COMPLETELY INTEGRABLE TORUS ACTIONS ON COMPLEX MANIFOLDS WITH FIXED POINTS

HIROAKI ISHIDA AND YAEL KARSHON

Abstract. We show that if a holomorphic n dimensional compact torus action on a compact connected complex manifold of complex dimension n has a fixed point then the manifold is equivariantly biholomorphic to a smooth toric variety.

1. Introduction

We begin by recalling some notions from the theory of toric varieties.

We work in the vector space $\mathrm{Lie}(S^1)^n \cong \mathbb{R}^n$ with the lattice $\mathrm{Hom}(S^1, (S^1)^n) \cong \mathbb{Z}^n$. Here, we identify $\mathrm{Lie}(S^1)$ with $\mathbb{R}$ such that the exponential map $\exp\colon \mathbb{R} \to S^1$ is $t \mapsto e^{2\pi it}$.

A *unimodular fan* is a finite set Δ of convex polyhedral cones with the following properties.

(1) A face of a cone in Δ is also a cone in Δ.
(2) The intersection of two cones in Δ is a common face.
(3) Every cone in Δ is unimodular, i.e., it has the form $\mathrm{pos}(\lambda_1, \ldots, \lambda_k)$ where $\lambda_1, \ldots, \lambda_k$ is part of a $\mathbb{Z}$-basis of the lattice. Here, pos denotes the positive span: the set of linear combinations with non-negative coefficients.[1]

A fan Δ is *complete* if the union of the cones in Δ is all of $\mathrm{Lie}(S^1)^n$.

The theory of toric varieties associates to a unimodular fan Δ a complex manifold M_Δ with a holomorphic $(\mathbb{C}^*)^n$-action with the following properties.

(1) The fixed points in M_Δ are in bijection with the n-dimensional cones in Δ.
(2) Let p be a fixed point in M_Δ. Then the isotropy weights at p are a $\mathbb{Z}$-basis to the lattice $\mathrm{Hom}((S^1)^n, S^1) \subset (\mathrm{Lie}(S^1)^n)^*$. Moreover, let $\lambda_1, \ldots, \lambda_n$ be the dual basis; then the cone in Δ that corresponds to p is $\mathrm{pos}(\lambda_1, \ldots, \lambda_n)$.
(3) The manifold M_Δ is compact if and only if the fan Δ is complete.

Date: March 19, 2012.

2010 *Mathematics Subject Classification.* Primary 14M25, Secondary 32M05, 57S25.

Key words and phrases. Torus action, complex manifold, toric manifold.

The first author is supported by JSPS Research Fellowships for Young Scientists. This work is partially supported by the JSPS Institutional Program for Young Researcher Overseas Visits "Promoting international young researchers in mathematics and mathematical sciences led by OCAMI".

The second author is partially supported by the Natural Sciences and Engineering Research Council of Canada.

[1] This property of a cone or a fan is also described in the literature by the adjectives *smooth*, *non-singular*, *regular*, and *Delzant*.

For the details of the construction and the proof of these properties, we refer the reader to the book [2] by Cox, Little, and Schenck.

In fact, M_Δ is an *algebraic* variety. Moreover, every smooth complex algebraic variety that is equipped with an algebraic $(\mathbb{C}^*)^n$-action with an open dense free orbit is isomorphic to some M_Δ. (The proof of this fact appeared in the book [6] by Kempf, Knudsen, Mumford, and Saint-Donat and in the article [9] by Miyake and Oda and relies on a lemma of Sumihiro [10]; see Corollary 3.1.8 in [2].) Our main theorem is a complex analytic variant of this result:

Theorem 1. *Let M be a connected complex manifold of complex dimension n, equipped with a faithful action of the torus $(S^1)^n$ by biholomorphisms. If M is compact and the action has fixed points, then there exists a unimodular fan Δ and an $(S^1)^n$-equivariant biholomorphism of M_Δ with M.*

Remark 2.

(1) Our theorem gives a negative answer to a question that was raised by Buchstaber and Panov in [1, Problem 5.23].

Let M be a closed $2n$ dimensional manifold with an $(S^1)^n$-action that is locally standard: every orbit has a neighbourhood that is equivariantly diffeomorphic, up to an automorphism of $(S^1)^n$, to an invariant open subset of $\mathbb{C}^n$ with the standard $(S^1)^n$-action. Also assume that the quotient $M/(S^1)^n$ is diffeomorphic, as a manifold with corners, to a simple convex polytope P in $\mathbb{R}^n$.[2] Such manifolds, introduced in [3] and studied in the toric topology community, are called *quasi-toric manifolds*[3].

The question of Buchstaber and Panov is whether there exists a non-toric quasitoric manifold that admits an $(S^1)^n$-invariant complex structure.

(2) Our theorem strengthens an earlier result of Ishida and Masuda, that if a closed complex manifold of complex dimension n admits an $(S^1)^n$-action, and if its odd-degree cohomology groups vanish, then the Todd genus of the manifold is equal to one. See [5, Theorem 1.1 and Remark 1.2].

(3) It is necessary to assume that the action has fixed points: the complex torus $\mathbb{C}^*/(z \sim 2z)$ has a holomorphic S^1-action, induced from multiplication on $\mathbb{C}^*$, but it is not a toric variety.

(4) It is necessary to assume that the manifold is compact: the open unit disc in $\mathbb{C}$ with the natural circle action has a fixed point, but it is not a toric variety: the circle action does not extend to a $\mathbb{C}^*$-action.

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[2] A map from $M/(S^1)^n$ to P is a diffeomorphism of manifolds with corners if and only if it is a homeomorphism and, for every real valued function on P, the function extends to a smooth function on $\mathbb{R}^n$ if and only if its pullback to M is smooth. For every $k \in \{0, \ldots, n\}$, a diffeomorphism carries the k dimensional orbits in M to the relative interiors of the k dimensional faces of P.

[3] Davis-Januszkiewicz [3] used the term *toric manifold*, but this term was already used in the literature to mean a smooth toric variety, so Buchstaber-Panov [1] introduced instead the term *quasitoric manifold*.

2. The complexified action

Let the torus $(S^1)^n$ act on a complex manifold M by biholomorphisms. If the manifold M is compact, then the $(S^1)^n$-action extends to a $(\mathbb{C}^*)^n$-action that is holomorphic not only in the sense that each element of $(\mathbb{C}^*)^n$ acts by a biholomorphism but also in the sense that the action map $(\mathbb{C}^*)^n \times M \to M$ is holomorphic. See, e.g., [4, Theorem 4.4]. For the convenience of the reader, we briefly recall here some of the details of this standard construction.

Let $\xi_1, \ldots, \xi_n$ be the fundamental vector fields of the $(S^1)^n$-action with respect to the coordinate one-dimensional subtori. Let $J \colon TM \to TM$ be the multiplication by $\sqrt{-1}$. We claim that the vector fields $-J\xi_1, \ldots, -J\xi_n$ are holomorphic (in the sense that their flows preserve the complex structure) and commute with each other and with the vector fields ξ_i.

Because the $(S^1)^n$-action preserves J and ξ_j, it preserves $-J\xi_j$, for each j. So the vector fields $-J\xi_j$ commute with the vector fields ξ_i that generate this action. Because J is a complex structure, its Nijenhaus tensor, $N(Z, W) := 2\,([JZ, JW] - J[Z, JW] - J[JZ, W] - [Z, W])$, vanishes. Setting $Z = \xi_i$ and $W = \xi_j$, we get that $[J\xi_i, J\xi_j] = J[\xi_i, J\xi_j] + J[J\xi_i, \xi_j] + [\xi_i, \xi_j]$, and each of the three terms on the right hand side is zero. So the vector fields $-J\xi_j$ commute with each other. A vector field Y is holomorphic if and only if $[Y, JW] = J[Y, W]$ for each vector W; see [7, Proposition 2.10 in Chapter IX]. Set $Y := -J\xi_i$ and W arbitrary; because $JY (= \xi_i)$ is holomorphic, $[JY, JW] = J[JY, W]$; by the vanishing of the Nijenhaus tensor,

$$\begin{aligned} 0 = N(JY, W) &= 2\,([-Y, JW] - J[JY, JW] - J[-Y, W] - [JY, W]) \\ &= 2([-Y, JW] - J[-Y, W]), \end{aligned}$$

so Y is holomorphic.

If M is compact, the vector fields $-J\xi_1, \ldots, -J\xi_n$ are complete, and we get an $\mathbb{R}^{2n}$-action, $\mathbb{R}^{2n} \times M \to M$, via

$$\left(\sum_{i=1}^{2n} a_i \mathbf{e}_i, x \right) \mapsto c_x(1),$$

where $c_x(r)$ is the integral curve of the vector field $\sum_{i=1}^n -a_i J\xi_i + a_{n+i}\xi_i$ with $c_x(0) = x$. This action descends to a $(\mathbb{C}^*)^n$-action by biholomorphisms that extends the given $(S^1)^n$-action. Finally, the action map $(\mathbb{C}^*)^n \times M \to M$ is holomorphic, because its differential, which at the point (z, m) is the map $\mathbb{C}^n \times T_m M \to T_{z \cdot m} M$ that takes $(2\pi(r_1 + i\theta_1, \ldots, r_n + i\theta_n), v)$ to $\sum_j -r_j J\xi_j|_{z \cdot m} + \theta_j \xi_j|_{z \cdot m} + z_* v$, is complex linear.

Remark 3. In the next section we will see that if there exists a fixed point then the extended $(\mathbb{C}^*)^n$-action is faithful. In general, the extended $(\mathbb{C}^*)^n$-action might not be faithful.

Example 4. Let $(S^1)^n$ act on $\mathbb{C}^n$ with weights $\alpha_1, \ldots, \alpha_n$:

$$g \cdot (z_1, \ldots, z_n) = (g^{\alpha_1} z_1, \ldots, g^{\alpha_n} z_n),$$

where $g^{\alpha_i} = g_1^{\alpha_{i1}} \dots g_n^{\alpha_{in}}$ for $g = (g_1, \dots, g_n) \in (S^1)^n$ and the isotropy weight $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{Z}^n$. Then the complexified action is given by the same formula applied to $g = (g_1, \dots, g_n) \in (\mathbb{C}^*)^n$.

3. Structures near fixed points

Let M be a complex manifold of complex dimension n. Let the torus $(S^1)^n$ act on M faithfully by biholomorphisms. Let p be a point in M that is fixed by the $(S^1)^n$-action. Let $\alpha_1, \dots, \alpha_n$ be the isotropy weights at p.

We begin with a local result:

Lemma 5. *There exists an $(S^1)^n$-invariant neighbourhood U_p of p in M, an $(S^1)^n$-invariant neighbourhood $\widetilde{U}_p$ of the origin in T_pM, and an $(S^1)^n$-equivariant biholomorphism $\varphi_p \colon U_p \to \widetilde{U}_p$ whose differential at p is the identity map on T_pM.*

Here, $\mathbb{C}_{\alpha_i}$ denotes the one dimensional complex vector space $\mathbb{C}$ with the $(S^1)^n$-action that is obtained by composing the homomorphism $(S^1)^n \to S^1$ that is encoded by the weight α_i with the standard action of S^1 on $\mathbb{C}$ by scalar multiplication.

Proof. Let $\varphi \colon U \to \widetilde{U} \subseteq \mathbb{C}^n$ be a local holomorphic chart near p with $\varphi(p) = 0$. Identifying $\mathbb{C}^n$ with T_pM via the differential

$$(d\varphi)_p \colon T_pM \to T_0\mathbb{C}^n \cong \mathbb{C}^n,$$

we get a biholomorphism

$$\varphi' \colon U \to \widetilde{U}' \subseteq T_pM$$

whose differential at p is the identity map on T_pM. We want to obtain such a biholomorphism that is also equivariant.

Set

$$U' := \bigcap_{g \in (S^1)^n} gU.$$

Clearly, U' is invariant and contains p. We now show that U' is open. The complement of U' is the image of the closed subset $(S^1)^n \times (M \smallsetminus U)$ of $(S^1)^n \times M$ under the action map $(S^1)^n \times M \to M$. Because $(S^1)^n$ is compact, the action map is proper. Being proper means that the preimage of every compact set is compact; when the target space M is a manifold[4] it implies that the map is closed. Thus, the complement $M \smallsetminus U'$ is closed, and so U' is open.

To obtain an equivariant chart, we average φ': let

$$\widetilde{\varphi} := \int_{g \in (S^1)^n} (g \circ \varphi' \circ g^{-1})\, dg \colon\ U' \ \to\ T_pM,$$

[4] In fact, it is enough to assume that the target space is Hausdorff and compactly generated. Compactly generated means that a subset is closed if and only if its intersection with every compact set K is closed in K; this property holds if the space is locally compact or if the space is metrizable.

where dg is Haar measure on $(S^1)^n$. The map $\widetilde{\varphi}$ is holomorphic and $(S^1)^n$-equivariant. Moreover, its differential at p is the identity map on T_pM. By the implicit function theorem, $\widetilde{\varphi}$ restricts to a biholomorphism from some smaller open neighbourhood U'' of p in M to an open neighbourhood of the origin in T_pM. The restriction of $\widetilde{\varphi}$ to the invariant neighbourhood $U_p := \bigcap_{g \in (S^1)^n} g \cdot U''$ of p in M satisfies the requirements of the lemma. □

Corollary 6. *There exists an $(S^1)^n$-equivariant local holomorphic chart*

$$\varphi_p \colon U_p \to \mathbb{D}^n$$

from an invariant open neighbourhood U_p of p to a polydisc $\mathbb{D}^n$ in $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$.

Proof. By the definition of the isotropy weights, there exists a complex linear $(S^1)^n$-equivariant isomorphism between the tangent space T_pM and the representation $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$. Corollary 6 then follows from Lemma 5 by restricting the chart to the preimage of a polydisc. □

We would like to extend the chart of Corollary 6 to a chart whose image is all of $\mathbb{C}^n$. We can do this when the $(S^1)^n$ extends to a $(\mathbb{C}^*)^n$-action; for example, if the manifold is compact; by "sweeping" by the $(\mathbb{C}^*)^n$-action.

Lemma 7. *Suppose that the $(S^1)^n$-action extends to a $(\mathbb{C}^*)^n$-action. Then there exists an invariant open neighbourhood V_p of p in M and an $(S^1)^n$-equivariant biholomorphism of V_p with $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$.*

Proof. Let $\varphi_p \colon U_p \to \mathbb{D}^n$ be an $(S^1)^n$-equivariant holomorphic local chart, as in Corollary 6. Because φ_p is $(S^1)^n$-equivariant and holomorphic, it intertwines the restriction to U_p of the vector fields that generate the complexified $(\mathbb{C}^*)^n$-action on M with the restriction to $\mathbb{D}^n$ of the vector fields that generate the complexified $(\mathbb{C}^*)^n$-action on $\mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$. This, and the fact that φ_p is a diffeomorphism between U_p and $\mathbb{D}^n$, implies that φ_p also intertwines the partial flows on U_p and on $\mathbb{D}^n$ that are generated by these vector fields; in particular it intertwines the domains of definition of these partial flows.

For each $t \in \mathbb{R}$, let g_t be the element of $(\mathbb{C}^*)^n$ that acts on $\mathbb{C}^n$ as scalar multiplication by e^{-t}, and let $\eta \in \mathrm{Lie}(\mathbb{C}^*)^n$ be the generator of the one-parameter subgroup $t \mapsto g_t$. Because $e^{-t}\mathbb{D}^n \subset \mathbb{D}^n$ for all $t \geq 0$, and because φ_p intertwines the domains of definition of the partial flows on U_p and on $\mathbb{D}^n$ that correspond to η, we get that $g_t U_p \subset U_p$ for all $t \geq 0$. So, for every $t \geq 0$, the domain of definition of the $(S^1)^n$-equivariant biholomorphism

$$\varphi_p^{(t)} := (g_t)^{-1} \circ \varphi_p \circ g_t \; : \; g_{-t} U_p \to e^t \mathbb{D}^n$$

contains U_p. Here, $g_t \colon g_{-t} U_p \to U_p$ and $g_t \colon e^t \mathbb{D}^n \to \mathbb{D}^n$ are given by the complexified actions on M and on $\mathbb{C}^n$. By the choice of g_t, the latter map is multiplication by e^{-t}.

Moreover, because φ_p intertwines the partial flows that correspond to η and these partial flows are defined for all $t \geq 0$, the restriction to U_p of $\varphi_p^{(t)}$ coincides with φ_p for all $t \geq 0$. Substituting $t - s$ instead of t, we get that the maps $\varphi_p^{(t)}$ and $\varphi_p^{(s)}$ agree whenever they are both defined. Thus, all these maps fit together into a map

$$\bigcup_{t\geq 0} \varphi_p^{(t)} \colon V_p \to \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n},$$

where $V_p = \bigcup_{t\geq 0} g_{-t} U_p$. This map is onto, because its image is the union of the sets $e^t\mathbb{D}^n$ over all $t \geq 0$. The map is one to one, because it is one to one on each $g_{-t}U_p$, and for every two points in the domain there exists a $t \geq 0$ such that the points are both in $g_{-t}U_p$. Because V_p is covered by $(S^1)^n$-invariant open sets $g_{-t}U_p$ on which the map is an $(S^1)^n$-equivariant biholomorphism, we deduce that the map is itself an $(S^1)^n$-equivariant biholomorphism, as required. □

4. Obtaining a fan

Let M be a complex manifold of complex dimension n, let the torus $(S^1)^n$ act on M faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. Moreover, assume that the action has at least one fixed point.

In Lemma 7 we assigned to every fixed point p in M an open subset V_p that is biholomorphic to $\mathbb{C}^n$. By assumption, there exists at least one fixed point. So the union of the sets V_p over these fixed points,

$$\bigcup_{p\in M^{(S^1)^n}} V_p,$$

is nonempty. We fix a connected component of this union and denote it X.

Remark 8. We would like to know that if M connected then the union of the sets V_p is all of M. We do not know how to prove this directly; we do not even know if it is always true. We will eventually show that if M is compact and connected then X is compact; so in this case X must coincides with M, and the union of the sets V_p is indeed all of M.

The connected components of the fixed point sets of the circle subgroups of $(S^1)^n$ are closed complex submanifolds of X. If such a submanifold has complex codimension one, then, in analogy with the toric topology literature, we call it a *characteristic submanifold* of X (cf. [8, p. 240]).

Because X is a union of finitely many V_ps and each V_p has only finitely many characteristic submanifolds, there are only finitely many characteristic submanifolds in X. Denote them

$$X_1, \ldots, X_m.$$

Let T_i be the subgroup of T that fixes X_i. If a compact group acts faithfully on a connected manifold then at every fixed point the linear isotropy representation is faithful. Therefore, the linear isotropy representation of T_i at any point q of X_i is faithful. Because T_i acts holomorphically and fixes X_i, we get a faithful representation of T_i on the one dimensional complex space T_qX/T_qX_i. This gives an injection $T_i \to S^1$, where S^1 acts on T_qX/T_qX_i by scalar multiplication. By continuity, this injection is independent of the choice of point q in X_i. Because, by assumption, T_i contains a circle subgroup of T, this injection is an isomorphism. Let

$$\lambda_i \colon S^1 \to T_i \subset (S^1)^n$$

be the inverse of this isomorphism, composed with the inclusion map into $(S^1)^n$.

We define an abstract simplicial complex:

$$\Sigma := \left\{ I \subseteq \{1, \ldots, m\} \;\Big|\; X_I := \bigcap_{i \in I} X_i \neq \emptyset \right\}.$$

To each simplex $I \in \Sigma$ we assign the cone

$$C_I := \operatorname{pos}(\lambda_i \mid i \in I) := \left\{ \sum_{i \in I} a_i \lambda_i \;\Big|\; a_i \geq 0 \right\}$$

in $\operatorname{Lie}(S^1)^n$.

Example 9. Take $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Let $(S^1)^n$ act on it with weights $\alpha_1, \ldots, \alpha_n \in \operatorname{Hom}((S^1)^n, S^1) \subset (\operatorname{Lie}(S^1)^n)^*$. Suppose that the action is faithful; then $\alpha_1, \ldots, \alpha_n$ are a $\mathbb{Z}$-basis of $\operatorname{Hom}((S^1)^n, S^1)$. The characteristic submanifolds are the coordinate hyperplanes $\{z_i = 0\}$ for $i = 1, \ldots, n$. The homomorphisms $\lambda_1, \ldots, \lambda_n$ are the basis to $\operatorname{Hom}(S^1, (S^1)^n) \subset \operatorname{Lie}(S^1)^n$ that is dual to $\alpha_1, \ldots, \alpha_n$.

Recall that a cone in $\operatorname{Lie}(S^1)^n$ is *unimodular* if it is generated by part of a $\mathbb{Z}$-basis of $\operatorname{Hom}(S^1, (S^1)^n)$.

Returning to our general case –

Lemma 10. *The cones C_I, for $I \in \Sigma$, are unimodular.*

Proof. Let $I \in \Sigma$. By the definition of Σ, this means that the intersection $\bigcap_{i \in I} X_i$ is nonempty. Let q be a point in this intersection. Let p be a fixed point such that $q \in V_p$. Because V_p is isomorphic to some $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$ on which the action is faithful, the lemma follows from Example 9. □

Every V_p contains an open dense free $(\mathbb{C}^*)^n$ orbit. For any two V_ps that are in the connected component X, these orbits coincide. Thus, there exists a unique free $(\mathbb{C}^*)^n$ orbit in X, it is open and dense, and it is contained in every V_p that is contained in X.

Fix a point q in the free $(\mathbb{C}^*)^n$ orbit in X. For any $\xi \in \operatorname{Lie}(S^1)^n$, consider the curve

$$c_q^\xi \colon \mathbb{R} \to X$$

that is given by

$$c_q^\xi(r) := \exp(-rJ\xi) \cdot q \qquad \text{for } r \in \mathbb{R}$$

where $\exp\colon \operatorname{Lie}(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is the exponential map and where J denotes multiplication by i in $\operatorname{Lie}(\mathbb{C}^*)^n$.

Denote by C_I^0 the relative interior of the cone C_I. Denote

$$X_I^0 = \bigcap_{i\in I} X_i \smallsetminus \bigcap_{j\notin I} X_j.$$

Lemma 11. *Let $\xi \in \mathrm{Lie}(S^1)^n$ and $I \in \Sigma$. Then $\xi \in C_I^0$ if and only if the curve $c_q^\xi(r)$ converges as $r \to -\infty$ to a point q' in X_I^0. Moreover, in this case the limit point q' belongs to V_p for every p such that $V_p \cap X_I \neq \emptyset$.*

Proof. Suppose that $\xi \in C_I^0$. By the definition of Σ, X_I is nonempty. Let p be such that V_p meets X_I. Without loss of generality assume that $I = \{1, \dots, k\}$ and that the characteristic submanifolds that meet V_p are $X_1, \dots, X_n$. Let $\alpha_1, \dots, \alpha_n$ denote the isotropy weights at p. The assumption that $\xi \in C_I^0$ exactly means that $\langle \xi, \alpha_i\rangle$ is positive for $i = 1, \dots, k$ and zero for $i = k+1, \dots, n$. Fix an isomorphism $(z_1, \dots, z_n)\colon V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \dots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all i. In these coordinates, the curve $c_q^\xi(r)$ is represented as

$$(z_1, \dots, z_n)(c_q(r)) = (e^{2\pi r\langle \xi,\alpha_1\rangle}, \dots, e^{2\pi r\langle \xi,\alpha_n\rangle}).$$

As r approaches $-\infty$, the curve in $\mathbb{C}^n$ approaches the point $(\underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{n-k})$. On the other hand, the coordinates take each intersection $V_p \cap X_i$ to the coordinate hyperplane $\{(z_1, \dots, z_n) \mid z_i = 0\}$, and they take the intersection $V_p \cap X_I^0$ to the set $\{(z_1, \dots, z_n) \mid z_i = 0 \text{ iff } 1 \le i \le k\}$. So the curve approaches a point in $V_p \cap X_I^0$, as required.

Now suppose that the curve $c_q^\xi(r)$ converges as $r \to -\infty$ to a point in X_I^0. Let p be such that this limit point is contained in V_p. As before, without loss of generality assume that $I = \{1, \dots, k\}$ and that the characteristic submanifolds that meet V_p are exactly $X_1, \dots, X_n$; fix an isomorphism $(z_1, \dots, z_n)\colon V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \dots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all i; the curve $c_q^\xi(r)$ is represented as $(z_1, \dots, z_n)(c_q(r)) = (e^{2\pi r\langle \xi,\alpha_1\rangle}, \dots, e^{2\pi r\langle \xi,\alpha_n\rangle})$. Because the curve approaches a limit as $r \to -\infty$, the pairings $\langle \xi, \alpha_i\rangle$ are nonnegative for all $i = 1, \dots, n$. Because this limit is in X_I^0, the pairings are positive for every $i \in I$ and they vanish for every $i \in \{1, \dots, n\} \smallsetminus I$. Thus, $\xi \in C_I^0$ as required. □

Corollary 12. (1) *For every $I, J \in \Sigma$, if $I \neq J$, then $C_I^0 \cap C_J^0 = \emptyset$.*

(2) *For every $I, J \in \Sigma$,*

$$C_I \cap C_J = C_{I\cap J}.$$

(3) *The collection of cones*

$$\Delta := \left\{ C_I \,\middle|\, I \in \Sigma \right\}$$

is a fan, that is, every face of every cone in Δ is itself in Δ, and the intersection of every two cones in Δ is a common face.

Proof. Part (1) follows from Lemma 11 because the sets X_I^0 are disjoint. Part (3) follows from Part (2).

For Part (2), we only need to show the inclusion $C_I \cap C_J \subseteq C_{I\cap J}$, because the opposite inclusion is trivial. Let $\xi \in C_I \cap C_J$. Let $I' \subset I$ and $J' \subset J$ be the subsets such that $\xi \in C_{I'}^0$

and $\xi \in C^0_{J'}$. Then $C^0_{I'} \cap C^0_{J'} \neq \emptyset$. By Part (1), $I' = J'$. Let $L = I' = J'$. Then $L \subset I \cap J$, and $\xi \in C^0_L \subset C_{I \cap J}$. □

Lemma 13. *For every $I \in \Sigma$, the set X_I is an $(S^1)^n$-invariant smooth closed complex submanifold of X of complex codimension $|I|$, it is connected, and it contains a fixed point.*

Proof. Fix $I \in \Sigma$.

Because each of the sets X_i, for $i \in I$, is closed in X, so is the intersection X_I of these sets.

Because X is the union of open subsets V_p, and because every intersection $V_p \cap X_I$ is an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in V_p, we deduce that X_I is itself an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in X. It remains to show that X_I is connected and contains a fixed point.

Choose any $\xi \in C^0_I$ (for example, we may take $\xi = \sum_{i \in I} \lambda_i$), and choose any q in the free $(\mathbb{C}^*)^n$ orbit in X. By Lemma 11, the curve $c^\xi_q(r)$ converges as $r \to -\infty$; let q' be its limit. Also by Lemma 11, for every p such that $V_p \cap X_I \neq \emptyset$, the limit point q' belongs to V_p. Because X_I is the union over such p of the subsets $V_p \cap X_I$, and because each of these subsets is connected and contains q', the union X_I is connected. Also, every p such that $V_p \cap X_I \neq \emptyset$ belongs to $V_p \cap X_I$; because the set of such ps is nonempty, X_I contains a fixed point. □

Corollary 14. *In the fan Δ, every cone is contained in an n dimensional cone.*

Proof. Every cone in the fan has the form C_I for some $I \in \Sigma$. By Lemma 13, the set X_I contains a fixed point; let p be such a fixed point. Since V_p was chosen as in Lemma 7, by Example 9 there exist exactly n characteristic submanifolds, say, X_j for $j \in J \subset \{1, \ldots, m\}$ with $|J| = n$, that pass through p. Then $J \in \Sigma$, and C_J is an n dimensional cone in Δ that contains C_I. □

5. Isomorphism of the subset X with a toric manifold

Let M be a complex manifold of complex dimension n, let the torus $(S^1)^n$ act on M faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. Moreover, assume that the action has at least one fixed point.

In Section 4 we described an open subset X of M and a unimodular fan Δ. Let M_Δ be the toric variety that is associated to the fan Δ.

Lemma 15. *There exists an $(S^1)^n$-equivariant biholomorphism between M_Δ and X.*

We recall some properties of the set X and the fan Δ. Let F denote the fixed point set in X. For every fixed point $p \in F$, let $\alpha_{p,1}, \ldots, \alpha_{p,n}$ denote the isotropy weights of the torus action at p.

(1) The set X is the union over $p \in F$ of subsets V_p, such that each V_p is an invariant open neighbourhood of p that is equivariantly biholomorphic to the linear representation $\mathbb{C}_{\alpha_{p,1}}, \ldots, \mathbb{C}_{\alpha_{p,n}}$.

(2) The n-dimensional cones in Δ are in bijection with the fixed point sets $p \in F$, and the cone corresponding to the fixed point p is $\text{pos}(\lambda_{i_1}, \ldots, \lambda_{i_n})$, where $\lambda_{i_1}, \ldots, \lambda_{i_n}$ is a basis of $\text{Lie}(S^1)^n$ that is dual to the basis $\alpha_{p,1}, \ldots, \alpha_{p,n}$ of $(\text{Lie}(S^1)^n)^*$.

The toric variety M_Δ that is associated to the fan Δ has similar properties: it is the union over $p \in F$ of invariant subsets V'_p, and every V'_p is equivariantly biholomorphic to $\mathbb{C}_{\alpha_{p,1}} \oplus \ldots \oplus \mathbb{C}_{\alpha_{p,n}}$.

Lemma 15 follows immediately from these properties of X and M_Δ, by the following lemma.

Lemma 16. *Let X and X' be complex manifolds of complex dimension n, equipped with holomorphic $(\mathbb{C}^*)^n$-actions. Suppose that there exist open dense $(\mathbb{C}^*)^n$ orbits O in X and O' in X'. Suppose that there exist invariant open subsets V_p in X and V'_p in X', both indexed by $p \in F$, such that X is the union of the sets V_p and X' is the union of the sets V'_p, and that for each $p \in F$ there exists an equivariant biholomorphism $\varphi_p \colon V_p \to V'_p$. Then X is equivariantly biholomorphic to X'.*

Proof. Necessarily, O is contained in each V_p and O' is contained in each V'_p. Fix a point q in O and a point q' in O'. After possibly composing each φ_p by the action of an element of $(\mathbb{C}^*)^n$, we may assume that $\varphi_p(q) = q'$ for each $p \in F$. So, for each p and $\tilde{p} \in F$, the maps φ_p and $\varphi_{\tilde{p}}$ coincide at the point q. By equivariance, φ_p and $\varphi_{\tilde{p}}$ coincide on all of O; by continuity, they coincide on the entire overlap $V_p \cap V_{\tilde{p}}$. Thus, the φ_p fit together into a map

$$\varphi = \bigcup_p \varphi_p \colon X \to X'.$$

This map is holomorphic, equivariant, and onto. Similarly, the inverses $\psi_p := {\varphi_p}^{-1}$ fit together into a map

$$\psi = \bigcup_p \psi_p \colon X' \to X.$$

We have that $\psi \circ \varphi = \text{id}_X$ and $\varphi \circ \psi = \text{id}_{X'}$; thus, $\varphi \colon X \to X'$ is an equivariant biholomorphism, as required. ◻

6. The compact case

Let M be a complex manifold of complex dimension n, with a faithful $(S^1)^n$-action, with fixed points.

Suppose that M is compact. In Section 2 we extended the $(S^1)^n$-action to a holomorphic $(\mathbb{C}^*)^n$-action. In Section 4 we chose an open subset X of M of a particular form and we associated to it a fan Δ.

Lemma 17. *The fan Δ is complete.*

We begin by proving a special case:

Lemma 18. *Let M' be a complex manifold of complex dimension one, equipped with a faithful holomorphic action of S^1 with at least one fixed point. Suppose that M' is compact and connected. Then M' is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard $\mathbb{C}^*$-action.*

Proof. Consider the S^1-action on M'. Near a fixed point, it is isomorphic to the restriction of either the standard S^1-action on $\mathbb{C}$ or the opposite S^1-action on $\mathbb{C}$ to an invariant neighbourhood of the origin in $\mathbb{C}$.

Consider the flow that is generated by $-J\xi$, where ξ generates the S^1-action. If the S^1-action near a fixed point is standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $-\infty$. If the S^1-action near a fixed point is opposite from standard, then the trajectories of this flow converge to the fixed point as their parameter approaches ∞.

Outside the fixed point set, the action is free. The quotient M'/S^1 is[5] a real one-manifold with boundary; its boundary is exactly the image of the fixed point set. Because M' is compact and contains a fixed point, and by the classification of one-manifolds, the quotient M'/S^1 must be a closed segment.

The flow on M' that is generated by $-J\xi$ descends to a flow on the interior of M'/S^1 that does not have fixed points. For each boundary component, the flow approaches that component either as its parameter approaches ∞ or as the parameter approaches $-\infty$. Necessarily, it approaches one boundary component when the parameter approaches ∞ and it approaches the other boundary component when the parameter approaches $-\infty$.

The corresponding fan must then be equal to the fan of $\mathbb{CP}^1$, and the manifold is equivariantly biholomorphic to $\mathbb{CP}^1$ by Lemma 16. □

We now return to the setup of Lemma 17: We have a complex manifold M of complex dimension n, with a faithful $(S^1)^n$-action, with fixed points. We assume that M is compact. We chose an open subset X of M of a particular form and we associated to it a fan Δ.

Lemma 19. *Every $n-1$ dimensional cone in Δ is a common face of two n dimensional cones in Δ.*

Proof. Let C_I be an $n-1$ dimensional cone in Δ, corresponding to the subset $I = \{i_1, \dots, i_{n-1}\}$ of $\{1, \dots, m\}$.

Let T_I be the codimension one subtorus of $(S^1)^n$ that is generated by the circles T_i for $i \in I$. By Lemma 13, X_I is a connected complex manifold of dimension one, equipped with a faithful holomorphic action of the circle $(S^1)^n/T_I$ with at least one fixed point. We will now show that X_I is compact, and will deduce Lemma 19 from Lemma 18.

First note that X_I is a connected component of the fixed point set of T_I in X. This follows from the facts that X_I is connected (by Lemma 13) and that, for every V_p in X, if the intersection $V_p \cap X_I$ is nonempty then it is a connected component of the fixed point set of T_I in V_p. Let N denote the connected component of the fixed point set of T_I in M that contains X_I. As in any holomorphic torus action on a complex manifold, N is a

[5] Here, "is" means that there exists a unique manifold-with-boundary structure on M'/S^1 such that a function is smooth if and only if its pullback to M' is smooth.

$(S^1)^n$-invariant closed complex submanifold of M. By examining N near a point of X_I, we deduce that N has complex dimension one. Because N is closed in M and M is compact, N is compact. By Lemma 18, N is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard action of the circle $(S^1)^n/T_I$. In particular, N contains two fixed points; denote them p' and p''. At least one of these fixed points is in X_I, by Lemma 13. The intersection $V_{p'} \cap N$, being a $(\mathbb{C}^*)^n$-invariant neighbourhood of p' in N, must be all of $N \smallsetminus \{p''\}$. Similarly, the intersection $V_{p''} \cap N$, is all of $N \smallsetminus \{p'\}$. Thus, the intersection $V_{p'} \cap V_{p''}$ is nonempty. Because at least one of the sets $V_{p'}$ and $V_{p''}$ is contained in X, and because X is a connected component of the union of the sets V_p, we deduce that X contains both $V_{p'}$ and $V_{p''}$. Thus, N is entirely contained in X, and so N must be equal to X_I. Thus, X_I is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard action of the circle $(S^1)^n/T_I$. This implies the result of Lemma 19. □

We are now ready to prove Lemma 17.

Proof of Lemma 17. Let $|\Delta|$ denote the union of the cones in Δ, and let $|\Delta^{n-2}|$ denote the union of the cones in Δ that have codimension ≥ 2. The complement $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$ is connected, open, and dense in $\mathrm{Lie}(S^1)^n$.

By Lemma 19, the union of the relative interiors of the faces of Δ of dimension $(n-1)$ and of dimension n is open in $\mathrm{Lie}(S^1)^n$. This union is $|\Delta| \smallsetminus |\Delta^{n-2}|$. Thus, $|\Delta| \smallsetminus |\Delta^{n-2}|$ is also open in $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$.

But because $|\Delta|$ is closed in $\mathrm{Lie}(S^1)^n$, we also have that $|\Delta| \smallsetminus |\Delta^{n-2}|$ is closed in $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$.

Because $|\Delta| \smallsetminus |\Delta^{n-2}|$ is open and closed in $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$ and $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$ is connected, we deduce that $|\Delta| \smallsetminus |\Delta^{n-2}|$ is either empty or is equal to all of $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$.

Because, by assumption, M has a fixed point, Δ has at least one n dimensional cone, so $|\Delta| \smallsetminus |\Delta^{n-2}|$ is not empty. So $|\Delta| \smallsetminus |\Delta^{n-2}|$ is equal to all of $\mathrm{Lie}(S^1)^n \smallsetminus |\Delta^{n-2}|$. Taking the closures, we deduce that $|\Delta| = \mathrm{Lie}(S^1)^n$, as required. □

We are now ready to prove our main theorem.

Proof of Theorem 1. Lemma 16 gives an equivariant biholomorphism

$$\varphi\colon M_\Delta \to X.$$

By Lemma 17, the fan Δ is complete. This implies that the toric variety M_Δ is compact. So X must be compact. Because M is Hausdorff and connected, and X is a subset that is both compact and open, X is all of M. So φ defines an equivariant biholomorphism from M_Δ to M, as required. □

Acknowledgment. We thank Ignasi Mundet i Riera for a counterexample to a statement that, if true, would have simplified our proof (see Remark 8).

References

[1] V. E. Buchstaber and T. E. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture series, vol. 24, Amer. Math. Soc., Providence, R.I., 2002.

[2] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics **124**, American Mathematical Society, 2011.
[3] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds, and torus actions*, Duke Math. J. vol. **62** no. 2 (1991), 417–451.
[4] Victor Guillemin and Shlomo Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), 515–538.
[5] H. Ishida and M. Masuda, *Todd genera of complex torus manifolds*, `arXiv:1203.3129`.
[6] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal Embeddings I*, Lecture Notes in Math. **339**, Springer, Berlin, 1973.
[7] S. Kobayashi and K. Nomizu, *Foundation of Differential Geometry Volume II*, Interscience Publishers, John Willey & Sons, Inc., New York-London-Sydney, 1969.
[8] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. 51 (1999), 237–265.
[9] K. Miyake and T. Oda, *Almost homogeneous algebraic varieties under algebraic torus action*, in: Manifolds – Tokyo 1973 (Proc. Internat. Conf. Tokyo, 1973), (A. Hattori, Ed.)), Univ. Tokyo Press, Tokyo, 1975, 373–381.
[10] H. Sumihiro, *Equivariant completion, I, II*, J. Math. Kyoto Univ. **14**, 1–28; **15** (1975), 573–605.

Osaka City University Advanced Mathematical Institute, Sumiyoshi-ku, Osaka 558-8585, Japan.
E-mail address: ishida@sci.osaka-cu.ac.jp

Dept. of Mathematics, University of Toronto, 40 St.George Street, Toronto Ontario M5S 2E4, Canada
E-mail address: karshon@math.toronto.edu