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with $0 \in \Omega$, $\alpha \geq 0$ is a constant and $p > 1$. Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$S_p = \inf \left\{ \int_\Omega |\nabla v|^2 dx \mid v \in H_0^1(\Omega), \int_\Omega |x|^{2\alpha} |v|^{p+1} dx = 1 \right\}. \tag{1.2}$$

Standard variational method implies that S_p is achieved by a positive function $v_p \in H_0^1(\Omega)$ and then $u_p = S_p^{1/(p-1)} v_p$ solves (1.1). We call u_p a least energy solution to the problem (1.1).

When $\alpha = 0$, several studies on the asymptotic behavior of least energy solutions u_p as $p \to \infty$ have been done in [4], [5], [3] and [1]. Recently, Chunyi Zhao [9] extended the study to the case when $\alpha > 0$, and obtained some results. First he showed that for any $\alpha > 0$, there exists $\delta > 0$ such that the least energy solution u_p satisfies $1 - \delta \le \|u_p\|_\infty \le \sqrt{e} + \delta$ for p large enough. To state his results further, we introduce some notations: Let x_p be a global maximum point of u_p and define $\varepsilon_p > 0$ by the relation

$$\varepsilon_p^2 |x_p|^{2\alpha} p \|u_p\|_\infty^{p-1} = 1.$$

Also define the function $\tilde{u}_p : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \to \mathbb{R}$ such that

$$\tilde{u}_p(y) = \frac{p}{\|u_p\|_\infty} \left\{ u_p\left(\varepsilon_p y + x_p\right) - u_p(x_p) \right\}.$$

By using these symbols, the main result of C. Zhao reads as follows:

Theorem 1 *(Chunyi Zhao [9]) Assume $\alpha > e - 1$. Then $\varepsilon_p \to 0$ and $\Omega_p \to \mathbb{R}^2$ as $p \to \infty$. Also for any sequence $p_n \to \infty$ as $n \to \infty$, there exists a subsequence (again denoted by the same symbol) such that*

$$\tilde{u}_{p_n}(y) \to U(y) := -2\log\left(1 + \frac{|y|^2}{8}\right) \quad in\ C^2_{loc}(\mathbb{R}^2), \tag{1.3}$$

$$|x_{p_n}|^{2+2\alpha} p_n \|u_{p_n}\|_\infty^{p_n-1} \to \infty, \quad dist(x_{p_n}, \partial\Omega)^2 p_n \|u_{p_n}\|_\infty^{p_n-1} \to \infty \tag{1.4}$$

as $n \to \infty$. Moreover, the least energy solution u_p has at most two global maximum points in Ω for large p.

After obtaining these results, Zhao conjectured that u_p has only one global maximum point when p large in Theorem 1.

Main purpose of this note is to answer the conjecture affirmatively.

Theorem 2 *Under the assumption of Theorem 1, the number of global maximum points of least energy solution u_p is exactly 1 for p large enough.*

For the proof, we will use the Morse index characterization of least energy solutions and an argument of [6]. Relations between the number of blowing-up points and the Morse indices of blowing-up solutions to a two-dimensional Liouville equation have been studied in [7], [8].

2. Proof of Theorem 2.

As in §1, let v_p denote a solution of (1.2), which may be chosen positive. Then v_p solves the equation

$$-\Delta v_p = S_p|x|^{2\alpha}v_p^p \quad x \in \Omega, \quad v_p > 0 \quad x \in \Omega, \quad v_p = 0 \quad x \in \partial\Omega.$$

Let $u_p = S_p^{1/(p-1)}v_p$ be a least energy solution to (1.1). First, we recall the well-known fact, which says that the Morse index of u_p is less than or equal to 1 for any $p > 1$.

Lemma 3 *Let $L_p = -\Delta_x - p|x|^{2\alpha}u_p^{p-1}(x) : H_0^1(\Omega) \to H^{-1}(\Omega)$ denote the linearized operator around u_p. Then the second eigenvalue of L_p, denoted by $\lambda_2(L_p, \Omega)$, is nonnegative.*

Proof. When $\alpha = 0$, a proof of this lemma is shown, for example, in [2]. Proof in the case of $\alpha > 0$ is similar. Here we recall it for the sake of completeness.

Let $\overline{L}_p = -\Delta_x - pS_p|x|^{2\alpha}v_p^{p-1}(x) : H_0^1(\Omega) \to H^{-1}(\Omega)$ denote the linearized operator around v_p. Since $S_pv_p^{p-1} = u_p^{p-1}$, it is enough to show that the second eigenvalue of $\overline{L}_p$, denoted by $\lambda_2(\overline{L}_p, \Omega)$, is nonnegative. For this purpose, let us define

$$f(t) = \frac{\int_\Omega |\nabla(v_p + t\varphi)|^2 dx}{\left(\int_\Omega |x|^{2\alpha}|v_p + t\varphi|^{p+1}dx\right)^{\frac{2}{p+1}}}$$

for any $\varphi \in H_0^1(\Omega)$. By the minimality of v_p, we have $f'(0) = 0$ and $f''(0) \geq 0$. Calculation using $\int_\Omega |x|^{2\alpha}v_p^{p+1}dx = 1$ and $\int_\Omega |\nabla v_p|^2 dx = S_p$ shows that

$$\begin{aligned} f''(0) &= 2\left\{\int_\Omega |\nabla\varphi|^2 dx - pS_p\int_\Omega |x|^{2\alpha}v_p^{p-1}\varphi^2 dx + (p-1)S_p\left(\int_\Omega |x|^{2\alpha}v_p^p\varphi dx\right)^2\right\} \\ &= 2(\overline{L}_p\varphi, \varphi)_{L^2(\Omega)} + 2(p-1)S_p\left(\int_\Omega |x|^{2\alpha}v_p^p\varphi dx\right)^2. \end{aligned}$$

Combining this to a variational characterization of $\lambda_2(\overline{L}_p,\Omega)$, we have

$$\lambda_2(\overline{L}_p,\Omega) = \sup_{L\subset H_0^1(\Omega),\, codim L=1} \inf_{\varphi\in L} \frac{(\overline{L}_p\varphi,\varphi)_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\Omega)}}$$
$$\geq \inf_{\varphi\in H_0^1(\Omega),\, \varphi\perp|x|^{2\alpha}v_p^p} \frac{(\overline{L}_p\varphi,\varphi)_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\Omega)}} = \inf_{\varphi\in H_0^1(\Omega),\, \varphi\perp|x|^{2\alpha}v_p^p} \frac{1}{2}\frac{f''(0)}{\|\varphi\|^2_{L^2(\Omega)}} \geq 0.$$

□

By using this fact, we prove Theorem 2 by a contradiction argument.

Proof of Theorem 2. Assume the contrary that there exist two global maximum points $x^1_{p_n}, x^2_{p_n}$ of u_{p_n}, $x^i_{p_n}\in\Omega$, $\|u_{p_n}\|_\infty = u_{p_n}(x^i_{p_n}), (i=1,2)$ for some sequence $p_n\to\infty$. Define $\varepsilon^i_{p_n}>0$ by the relation

$$(\varepsilon^i_{p_n})^2|x^i_{p_n}|^{2\alpha}p_n\|u_{p_n}\|_\infty^{p_n-1} = 1, \tag{2.1}$$

and the scaled functions

$$\tilde{u}^i_{p_n}(y) = \frac{p_n}{\|u_{p_n}\|_\infty}\left\{u_{p_n}\left(\varepsilon^i_{p_n}y + x^i_p\right) - u_{p_n}(x^i_{p_n})\right\}, \tag{2.2}$$
$$y\in\Omega^i_{p_n} := \frac{\Omega - x^i_{p_n}}{\varepsilon^i_{p_n}}$$

for $i=1,2$. Now we assume $\alpha > e-1$, so all results of Theorem 1 hold true for $\tilde{u}^i_{p_n}, (i=1,2)$. In particular, there exists a subsequence (denoted by the same symbol again) such that (1.3) holds for both $\tilde{u}^i_{p_n}$.

Next, we define elliptic operators

$$\tilde{L}^i_{p_n} = -\Delta_y - \left|\frac{\varepsilon^i_{p_n}}{|x^i_{p_n}|}y + \frac{x^i_{p_n}}{|x^i_{p_n}|}\right|^{2\alpha}\left(1+\frac{\tilde{u}^i_{p_n}}{p_n}(y)\right)^{p_n-1}, \quad (i=1,2) \tag{2.3}$$

acting on $H^1_0(\Omega^i_{p_n})$. These operators are related to the operator L_{p_n} by the formula

$$(\varepsilon^i_{p_n})^2 L_{p_n}\Big|_{x=\varepsilon^i_{p_n}y+x^i_{p_n}} = \tilde{L}^i_{p_n}, \quad y\in\Omega^i_{p_n},$$

for $i=1,2$. Also, eigenvalues are related with each other by the formula

$$(\varepsilon^i_{p_n})^2\lambda_j(L_{p_n},D) = \lambda_j(\tilde{L}^i_{p_n},D^i_{p_n}), \quad D^i_{p_n} = \frac{D-x^i_{p_n}}{\varepsilon^i_{p_n}}, \tag{2.4}$$

where $\lambda_j(L_{p_n}, D)$ will denote a j-th eigenvalue of the operator L_{p_n} acting on $H_0^1(D)$ for a domain D, etc.

Let $B(a,R) = B_R(a)$ denote an open ball of center $a \in \mathbb{R}^2$ with radius R. We prove the following:

Lemma 4 *There exist disjoint balls B^i $(i = 1, 2)$, each ball is of the form $B(x^i_{p_n}, \varepsilon^i_{p_n} R)$ for some $R > 0$, such that $\lambda_1(L_{p_n}, B^i) < 0$ for $i = 1, 2$ when n sufficiently large.*

Proof. For $R > 0$, we define

$$w_R(y) = 2 \log \frac{8 + R^2}{8 + |y|^2}.$$

Since $w_R = 0$ on $\partial B_R(0)$, we see $w_R \in H_0^1(B_R(0))$.

We will prove that $(\tilde{L}^i_{p_n} w_R, w_R)_{L^2(B_R(0))} < 0$ for $R > 0$ sufficiently large and $B_R(0) \subset \Omega^i_{p_n}$. Indeed,

$$\begin{aligned}
(\tilde{L}^i_{p_n} w_R, w_R)_{L^2(B_R(0))} &= \int_{B_R(0)} |\nabla w_R|^2 dy \\
&\quad - \int_{B_R(0)} \left| \frac{\varepsilon^i_{p_n}}{|x^i_{p_n}|} y + \frac{x^i_{p_n}}{|x^i_{p_n}|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}^i_{p_n}}{p_n}(y) \right)^{p_n - 1} w_R^2(y) dy \\
&=: I_1 - I_2.
\end{aligned}$$

We see

$$I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8 + |y|^2)^2} dy = 2\pi \int_0^R \frac{16 r^2}{(8 + r^2)^2} r dr = 32\pi \left[\log R + o_R(1) \right],$$

where $o_R(1) \to 0$ as $R \to \infty$. As for I_2, (1.4) implies $\frac{\varepsilon^i_{p_n}}{|x^i_{p_n}|} \to 0$ as $n \to \infty$ even if $x^i_{p_n} \to 0$. Also $\frac{x^i_{p_n}}{|x^i_{p_n}|} \to \exists y_0$, $|y_0| = 1$ for a subsequence. Therefore by choosing a subsequence, we have

$$\left| \frac{\varepsilon^i_{p_n}}{|x^i_{p_n}|} y + \frac{x^i_{p_n}}{|x^i_{p_n}|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}^i_{p_n}}{p_n}(y) \right)^{p_n - 1} \to e^{U(y)}$$

in $C^2_{loc}(\mathbb{R}^2)$ by (1.3). Thus,

$$
\begin{aligned}
I_2 &= \int_{B_R(0)} \left| \frac{\varepsilon^i_{p_n}}{|x^i_{p_n}|} y + \frac{x^i_{p_n}}{|x^i_{p_n}|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}^i_{p_n}}{p_n}(y) \right)^{p_n - 1} w_R^2(y) dy \\
&= \int_{B_R(0)} \frac{1}{\left(1 + \frac{|y|^2}{8}\right)^2} \left\{ \log \frac{8 + R^2}{8 + |y|^2} \right\}^2 dy + o_n(1) \\
&= 2\pi \int_0^R \frac{r}{\left(1 + \frac{r^2}{8}\right)^2} \left\{ \log(8 + R^2) - \log(8 + r^2) \right\}^2 dr + o_n(1) \\
&= 2\pi \cdot 8^2 \left\{ \log(8 + R^2) \right\}^2 \left[\frac{1}{16} + o_R(1) \right] + o_n(1) \\
&= 32\pi \left(\log R \right)^2 \left[1 + o_R(1) \right] + o_n(1),
\end{aligned}
$$

where we have used $\int_0^\infty \frac{r}{(8+r^2)^2} dr = \frac{1}{16}$. Hence we obtain

$$
(\tilde{L}^i_{p_n} w_R, w_R)_{L^2(B_R(0))} = I_1 - I_2 = -32\pi \left(\log R \right)^2 \left[1 + o_R(1) \right] < 0
$$

by taking n sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset \Omega^i_{p_n}$. This implies that the first eigenvalue of the operator $\tilde{L}^i_{p_n}$ on B_R is negative: $\lambda_1(\tilde{L}^i_{p_n}, B_R) < 0$. By this and the scaling formula (2.4) proves the first half part of the Lemma.

Recall that, under the assumption considered here, the following estimate is proved in [9] Lemma 5.1:

$$
\frac{|x^1_p - x^2_p|}{\max\{\varepsilon^1_p, \varepsilon^2_p\}} \to \infty
$$

as $p \to \infty$. This implies that these two balls $B(x^i_{p_n}, \varepsilon^i_{p_n} R)$ are disjoint for n sufficiently large. □

By Lemma 4, we have

$$
\lambda_1(L_{p_n}, B^i) < 0 \quad i = 1, 2 \tag{2.5}
$$

for n sufficiently large. On the other hand, a well known estimate of $\lambda_2(L_{p_n}, \Omega)$ claims

$$
\lambda_2(L_{p_n}, \Omega) \le \sum_{i=1}^2 \lambda_1(L_{p_n}, B^i). \tag{2.6}
$$

See, for example, [7] Appendix. From (2.5) and (2.6), we have $\lambda_2(L_{p_n}, \Omega) < 0$ for n large, which contradicts to Lemma 3. $\square$

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