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Certain Lagrangian Submanifolds in Hermitian Symmetric Spaces and Hamiltonian Stability Problems

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Abstract. In this article we shall provide an exposition on our recent works and related topics in geometry of compact homogeneous Lagrangian submanifolds in Hermitian symmetric spaces such as complex Euclidean spaces, complex projective spaces, complex hyperquadrics, and so on.

1 Introduction

Submanifold geometry is one of major subjects in classical and modern differential geometry. In submanifold geometry it is of especial interest to study various types of submanifolds in symmetric spaces and related geometric variational problems such as *minimal submanifold theory* and *harmonic map theory.*

In this article we shall focus on the Lie theoretic approach to *Lagrangian submanifolds* in Kähler manifolds and related Hamiltonian variational problem. The purpose of this article is to provide an exposition on recent results and related research problems in geometry of compact homogeneous Lagrangian submanifolds in Hermitian symmetric spaces such as complex Euclidean spaces, complex projective spaces, complex hyperquadrics, and so on.

About 1990's ([36], [37]) Y.-G. Oh first considered the volume minimizing property of Lagrangian submanifolds in a Kähler manifold under Hamiltonian deformations. He introduced and investigated the notions of *Hamiltonian minimality* and

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Hamiltonian stability for Lagrangian submanifolds.

A Lagrangian submanifold is called *homogeneous* if it is obtained as an orbit under the Hamiltonian group action of an analytic subgroup of the automorphism group of a Kähler manifold. Since we know that any compact homogeneous Lagrangian submanifold in a Kähler manifold is always Hamiltonian minimal, such Lagrangian submanifolds in Kähler manifolds provide many good examples of Hamiltonian minimal Lagrangian submanifold. *Theoretically* it is possible to analyze the second variations of compact homogeneous Lagrangian submanifolds by *Harmonic Analysis over Compact Homogeneous Spaces* in order to determine their Hamiltonian stability. It is a quite natural and interesting problem to classify compact homogeneous Lagrangian submanifolds in specific Kähler manifolds and to determine their Hamiltonian stability.

Problem. Classify compact homogeneous Lagrangian submanifolds in specific Kähler manifolds such as Hermitian symmetric spaces, generalized flag manifolds with invariant Kähler metrics, toric manifolds and so on.

Problem. Determine the Hamiltonian stability of compact homogeneous Lagrangian submanifolds in such Kähler manifolds.

Compact homogeneous Lagrangian submanifolds with parallel second fundamental form in complex space forms $\mathbb{C}^n$, $\mathbb{C}P^n$, $\mathbb{C}H^n$ have already been classified completely by H. Naitoh and M. Takeuchi ([33], [34], [35]) and from their classification they all can be constructed by using the standard embeddings of symmetric R-spaces (of type $U(r)$). There exists compact homogeneous Lagrangian submanifolds with non-parallel second fundamental form in complex projective spaces, more generally in complex space forms. The classification problem of compact homogeneous Lagrangian submanifolds in complex projective spaces can be converted to a problem on *prehomogeneous vector spaces*. L. Bedulli and A. Gori ([8]) classified compact homogeneous Lagrangian submanifolds obtained as orbits of simple compact Lie groups, by using the classification theory of prehomogeneous vector spaces due to Mikio Sato and Tatsuo Kimura ([45]). The classification of Bedulli and Gori includes so many new compact homogeneous Lagrangian submanifolds with non-parallel second fundamental form in complex projective spaces. We shall explain what are known about the Hamiltonian stability results for those compact homogeneous Lagrangian submanifolds in complex space forms.

Real forms of compact Hermitian symmetric spaces form a nice class of Lagrangian submanifolds in Hermitian symmetric spaces of rank greater than 1, which are nothing but totally geodesic Lagrangian submanifolds and are canonically embedded symmetric R-spaces ([48]). The stability of each real form as minimal submanifolds has been determined previously by M. Takeuchi ([48]). By using his results we can determine the Hamiltonian stability of each real form and we observe three types of examples of *Hamiltonian unstable* real forms in some Hermitian symmetric spaces of rank greater than 1. We shall also mention recent progress in research by the Japanese differential geometry group on real forms of compact Hermitian symmetric spaces as Lagrangian submanifolds. From the classification results of *extrincic* symmetric submanifolds in Riemannian symmetric spaces (J. Berndt, J. Eschenburg, H. Naitoh and T. Tsukada [9]) we know that all Lagrangian submanifolds in Hermitian symmetric spaces of rank greater than 1 with (non-totally geodesic) parallel second fundamental form are explicitly expressed as *canoincal* Lagrangian deformation of some real forms.

Therefore it is an interesting and important problem to construct and classify compact homogeneous Lagrangian submanifolds in Hermitian symmetric spaces which do not necessarily have parallel second fundamental form.

The complex hyperquadric is one of compact rank 2 Hermitian symmetric spaces and it can be canonically isometric to the real Grassmann manifold of oriented 2-dimensional vector subspaces. The *Gauss map* of each oriented hypersurface in the unit standard sphere provides a Lagrangian immersion into the complex hyperquadric. We observe that the Gauss map of oriented hypersurface with constant principal curvatures in the unit standard sphere, so called *isoparametric hypersurface*, is a minimal Lagrangian immersion and the image of the Gauss map of compact isoparametric hypersurface in the unit standard sphere is a compact *smooth* minimal Lagrangian submanifold *embedded* in the complex hyperquadric. The structure and classification theory of isoparametric hypersurfaces in the standard sphere are well-developed since Elie Cartan's work. The Gauss images of compact isoparametric hypersurfaces in the unit standard sphere form a nice class of compact minimal Lagrangian submanifolds embedded in the complex hyperquadric. We can observe that the homogeneity of original isoparametric hypersurfaces in the standard sphere is equivalent to the homogeneity of its Gauss image in the complex hyperquadric.

All homogeneous isoparametric hypersurfaces are known to be obtained as principal orbits of the isotropy representation of Riemannian symmetric spaces of rank 2 (W.-Y. Hsiang and H. B. Lawson, Jr. [19], R. Takagi and T. Takahashi [46]). Non-homogeneous isoparametric hypersurfaces in the standard sphere were discovered first by H. Ozeki and M. Takeuchi ([41]) and generalized by D. Ferus, H. Karcher and H. F. Münzner ([14]). They can be constructed by the representations of Clifford algebras (isoparametric hypersurfaces of *OT-FKM type*).

We shall discuss the properties of compact minimal Lagrangian submanifolds embedded in the complex hyperquadric obtained as the Guass images of compact isoparametric hypersurfaces in the unit standard sphere. Especially we classified all compact homogeneous Lagrangian submanifolds in the complex hyperquadric and determined the Hamiltonian stability of the Guass images of compact homogeneous isoparametric hypersurfaces in the unit standard sphere ([24], [26]).

There are several related questions and further problems to be studied in future.

This article is organized as follows: In Section 2 we shall recall the fundamental definitions and properties on Lagrangian immersions, Hamiltonian deformations and moment maps. In Section 3 we shall treat Lagrangian submanifolds in Kähler manifolds and explain the notions of the mean curvature form, Hamiltonian minimality, Hamiltonian stability, strictly Hamiltonian stability and globally Hamiltonian stability. In Section 4 we shall discuss fundamental examples of compact homogeneous Lagrangian submanifolds in complex Euclidean spaces, complex projective spaces, complex space forms and compact Hermitian symmetric spaces, and their Hamiltonian stability. In Section 5 we shall explain the Gauss map construction of compact minimal Lagrangian submanifolds in complex hyperquadrics from isoparametric hypersurfaces in the unit standard sphere. In Section 6 we shall describe our classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics. In Section 7 we shall describe our Hamiltonian stability results of all compact homogeneous minimal Lagrangian submanifolds embedded in complex hyperquadrics, which are obtained as the Gauss images of compact *homogeneous* isoparametric hyersurfaces in the unit standard sphere. In Section 8 we shall mention related questions and further problems to be studied.

2 Lagrangian submanifolds, Hamiltonian deformations and moment maps.

Let (M, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω. A *Lagrangian immersion* $\varphi : L \longrightarrow M$ is defined as a smooth immersion of an n-dimensional smooth manifold L into M satisfying the condition $\varphi^*\omega = 0$.

The most elementary example of a symplectic manifold is a plane $M = \mathbb{R}^2$ equipped with the standard area from $\omega = dx \wedge dy$. In this case a 1-dimensional Lagrangian submanifold is a plane curve and a 1-dimensional compact Lagrangian submanifold is nothing but a closed plane curve.

The normal bundle $\varphi^{-1}TM/\varphi_*TL$ of a Lagrangian immersion $\varphi : L \longrightarrow M$ can be identified with the cotangent bundle T^*L of L :

$$\varphi^{-1}TM/\varphi_*TL \ni v \longmapsto \alpha_v := \omega(v, \cdot) \in T^*L$$

A *Lagrangian deformation* is defined as a smooth one-parameter family of Lagrangian immersions $\varphi_t : L \longrightarrow M$ with $\varphi = \varphi_0$. Let $\alpha_{V_t} := \omega(V_t, d\varphi(\cdot))$ be 1-forms on L corresponding to its variational vector field $V_t := \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi^{-1}TM)$. The Lagrangian deformation is characterized by the closedness of α_{V_t}, for each t. Furthermore, if α_{V_t} is exact for each t, then $\{\varphi_t\}$ is called a *Hamiltonian deformation* of $\varphi = \varphi_0$.

Suppose that a connected Lie group K has the *Hamiltonian group action* on (M, ω) with the *moment map* $\mu_K : M \longrightarrow \mathfrak{k}^*$. If a Lagrangian submanifold L in M is preserved by the group action of K, then

$$L \subset \mu_K^{-1}(\alpha) \quad \text{for } \exists\, \alpha \in \mathfrak{z}(\mathfrak{k}^*),$$

where

$$\mathfrak{z}(\mathfrak{k}^*) := \{\alpha \in \mathfrak{k}^* \mid \mathrm{Ad}^*(a)\alpha = \alpha \text{ for all } a \in K\}.$$

Assume that M and K are compact. Then an orbit $L = K \cdot x$ of K is Lagrangian if and only if

$$L = K \cdot x = \mu_K^{-1}(\alpha) \quad \text{for } \exists\, \alpha \in \mathfrak{z}(\mathfrak{k}^*) \cong \mathfrak{c}(\mathfrak{k}),$$

where $\mathfrak{c}(\mathfrak{k})$ denotes the center of $\mathfrak{k}$.

3 Lagrangian submanifolds in Kähler manifolds and Hamiltonian stability

Assume that (M, ω, J, g) is a Kähler manifold with a complex structure J and a Kähler metric g. Here ω is the Kähler form defined by $\omega(X, Y) = g(JX, Y)$ for each $X, Y \in TM$.

Let $\varphi : L \to M$ be a Lagrangian immersion into a Kähler manifold. Let

$$S(X, Y, Z) := \omega(B(X, Y), Z) = g(JB(X, Y), Z) \quad (X, Y, Z \in TL)$$

be a symmetric 3-tensor field on L defined by the second fundamental form B of L in M. Let

$$H = \operatorname{tr} B = \sum_{i=1}^{n} B(e_i, e_i)$$

denote the mean curvature vector field of φ, where $\{e_i\}$ is an orthonormal basis of $T_x L$. The corresponding 1-form $\alpha_H \in \Omega^1(L)$ is called the *mean curvature form* of φ. It is well-known that the mean curvature form α_H satisfies the following identity

$$d\alpha_H = \varphi^* \rho_M,$$

where ρ_M denotes the Ricci form of M. In particular, if M is Einstein-Kähler, then α_H is closed.

In Riemannian geometry a submanifold vanishing the mean curvature vector field, i.e. $H = 0$, is called a *minimal* submanifold. A 1-dimensional minimal submanifold is nothing but a geodesic. It is well-known that there is no compact minimal submanifold in a Euclidean space, more generally in a simply connected complete Riemannian manifold of nonnegative sectional curvatures (cf. [22, Vol. II, p.379, Note. 27]).

Let $\operatorname{Aut}(M, \omega, J, g)$ be the automorphism group of the Kähler structure of (M, ω, J, g). If a Lagrangian submanifold L embedded in M is obtained as an orbit of a connected Lie subgroup K of $\operatorname{Aut}(M, \omega, J, g)$, then we call L a *homogeneous Lagrangian submanifold* of a Kähler manifold M. Let $\mathfrak{k}$ denote the Lie algebra of K. Then it is elementary that the mean curvature form α_H of a Lagrangian orbit $L = K \cdot p$ of K in M is given by the formula

$$\alpha_H(\tilde{X}) = -\frac{1}{2} \operatorname{div}(J\tilde{X})|_L \tag{3.1}$$

for each $X \in \mathfrak{k}$.

Assume that (M, ω, g, J) is an Einstein-Kähler manifold with nonzero Einstein constant $\kappa \neq 0$. It is known that the map $\tilde{\mu} : M \to \mathfrak{k}^*$ defined by

$$\langle \tilde{\mu}, X \rangle : M \ni x \longmapsto \frac{1}{2\kappa} (\operatorname{div} JX)(x) \in \mathbf{R} \quad (X \in \mathfrak{k}) \tag{3.2}$$

is a moment map of the group action of K with respect to ω. We call $\tilde{\mu}$ the *canonical moment map* for the group action of a Lie subgroup K. Therefore we know that

Lemma 3.1. *the mean curvature form α_H of a Lagrangian orbit of K in M is expressed as*

$$\alpha_H(\tilde{X}) = -\kappa \langle \tilde{\mu}, X\rangle|_L \quad (X \in \mathfrak{k}). \tag{3.3}$$

In particular $L = K \cdot p$ is a minimal Lagrangian orbit if and only if $L = K \cdot p \subset \tilde{\mu}^{-1}(0)$.

The notion of Hamiltonian minimality and Hamiltonian stability was introduced and discussed first by Y. G. Oh ([36]).

For the simplicity, suppose that L is compact without boundary.

Definition. A Lagrangian immersion φ is called *Hamiltonian minimal* (shortly, *H-minimal*) or *Hamiltonian stationary* if under every Hamiltonian deformation $\{\varphi_t\}$ the first variation of the volume vanishes, that is

$$\frac{d}{dt}\mathrm{Vol}(L, \varphi^* g)|_{t=0} = 0.$$

The *H-minimal Lagrangian submanifold equation* is

$$\delta\alpha_H = 0,$$

where δ denotes the codifferential operator relative to the induced metric $\varphi^* g$ on L.

Definition. An H-minimal Lagrangian immersion φ is called *Hamiltonian stable* (shortly, *H-stable*) if under every Hamiltonian deformation $\{\varphi_t\}$ the second variation of the volume is nonnegative, that is

$$\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi^* g)|_{t=0} \geq 0.$$

The second variational formula was given as follows ([37]):

$$\begin{aligned} &\frac{d^2}{dt^2}\mathrm{V}ol(L, \varphi^* g)|_{t=0} \\ &= \int_L \left(\langle \Delta^1_L \alpha, \alpha\rangle - \langle \bar{R}\alpha, \alpha\rangle + 2\langle \alpha \otimes \alpha \otimes \alpha_H, S\alpha\rangle + \langle \alpha_H, \alpha\rangle^2\right) dv, \end{aligned}$$

where we set $\alpha = \alpha_{V_0} \in B_1(L)$. Here

$$\langle \bar{R}\alpha, \alpha\rangle = \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j).$$

Definition. An H-minimal Lagrangian immersion φ is called *strictly Hamiltonian stable* (shortly, *strictly H-stable*) if the following two conditions are satisfied:

(a) φ is Hamiltonian stable.

(b) The null space of the second variation on Hamiltonian deformations coincides with the vector subspace spanned by infinitesimal holomorphic isometries of M (assume that M is simply connected, or more generally $H^1(M, \mathbb{R}) = \{0\}$).

Remark 3.2. If L is strictly Hamiltonian stable, then L has local minimum volume under every Hamiltonian deformation.

Corollary 3.3. *Suppose that L is minimal Lagrangian and M is Einstein-Kähler with Einstein constant κ. Then L is Hamiltonian stable if and only if the first (positive) eigenvalue λ_1 of the Laplacian Δ_L^0 of L acting on on functions satisfies $\lambda_1 \geq \kappa$.*

Proposition 3.4 (cf. [25])**.** *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is always H-minimal.*

Theoretically it is possible to analyze the second variations of compact homogeneous Lagrangian submanifolds by *Harmonic Analysis over Compact Homogeneous Spaces* in order to determine their (strictly) Hamiltonian stability. It is a quite natural and interesting problem to classify compact homogeneous Lagrangian submanifolds in specific Kähler manifolds and to determine their Hamiltonian stability.

4 Examples

Not so many examples of compact Hamiltonian stable Lagrangian submanifolds are known.

4.1 Hamiltonian stable Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$

Example 1. A circle $S^1(r) \subset \mathbb{C} = \mathbb{R}^2$ with radius $r > 0$ on the Euclidean plane is a compact 1-dimensional H-minimal (not minimal!) Lagrangian submanifold in $\mathbb{R}^2$ which is globally strictly H-stable, because of the isoperimetric inequality on $\mathbb{R}^2$.

Example 2. A great or small circle $S^1(r) \subset S^2(1)$ with radius $0 < r \leq 1$ on the 2-dimensional standard unit sphere is a compact 1-dimensional H-minimal (minimal if and only if $r = 1$) Lagrangian submanifold in $S^2(1)$, which is globally strictly H-stable, because of the isoperimetric inequality on $S^2(1)$.

Example 3. The product of two circles in Example 1 $S^1(r_1) \times S^1(r_2) \subset \mathbb{C}^2 = \mathbb{R}^2$ is also a compact 2-dimensional H-minimal Lagrangian submanifold in $\mathbb{R}^2$, which is strictly H-stable. Its *global* H-stability is conjectured by Y. G. Oh and it is still open. Example 2 is constructed from this product example as follows: For each $r_1 > 0$, $r_2 > 0$ with $(r_1)^2 + (r_2)^2 = 1$,

$$\begin{tikzcd} S^1(r_1) \times S^1(r_2) \arrow[d, "\pi"', "S^1"] & \subset & S^3(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathbb{C}^2 \\ S^1(r) & \subset & \mathbb{C}P^1 = S^2(\frac{1}{2}) & & \end{tikzcd}$$

Note that $r_1 = r_2$ if and only if $S^1 \subset \mathbb{C}P^1 = S^2(\frac{1}{2})$ is a great circle (i.e. a geodesic).

More generally, the product of $n+1$ circles in Example 1 $S^1(r_1)\times\cdots\times S^1(r_{n+1}) \subset \mathbb{C}^{n+1} = \mathbb{R}^{2(n+1)}$ is also a compact $(n+1)$-dimensional H-minimal (never minimal!) Lagrangian submanifold in $\mathbb{C}^{n+1} = \mathbb{R}^{2(n+1)}$, which is strictly Hamiltonian stable ([37]). The above construction can be generalized as follows:

$$\begin{tikzcd} S^1(r_1) \times \cdots \times S^1(r_{n+1}) \arrow[d, "\pi"', "S^1"] & \subset & S^{2n+1}(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathbb{C}^{n+1} \\ T^n & \subset & \mathbb{C}P^n & & \end{tikzcd}$$

Then $T^n \subset \mathbb{C}P^n$ is a compact H-minimal Lagrangian submanifold which is strictly H-stable ([37]). Note that it is minimal in $\mathbb{C}P^n$ if and only if $r_1 = \cdots = r_{n+1}$.

Example 4. A great circle $S^1(1) \subset S^2(1)$ in Example 2 can be considered as a real projective line $\mathbb{R}P^1 \subset \mathbb{C}P^1$. Generally, the real projective subspaces $\mathbb{R}P^n \subset \mathbb{C}P^n$, which exhaust all totally geodesic (by definition $S = 0$) Lagrangian submanifolds in $\mathbb{C}P^n$, are compact strictly Hamiltonian stable and globally Hamiltonian stable minimal (in this case totally geodesic) Lagrangian submanifold ([36]).

$$\begin{tikzcd} S^1 \cdot S^n \arrow[d, "\pi"', "S^1"] & \subset & S^{2n+1}(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathbb{C}^{n+1} \\ \mathbb{R}P^n & \subset & \mathbb{C}P^n & & \end{tikzcd}$$

Here the submanifold $S^1 \cdot S^n$ is the inverse image of $\mathbb{R}P^n$ by the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ and we have

$$S^1 \cdot S^n = \bigcup_{c \in S^1} c\,\mathbb{R}^{n+1} \cap S^{2n+1}(1) \cong (S^1 \times S^n)/\mathbb{Z}_2,$$

which is isometric to a real hyperquadric

$$Q_{1,n+1}(\mathbb{R}) \cong (SO(2) \times SO(n+1))/S'(O(1) \times O(n)),$$

where

$$S'(O(1) \times O(n)) := \left\{ \left(\begin{pmatrix} \varepsilon & 0 \\ 0 & A \end{pmatrix} \quad 0 \atop 0 \quad \begin{pmatrix} B & 0 \\ 0 & \varepsilon \end{pmatrix} \right) \mid \varepsilon = \pm 1, A \in O(1), B \in O(n) \right\}.$$

Then $S^1 \cdot S^n$ is a compact minimal submanifold in $S^{2n+1}(1)$ and a compact Hamiltonian minimal (never minimal) Lagrangian submanifold in $\mathbb{C}^{n+1}$ which is strictly Hamiltonian stable ([2], [4], [5]).

Example 5. Let $U(p)$ and $SU(p)$ be the unitary group and special unitary group of degree p. Let $\mathrm{M}(p, \mathbb{C})$ denotes the complex vector space of all complex $p \times p$ matrices. Then $U(p)$ and $SU(p)$ are naturally embedded into the vector space $\mathrm{M}(p, \mathbb{C})$. Here set $n = p^2 - 1$.

$$\begin{tikzcd}[column sep=small]
U(p) = S^1 \cdot SU(p) \arrow[d, "\pi"', "S^1"] & \subset & S^{2p^2-1}(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathrm{M}(p,\mathbb{C}) = \mathbb{C}^{p^2} \\
SU(p)/\mathbb{Z}_p & \subset & \mathbb{C}P^{p^2-1} & &
\end{tikzcd}$$

Then the irreducible symmetric space $SU(p)/\mathbb{Z}_p$ is a compact minimal Lagrangian submanifold in $\mathbb{C}P^n$ which is strictly Hamiltonian stable ([3]). $U(p)$ is a compact minimal submanifold in $S^{2n+1}(1)$ and a compact Hamiltonian minimal (never minimal) Lagrangian submanifold in $\mathbb{C}^{n+1}$ which is strictly H-stable ([2], [4], [5]).

Example 6. Let $\mathrm{S}(p,\mathbb{C})$ denotes the complex vector space of all complex symmetric $p \times p$ matrices. The symmetric space $U(p)/O(p)$ is standardly embedded into the vector space $\mathrm{S}(p,\mathbb{C})$ as follows:

$$U(p)/O(p) \ni a\,O(p) \longmapsto {}^t aa \in \mathrm{S}(p,\mathbb{C})$$

Here set $n = p(p+1)/2 - 1$.

$$\begin{tikzcd}[column sep=small]
U(p)/O(p) = S^1 \cdot SU(p)/SO(p) \arrow[d, "\pi"', "S^1"] & \subset & S^{p(p+1)-1}(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathrm{S}(p,\mathbb{C}) = \mathbb{C}^{p(p+1)/2} \\
SU(p)/(SO(p)\mathbb{Z}_p) & \subset & \mathbb{C}P^{(p-1)(p+2)/2} & &
\end{tikzcd}$$

Then the irreducible symmetric space $SU(p)/(SO(p)\mathbb{Z}_p)$ is a compact minimal Lagrangian submanifold in $\mathbb{C}P^n$ which is strictly Hamiltonian stable ([3]). $U(p)/O(p)$ is a compact minimal submanifold in $S^{2n+1}(1)$ and a compact Hamiltonian minimal (never minimal) Lagrangian submanifold in $\mathbb{C}^{n+1}$ which is strictly Hamiltonian stable ([2], [4], [5]).

Example 7. Let $\mathrm{AS}(2p,\mathbb{C})$ denote the complex vector space of all complex skew-symmetric $2p \times 2p$ matrices. The symmetric space $U(2p)/Sp(p)$ is standardly embedded into the vector space $\mathrm{AS}(2p,\mathbb{C})$ as follows:

$$U(2p)/Sp(p) \ni a\,Sp(p) \longmapsto {}^t aJ_p a \in \mathrm{AS}(2p,\mathbb{C}),$$

where $J_p = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$ and I_p is the identity matrix of degree p. Here set $n = p(2p-1) - 1$.

$$\begin{tikzcd} U(2p)/Sp(p) = S^1 \cdot SU(2p)/Sp(p) \arrow[d, "\pi"', "S^1"] & \subset & S^{p(p+1)-1}(1) \arrow[d, "\pi"', "S^1"] & \subset & \mathrm{AS}(2p, \mathbb{C}) = \mathbb{C}^{p(2p-1)} \\ SU(2p)/(Sp(p)\mathbb{Z}_{2p}) & \subset & \mathbb{C}P^{(2p+1)(p-1)} & & \end{tikzcd}$$

Then the irreducible symmetric space $SU(p)/\mathbb{Z}_p$ is a compact minimal Lagrangian submanifold in $\mathbb{C}P^n$ which is strictly H-stable ([3]). $U(2p)/Sp(p)$ is a compact minimal submanifold in $S^{2n+1}(1)$ and a compact H-minimal (never minimal) Lagrangian submanifold in $\mathbb{C}^{n+1}$ which is strictly H-stable ([2], [4], [5]).

Example 8. Let $\mathfrak{C}$ be the Cayley algebra over $\mathbb{R}$ and $\mathfrak{C}^{\mathbb{C}}$ be the complexification of $\mathfrak{C}$. Let $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ be the real vector space of all 3×3 Hermitian matrices over $\mathfrak{C}$ and $\mathfrak{J}^{\mathbb{C}}$ be the complexification of $\mathfrak{J}$. $\mathfrak{J}$ and $\mathfrak{J}^{\mathbb{C}}$ have the structures of the Jordan algebras as t he multiplication $\circ$, the cross product $\times$, the inner product $(\ ,\ )$, the cubic product $(\ ,\ ,\ )$, the determinant det and the Hermitian inner product on $\mathfrak{J}^{\mathbb{C}}$. The simply connected compact Lie groups of exceptional types E_6 and F_4 are defined as

$$\begin{aligned} E_6 &:= \{a \in GL(\mathfrak{C}^{\mathbb{C}}) \mid \det(a(X)) = \det(X), \langle a(X), a(Y) \rangle = \langle X, Y \rangle\}, \\ F_4 &:= \{a \in E_6 \mid \langle a(X), a(Y) \rangle = \langle X, Y \rangle\} = \{a \in E_6 \mid a(I_3) = I_3\}. \end{aligned}$$

Refer Ichiro Yokota's article [52] for an excellent article on exceptional Lie groups. The symmetric space $S^1 \cdot E_6/F_4$ is standardly embedded into the vector space $\mathfrak{J}^{\mathbb{C}}$ as follows:

$$S^1 \cdot E_6/F_4 \ni a\, F_4 \longmapsto a(I_3) \in \mathfrak{J}^{\mathbb{C}}.$$

Here set $n + 1 = 27$.

$$
\begin{array}{ccccc}
S^1 \cdot E_6/F_4 & \subset & S^{53}(1) & \subset & \mathfrak{J}^{\mathbb{C}} = \mathbb{C}^{27} \\
\pi \Big\downarrow S^1 & & \pi \Big\downarrow S^1 & & \\
E_6/(F_4\mathbb{Z}_3) & \subset & \mathbb{C}P^{26} & &
\end{array}
$$

Then the irreducible symmetric space $E_6/(F_4\mathbb{Z}_3)$ is a compact minimal Lagrangian submanifold embedded in $\mathbb{C}P^n$ which is strictly Hamiltonian stable ([3]). $S^1 \cdot E_6/F_4$ is a compact minimal submanifold embedded in $S^{2n+1}(1)$ and a compact Hamiltonian minimal (never minimal) Lagrangian submanifold embedded in $\mathbb{C}^{n+1}$ which is strictly Hamiltonian stable ([2], [4], [5]).

The compact symmetric spaces appearing in the above examples

$$S^1 \cdot S^n,\ U(p),\ U(p)/O(p),\ U(2p)/Sp(p),\ S^1 \cdot E_6/F_4$$

and their Lagrangian embedding in complex Euclidean spaces exhaust so-called *irreducible symmetric R-spaces of type* $U(r)$ and their standard embeddings (see [48]). A symmetric R-space can be characterized as a compact symmetric space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. All Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$ described in Examples 1-8 have parallel second fundamental form $\nabla S = 0$. Lagrangian submanifolds in complex space forms with $\nabla S = 0$ are already completely classified by Hiroo Naitoh and Masaru Takeuchi ([33], [34], [35]). According to the classification theory, all compact Lagrangian submanifolds embedded in simply connected complete complex space forms of constant holomorphic sectional curvature c (complex Euclidean spaces $\mathbb{C}^n$, complex projective spaces $\mathbb{C}P^n$, complex hyperbolic spaces $\mathbb{C}H^n$) are given as follows (cf. [38]):

(1) Suppose that L is a compact Lagrangian submanifold embedded in $\mathbb{C}^n$ with $\nabla S = 0$. Then L is congruent to the standard embedding of a symmetric R-space of type $U(r)$, that is, an irreducible symmetric R-space of type $U(r)$ or a Riemannian product of symmetric R-spaces of type $U(r)$.

(2) Suppose that L is a compact Lagrangian submanifold embedded in $\mathbb{C}P^n$ with $\nabla S = 0$. Then L is locally isometric to a symmetric space $M_0 \times M_1 \times \cdots \times M_r$, where L_0 is of Euclidean type, $\dim M_0 \geq r - 1$ and M_i is one of the following irreducible symmetric spaces of compact type: (a) S^n, (b) $SU(p)$, (c) $SU(p)/SO(p)$, (d) $SU(2p)/Sp(p)$, (e) E_6/F_4. Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration. Then $\pi^{-1}(L)$ is also a Lagrangian submanifold embedded in $\mathbb{C}^{n+1}$ of the case (1).

(3) Suppose that L is a compact Lagrangian submanifold embedded in $\mathbb{C}H^n$ with $\nabla S = 0$. Then L is locally isometric to a symmetric space $M_0 \times M_1 \times$

$\cdots \times M_r$, where L_0 is of Euclidean type, $\dim M_0 \geq r$ and M_i is one of irreducible symmetric spaces of compact type (a)$\sim$(e) in the case (2). Let $\mathbb{C}^{n+1}_1 = \mathbb{C} \times \mathbb{C}^n$ be an $(n+1)$-dimensional complex vector space equipped with an indefinite Hermitian form $F(\mathbf{z}, \mathbf{w}) := -z_0\bar{w}_0 + \sum_{i=1}^n z_i\bar{w}_i$ for each $\mathbf{z} = (z_0, z_1, \cdots, z_n), \mathbf{w} = (w_0, w_1, \cdots, w_n) \in \mathbb{C}^{n+1}_1$. Set $H^{2n+1}(4/c) := \{\mathbf{z} \in \mathbb{C}^{n+1}_1 \mid F(\mathbf{z}, \mathbf{z}) = 4/c\}$ and $\pi : H^{2n+1}(4/c) \to \mathbb{H}P^n$ be the natural Riemannian submersion with fiber S^1. Then $\pi^{-1}(L) = S^1 \times \tilde{M}$. Here $S^1 = \{\mathbf{z} \in \mathbb{C} \mid F(\mathbf{z}, \mathbf{z}) = -r_0^2\}$ for some $r_0 > 0$, $r_1 > 0$ with $-{}_0^2 + r_1^2 = 4/c$ and $\tilde{L}$ is a Lagrangian submanifold embedded in $\mathbb{C}^n$ of the case (1).

Theorem 4.1 ([50], [38])**.** *Let $M = \tilde{M}(c)$ be a complex space form and L be a compact Lagrangian submanifold immersed in M. If L is H-minimal and has nonnegative sectional curvatures, then L has parallel second fundamental form, i.e. $\nabla S = 0$. The converse also holds.*

This statement does not hold in general for compact Lagrangian submanifolds of Hermitian symmetric spaces of rank greater than 1 (See Remark 7.1).

Theorem 4.2 ([51], [10])**.** *If L is a H-stable minimal Lagrangian torus immersed in $\mathbb{C}P^2$, then L is a Clifford torus in $\mathbb{C}P^2$ ($n = 2$, $r_1 = r_2 = r_3$ in Example 3), in particular $\nabla S = 0$.*

Theorem 4.3 ([4],[5])**.** *If L is a compact Lagrangian submanifold embedded in a simply connected complete complex space form (that is, $\mathbb{C}^n$, $\mathbb{C}P^n$, $\mathbb{C}H^n$) with $\nabla S = 0$, then L is H-stable.*

Example 9. Let V_3 be the vector space of complex homogeneous polynomials with two variables z_0, z_1 of degree 3 and ρ_3 be the irreducible unitary representation of $SU(2)$ on V_3. Then

$$\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3$$

is a 3-dimensional compact embedded strictly H-stable minimal Lagrangian submanifold with $\nabla S \neq 0$ (Bedulli and Gori [7], independently [39]).

Moreover Bedulli and Gori [8] provided a nice classification of compact homogeneous Lagrangian submanifolds in $\mathbb{C}P^n$ obtained as Lagrangian orbits of compact simple $K \subset SU(n+1)$ by applying the classification theory of *prehomogeneous vector spaces* (Mikio Sato and Tatsuo Kimura [45]). Such Lagrangian orbits were classified as

$$\begin{aligned} \text{16 types of examples } &= [\text{5 types of examples with} \nabla S = 0] \\ &\quad + [\text{11 types of examples with} \nabla S \neq 0] \end{aligned}$$

A 5-dimensional minimal Lagrangian orbit $(SU(2) \times SU(2))/T^1 \cdot \mathbb{Z}_4 \subset \mathbb{C}P^5$ of non-simple $SU(2) \times SU(2)$ with non-parallel second fundamental form is H-stable (Petrecca and Podesta [43]).

Recently a new and interesting construction of compact *non-homogeneous* H-minimal Lagrangian submanifolds embedded in $\mathbb{C}^n$ and $\mathbb{C}P^n$ is studied by the method of *toric topology* in A. E. Mironov and T. Panov ([30]).

At present I do not know any counter example to the following questions:

Problem. Is it true that a compact Hamiltonian minimal Lagrangian submanifold *embedded* in a complex projective space is Hamiltonian stable? Or is there a compact *Hamiltonian unstable* Hamiltonian minimal Lagrangian submanifold *embedded* in a complex projective space?

Problem. Is it true that a compact Hamiltonian minimal Lagrangian submanifold *embedded* in a complex Euclidean space is Hamiltonian stable? Or is there a compact *Hamiltonian unstable* Hamiltonian minimal Lagrangian submanifold *embedded* in a complex Euclidean space?

The following problem is also interesting but still open:

Problem. Classify all compact homogeneous Lagrangian submanifolds in complex hyperbolic space form $\mathbb{C}H^n$.

4.2 Hamiltonian stability of real forms of Hermitian symmetric spaces

The Hamiltonian stability of each compact Lagrangian submanifold embedded as a real form L in compact irreducible Hermitian symmetric spaces M of higher rank are known as below ([48]). Here each M is equipped with the standard Kähler metric of Einstein constant $\frac{1}{2}$ and λ_1 denotes the first eigenvalue of the Laplacian of L on functions.

Concerned with recent other related works on real forms of compact Hermitian symmetric spaces as Lagrangian submanifolds, Hiroshi Iriyeh, Takashi Sakai, Hiroyuki Tasaki and Makiko Tanaka make progress on the *Tightness* and *Lagrangian* *intersection Floer homology* for real forms for compact Hermitian symmetric spaces ([49], [21]).

From the classification results of *extrincic* symmetric submanifolds in Riemannian symmetric spaces (J. Berndt, J. Eschenburg, H. Naitoh and T. Tsukada [9]) we know that all Lagrangian submanifolds in Hermitian symmetric spaces of rank greater than 1 with (non-totally geodesic) parallel second fundamental form are explicitly expressed as a Lagrangian deformation of a real form which is a canonically embedded symmetric R-spaces of type $U(r)$.

M	L	Einstein	λ_1	H-stable	stable
$G_{p,q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{H})$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbf{C})$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$SO(2m)/U(m)$	$SO(m), m \geq 5$	Yes	$\frac{1}{2}$	Yes	No
$SO(4m)/U(2m), m \geq 3$	$U(2m)/Sp(m)$	No	$\frac{m}{4m-2}$	No	No
$Sp(2m)/U(2m)$	$Sp(m), m \geq 2$	Yes	$\frac{1}{2}$	Yes	Yes
$Sp(m)/U(m)$	$U(m)/O(m)$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbf{C}), q-p \geq 3$	$Q_{p,q}(\mathbf{R}),\ p \geq 2$	No	$\frac{p}{p+q-2}$	No	No
$Q_{p+q-2}(\mathbf{C}), 0 \leq q-p < 3$	$Q_{p,q}(\mathbf{R}), p \geq 2$	No	$\frac{1}{2}$	Yes	No
$Q_{q-1}(\mathbf{C}),\ q \geq 3$	$Q_{1,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/TSpin(10)$	$P_2(\mathbf{K})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/TSpin(10)$	$G_{2,2}(\mathbf{H})/\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
E_7/TE_6	$SU(8)/Sp(4)\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$\frac{1}{6}$	No	No

Podestá [44] classified compact homogeneous Lagrangian submanifolds L of positive Euler characteristic in complex Grassmann manifolds $\mathrm{Gr}_k(\mathbb{C})$ and he showed that they are totally geodesic Lagrangian submanifolds $L = \mathrm{Gr}_p(\mathbb{R}^n) \subset \mathrm{Gr}_p(\mathbb{C}^n)$ with $p(n-p)$ even, $L = \mathrm{Gr}_p(\mathbb{H}^m) \subset \mathrm{Gr}_{2p}(\mathbb{C}^{2m})$, $L = \mathrm{Gr}_2(\mathbb{R}^8) \cong SO(7)/U(3)\cdot\mathbb{Z}_2 \subset \mathrm{Gr}_2(\mathbb{C}^8)$ and $L = \mathrm{Gr}_2(\mathbb{R}^7) \cong G_2/U(2)\cdot\mathbb{Z}_2 \subset \mathrm{Gr}_2(\mathbb{C}^7)$.

5 Lagrangian submanifolds in complex hyperquadrics and hypersurface geometry in $S^{n+1}(1)$

Next we shall discuss Lagrangian submanifolds in complex hyperquadrics $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C}) \cong SO(n+2)/SO(2) \times SO(n)$, which is a compact irreducible

Hermitian symmetric space of rank 2. Here $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^{n+2})$ denotes the real Grassmann manifold of oriented 2-planes in $\mathbf{R}^{n+2}$ and $Q_n(\mathbf{C})$ a complex hypersurface of $\mathbf{C}P^{n+1}$ defined by $z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$.

Let $N^n \subset S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface immersed in the unit sphere. Now we denote by $\mathbf{x}$ its position vector of points p of N^n and by $\mathbf{n}$ the unit normal vector field to N^n in $S^{n+1}(1)$. Then the *Gauss map* is defined as

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C}).$$

Here $[\mathbf{x}(p) \wedge \mathbf{n}(p)]$ denotes an oriented 2-plane in $\mathbf{R}^{n+2}$ spanned by two vectors $\mathbf{x}(p)$ and $\mathbf{n}(p)$. Then $\mathcal{G}$ is always a Lagrangian immersion.

The mean curvature form formula was shown by [42] as follows:

$$\alpha_H = -d\left(\sum_{i=1}^{n} \operatorname{arc\,cot}\kappa_i\right) = d\left(\operatorname{Im}\left(\log \prod_{i=1}^{n}(1 + \sqrt{-1}\kappa_i)\right)\right),$$

where κ_i $(i = 1, \cdots, n)$ denotes the principal curvatures of $N^n \subset S^{n+1}(1)$.

Now suppose that $N^n \subset S^{n+1}(1)$ is a compact oriented hypersurface with constant principal curvatures, so called *isoparametric hypersurface*. Then by the above mean curvature form formula the Gauss map $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ is a minimal Lagrangian immersion. By Münzner's famous result ([31], [32]), the number g of distinct principal curvatures must be $g = 1, 2, 3, 4, 6$. Then the image of the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is a compact *embedded* minimal Lagrangian submanifold

$$L = \mathcal{G}(N^n)(\cong N^n/\mathbf{Z}_g) \subset Q_n(\mathbf{C}).$$

See [40] for the details. By Münzner's results ([31]) we know that $2n/g = m_1 + m_2$ if g is even and $2n/g = 2m_1$ if g is odd.

Note that $g = 1$ or 2 if and only if $\mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold in $Q_n(\mathbf{C})$.

Theorem 5.1 ([25], [40])**.** *$L = \mathcal{G}(N^n)$ is a monotone and cyclic Lagrangian submaifold in $Q_n(\mathbf{C})$ with minimal Maslov number $2n/g$. Moreover, $L = \mathcal{G}(N^n)$ is orientable (resp. non-orientable) if and only if $2n/g$ is an even (resp. odd) integer.*

Recently H. Li, H. Ma and G. Wei constructed a class of non-isoparametric and non-minimal compact oriented rotational hypersurfaces whose Gauss maps are minimal Lagrangian immersions into complex hyperquadrics for $n \geq 3$ ([23])

Classification Theory of isoparametric hypersurfaces of the standard sphere: All isoparametric hypersurfaces in $S^{n+1}(1)$ are classified into

- *Homogeneous* ones (Hsiang-Lawson [19], R. Takagi-T. Takahashi [46]) can be obtained as principal orbits of the isotropy representations of Riemannian symmetric pairs (U, K) of rank 2.

- $g = 1 : N^n = S^n$, a great or small sphere.
- $g = 2, N^n = S^{m_1} \times S^{m_2}$, $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$, the Clifford hypersurfaces.
- $g = 3$, N^n is homogeneous, $N^n = \frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}, \frac{SU(3)}{T^2}, \frac{Sp(3)}{Sp(1)^3}, \frac{F_4}{Spin(8)}$ (E. Cartan).
- $g = 6$: Only homogeneous examples are known now.
    - $g = 6, m_1 = m_2 = 1$: homogeneous (Dorfmeister-Neher, R. Miyaoka [27]).
    - $g = 6, m_1 = m_2 = 2$: homogeneous (R. Miyaoka [28]).

- *Non-homogenous* isoparametric hypersurfaces in the standard sphere were discovered first by H. Ozeki and M. Takeuchi ([41]) and generalized by D. Ferus, H. Karcher and H. F. Münzner ([14]). They can be constructed by the representations of Clifford algebras (*isoparametric hypersurfaces of OT-FKM type*).
    - $g = 4$: except for $(m_1, m_2) = (7, 8)$, either homogeneous or OT-FKM type (Cecil-Chi-Jensen [11], Immervoll [20], Chi [12, 13]).

6 Classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics

In [24] we classified compact homogeneous Lagrangian submanifolds in complex hyperquadrics. Here we briefly explain our classification theory.

First we observed that $N^n \subset S^{n+1}(1)$ is *homogeneous* (i.e. an orbit of a compact Lie subgroup $K \subset SO(n+2)$) if and only if $\mathcal{G}(N^n)$ is *homogeneous* in $Q_n(\mathbb{C})$ ([24, p.759, Proposition 3.1]). By Hsiang-Lawson [19] and Takagi-Takahashi [46], we know that all homogeneous isoparametric hypersurfaces $N^n \subset S^{n+1}(1)$ can be obtained as principal orbits of isotropy representation of Riemannian symmetric pairs (U, K) of rank 2. We should notice that the group action by K of (U, K) is the *maximal* group action of cohomogeneity 1 on the standard sphere ([19]).

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition as a symmetric Lie algebra and $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. For each regular element H of $\mathfrak{a} \cap S^{n+1}(1)$, we have a homogeneous isoparametric hypersurface in the unit sphere $N^n := (\mathrm{Ad}K)H \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} \cong \mathfrak{p}$. Its Gauss image is $\mathcal{G}(N^n) = (\mathrm{Ad}K)[\mathfrak{a}] \subset \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbb{C})$. Then the *canonical* moment map $\tilde{\mu}$ of the action of K on $Q_n(\mathbb{C})$ induced by the adjoint action of K on $\mathfrak{p}$ is given as follows :

$$\tilde{\mu} : Q_n(\mathbb{C}) \cong \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \ni [\mathbf{a} + \sqrt{-1}\mathbf{b}] = [W] \longmapsto -[\mathbf{a}, \mathbf{b}] \in \mathfrak{k} \cong \mathfrak{k}^*$$

where $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of $W \subset \mathfrak{p}$ compatible with its orientation. Hence we obtain

$$\mathcal{G}(N^n) = \tilde{\mu}^{-1}(0) .$$

g	Type	(U,K)	$\dim N^n$	m_1, m_2	$N^n = K/K_0$
1	$S^1\times$ BDII	$(S^1 \times SO(n+2), SO(n+1))$ $n \geq 1$, $[\mathbb{R} \oplus A_1]$	n	n	S^n
2	BDII$\times$ BDII	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $1 \leq p \leq n-1$, $[A_1 \oplus A_1]$	n	$p, n-p$	$S^p \times S^{n-p}$
3	AI_2	$(SU(3), SO(3))\,[A_2]$	3	$1, 1$	$\frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3) \times SU(3), SU(3))\,[A_2]$	6	$2, 2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))\,[A_2]$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	$(E_6, F_4)\,[A_2]$	24	$8, 8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5) \times SO(5), SO(5))\,[B_2]$	8	$2, 2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(2) \times U(m)))$ $m \geq 2$, $[BC_2](m \geq 3), [B_2](m=2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2), SO(2) \times SO(m))$ $m \geq 3$, $[B_2]$	$2m-2$	$1,$ $m-2$	$\frac{SO(2)\times SO(m)}{\mathbb{Z}_2\times SO(m-2)}$
4	CII_2	$(Sp(m+2), Sp(2) \times Sp(m))$ $m \geq 2$, $[BC_2](m \geq 3), [B_2](m=2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$
4	DIII_2	$(SO(10), U(5))\,[BC_2]$	18	$4, 5$	$\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$
4	EIII	$(E_6, U(1) \cdot Spin(10))\,[BC_2]$	30	$6, 9$	$\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$
6	$\mathfrak{g}_2$	$(G_2 \times G_2, G_2)\,[G_2]$	12	$2, 2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))\,[G_2]$	6	$1, 1$	$\frac{SO(4)}{\mathbb{Z}_2+\mathbb{Z}_2}$

Suppose that L is a compact homogeneous Lagrangian submanifold in $Q_n(\mathbf{C})$, which is obtained as $L = K' \cdot [V_0]$ for a compact connected Lie subgroup G of $SO(n+2)$. Then one can show that there is $v \in S^{n+1}(1)$ such that $N^n = K' \cdot v \subset S^{n+1}(1)$ is a homogeneous isoparametric hypersurface in $S^{n+1}(1)$. By Hsiang-Lawsonś theorem [19] there is a compact Riemannian symmetric pair (U, K) of rank 2 with connected compact K and the symmetric Lie algebra $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ such that $\mathfrak{p} = \mathbb{R}^{n+2}$, $K' \subset \mathrm{Ad}_{\mathfrak{p}}(K)$ and $N^n = \mathrm{Ad}_{\mathfrak{p}}(K)v$. By using the moment map argument and some results from the complete classification of cohomogeneity 1 compact group actions on spheres due to Tohl Asoh [6], we showed that $L = K' \cdot [V_0] = K \cdot [V_0]$. Notice that $K \cdot [V_0] = \tilde{\mu}^{-1}(\eta)$ for some $\eta \in \mathfrak{c}(\mathfrak{k})$, Hence by Lemma 3.1 we obtain

Theorem 6.1 ([24])**.** *Any compact homogeneous minimal Lagrangian submanifold*

L^n in $Q_n(\mathbf{C})$ is the Gauss image $\mathcal{G}(N^n)$ of a compact homogeneous isoparametric hypersurface N^n in $S^{n+1}(1)$.

Moreover, by the Lie algebraic and the moment map arguments, we showed

Lemma 6.2 (([24])). *(1) If (U,K) is (i) $(S^1 \times SO(3), SO(2))$, (ii) $(SO(3) \times SO(3), SO(2) \times SO(2))$, (iii) $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ or (iv) $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2)$, then $L = \mu^{-1}(\xi) \subset Q_n(\mathbf{C})$ for some $\xi \in \mathfrak{c}(\mathfrak{k}) \cap \mathrm{Im}(\mu) \neq \{0\}$. In this case we have non-trivial families of Lagrangian orbits in $Q_n(\mathbf{C})$.*

(2) Otherwise, $\mathfrak{c}(\mathfrak{k}) \cap \mathrm{Im}(\mu) = \{0\}$ and $L = \mathcal{G}(N^n) = \mu^{-1}(0) \subset Q_n(\mathbf{C})$, which is a minimal Lagrangian submanifold in $Q_n(\mathbf{C})$.

The non-trivial families of Lagrangian orbits in $Q_n(\mathbb{C})$ in each case of (1) are explicitly described as follows:

(i) If (U,K) is $(S^1 \times SO(3), SO(2))$, then L is a small or great circle in $Q_1(\mathbb{C}) \cong S^2$.

(ii) If (U,K) is $(SO(3) \times SO(3), SO(2) \times SO(2))$, then L is a product of small or great circles of S^2 in $Q_2(\mathbb{C}) \cong S^2 \times S^2$.

(iii) If (U,K) is $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 3)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic K-orbits satisfying

(a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold in $Q_n(\mathbb{C})$.

(b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,

$$K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbb{Z}_2 \cong Q_{2,n}(\mathbb{R})$$

is an H-minimal Lagrangian submanifold in $Q_n(\mathbb{C})$ with $\nabla S = 0$ and thus $\nabla \alpha_{\mathrm{H}} = 0$.

(c) $K \cdot [W_{\pm\sqrt{-1}}]$ are isotropic submanifolds in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = 0$ (points !).

(iv) If (U,K) is $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic orbits satisfying

(a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a minimal (NOT totally geodesic) Lagrangian submanifold in $Q_n(\mathbb{C})$.

(b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,

$$K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times SO(m-2))$$

is an H-minimal Lagrangian submanifold in $Q_n(\mathbb{C})$ with $\nabla S \neq 0$ and $\nabla \alpha_{\mathrm{H}} = 0$.

(c) $K \cdot [W_{\pm\sqrt{-1}}] \cong SO(m)/S(O(1) \times O(m-1)) \cong \mathbb{R}P^{m-1}$ are isotropic submanifolds in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = m - 1$.

7 Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces.

Suppose that N^n is a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. Palmer ([42]) showed that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$ $(g = 1)$.

Problem. Investigate the Hamiltonian stability of its Gauss image $L = \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g$ embedded in $Q_n(\mathbb{C})$ as a compact minimal Lagrangian submanifold.

In the case $g = 1$, $N^n = S^n$ is a great or small sphere and $\mathcal{G}(N^n) \cong S^n$ is strictly Hamiltonian stable. More strongly, it is stable as a minimal submanifold ([48]). In the case when n is even, it is real homologically volume-minimizing because it is a calibrated submanifold by an invariant n-form (Gluck, Morgan and Ziller [15]). It is impossible to be calibrated in the case when n is odd, because $H^n(Q_n(\mathbb{C}); \mathbb{R}) = \{0\}$. In the case when n is even, $H^n(Q_n(\mathbb{C}), \mathbb{R}) \cong \mathbb{R}$. In the case when n is odd, $\pi_n(Q_n(\mathbb{C})) \cong \mathbb{Z}_2$, and $H_n(Q_n(\mathbb{C}), F) \cong \{0\}$ for any F. The recent result of [21] implies that the totally geodesic Lagrangian submanifold $S^n \subset Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2})$ is Hamiltonian volume minimizing for general n.

In the case $g = 2$, L is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$, L is Hamiltonian stable but not strictly Hamiltonian stable if and only if $m_2 - m_1 = 2$, L is strictly Hamiltonian stable if and only if $m_2 - m_1 < 2$.

Notice that $g = 1$ or $g = 2$ if and only if the Gauss image $L = \mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold, that is, a real form $Q_{m_1+1,m_2+1}(\mathbb{R})$ of $Q_n(\mathbb{C})$.

In the case $g = 3$, $L = \mathcal{G}(N^n)$ is strictly Hamiltonian stable (H. Ma - O.[24]).

Remark 7.1. In the case when $g = 3$, the invariant metric on $\mathcal{G}(N^n)$ induced by $\mathcal{G}$ from $Q_n(\mathbb{C})$ is a normal homogeneous metric ([24]) and hence $\mathcal{G}(N^n)$ is a compact minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$ with nonnegative sectional curvatures but $\nabla S \neq 0$ (compare with Theorem 4.1).

Theorem 7.2 (Hui Ma-O.[26]). *Suppose that $g = 6$ and thus $L = SO(4)/(\mathbb{Z}_2 + \mathbb{Z}_2) \cdot \mathbb{Z}_6$ $(m_1 = m_2 = 1)$ or $L = G_2/T^2 \cdot \mathbb{Z}_6$ $(m_1 = m_2 = 2)$ is homogeneous. Then L is strictly Hamiltonian stable.*

In the case $g = 4$, There are homogeneous case and non-homogeneous cace as we mentioned (Ozeki-Takeuchi, Ferus-Karcher-Münzner, Cecil-Chi-Jensen, Immervoll).

Theorem 7.3 (Hui Ma and O. [26]). *Suppose that $g = 4$ and $L = \mathcal{G}(N^n)$ is homogeneous. Then*

(1) $L = SO(5)/T^2 \cdot \mathbb{Z}_4$ $(m_1 = m_2 = 2)$ is strictly Hamiltonian stable.

(2) $L = U(5)/(SU(2) \times SU(2) \times U(1)) \cdot \mathbb{Z}_4$ $(m_1 = 4, m_2 = 5)$ is strictly Hamiltonian stable.

(3) $L = (SO(2) \times SO(m))/(\mathbb{Z}_2 \times SO(m-2)) \cdot \mathbb{Z}_4$ $(m_1 = 1, m_2 = m-2, m \geq 3)$ If $m_2 - m_1 \geq 3$, then L is NOT Hamiltonian stable. If $m_2 - m_1 = 2$, then L is Hamiltonian stable but not strictly Hamiltonian stable. If $m_2 - m_1 = 1$ or 0, then L is strictly Hamiltonian stable.

(4) $L = S(U(2) \times U(m))/S(U(1) \times U(1) \times U(m-2))) \cdot \mathbb{Z}_4 (m_1 = 2, m_2 = 2m - 3, m \geq 2)$ If $m_2 - m_1 \geq 3$, then L is NOT Hamiltonian stable. If $m_2 - m_1 = 1$ or -1, then L is strictly Hamiltonian stable.

(5) $L = Sp(2) \times Sp(m)/(Sp(1) \times Sp(1) \times Sp(m-2))) \cdot \mathbb{Z}_4$ $(m_1 = 4, m_2 = 4m - 5, m \geq 2)$. If $m_2 - m_1 \geq 3$, then L is NOT Hamiltonian stable. If $m_2 - m_1 = -1$, then L is strictly Hamiltonian stable.

(6) Suppose that $g = 4$ is homogeneous and

$$L = U(1) \cdot Spin(10)/(S^1 \cdot Spin(6)) \cdot \mathbb{Z}_4 \ (m_1 = 6, m_2 = 9, \ \text{thus } m_2 - m_1 = 3!)$$

Then L is strictly Hamiltonian stable!

Theorem 7.4 (Hui Ma-O.[26]). *Suppose that (U, K) is not of type* EIII*, that is, $(U, K) \neq (E_6, U(1) \cdot Spin(10))$. Then $L = \mathcal{G}(N)$ is NOT Hamiltonian stable if and only if $|m_2 - m_1| \geq 3$. Moreover if (U, K) is of type* EIII*, that is, $(U, K) =$*

$(E_6, U(1)\cdot Spin(10))$, *then* $(m_1, m_2) = (6, 9)$ *but* $L = \mathcal{G}(N)$ *is strictly Hamiltonian stable.*

8 Related Questions and Further Problems.

8.1 Questions

1. Investigate the Hamiltonian stability and other properties of the Gauss images of compact non-homogenous isoparametric hypersurfaces, particular OT-FKM type, embedded in spheres with $g = 4$.

2. Investigate the relation between our Gauss image construction and Karigiannis-Min-Oo's results.

3. Are there similar constructions of Lagrangian submanifolds in compact Hermitian symmetric spaces other than $\mathbb{C}P^n$, $Q_n(\mathbb{C})$? What about in the cases of $\mathrm{Gr}_2(\mathbb{C})$, $\mathrm{Gr}_r(\mathbb{C})$?

8.2 Cohomogeneity 1 special Lagrangian submanifolds in tangent bundle over the standard sphere

Let $T^1S^{n+1} \cong V_2(\mathbf{R}^{n+2})$ denote the unit tangent bundle of the standard sphere S^{n+1}. The cones over Legendrian lifts of isoparametric hypersurfaces with g distinct principal curvatures to $T^1S^{n+1} \cong V_2(\mathbf{R}^{n+2})$ provide fundamental examples of special Lagrangian cones in the (non-flat) Ricci-flat Kähler cone. In the cases of $g = 1, 2$, Kaname Hashimoto (OCU, D3) and Takashi Sakai (TMU) ([18]) classified all cohomogeneity 1 special Lagrangian submanifolds in the tangent bundle TS^{n+1} with respect to the *Stenzel metric* deformed from such special Lagrangian cones.

It is an interesting problem to investigate such cohomogeneity 1 special Lagrangian submanifolds in the cases of $g = 3, 4, 6$.

8.3 Extension to the semi-Riemannian case

It is another interesting problem to study an extension of our Gauss map construction to the semi-Riemannian case.

More recently, my new Ph. D. student, Harunobu Sakurai (OCU, D1), studies an extension of Lagrangian propery of the Gauss map and the mean curvature form formula to oriented semi-Riemannian hypersurfaces in semi-Riemannain space forms. From J. Hahn's work ([16], [17]), we have many interesting examples of minimal Lagrangian submanifolds in semi-Riemannian real Grassmann manifolds of oriented 2-planes .

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