THE GRAPH COHOMOLOGY RING OF THE GKM GRAPH OF A FLAG MANIFOLD OF TYPE G_2

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ABSTRACT. Suppose a compact torus T acts on a closed smooth manifold M. Under certain conditions, Guillemin and Zara associate to (M, T) a labeled graph $\mathcal{G}_M$ where the labels lie in $H^2(BT)$. They also define the subring $H_T^*(\mathcal{G}_M)$ of $\bigoplus_{v \in V(\mathcal{G}_M)} H^*(BT)$, where $V(\mathcal{G}_M)$ is the set of vertices of $\mathcal{G}_M$ and we call $H_T^*(\mathcal{G}_M)$ the "graph cohomology" ring of $\mathcal{G}_M$. It is known that the equivariant cohomology ring of M can be described by using combinatorial data of the labeled graph. The main result of this paper is to determine the ring structure of equivariant cohomology ring of a flag manifold of type G_2 directly, using combinatorial techniques on the graph $\mathcal{G}_M$. This gives a new computation of the equivariant cohomology ring of a flag manifold of type G_2. (See [1].)

1. INTRODUCTION

Suppose that a closed smooth manifold M has an action of a compact torus T. If the T-action is "nice", then it is known that we can describe its equivariant cohomology ring by using combinatorial way, namely by using GKM theory. By "nice" we mean that the T-action on M is GKM, namely the fixed point set of the T-action M^T is a finite set and the equivariant one-skeleton of M^T is a union of points or 2-spheres. Then we construct a graph $\mathcal{G}_M$ by replacing fixed points with vertices and 2-spheres with edges. This graph equipped with more information is defined by Guillemin and Zara [6] to be the GKM graph associated with (M, T). The object we study in this manuscript, the flag manifold of type G_2 with a standard maximal torus action, is GKM.

Interestingly, we can describe the equivariant cohomology ring of M by using the data of the GKM graph. The equivariant cohomology ring of M is defined to be the ordinary cohomology ring of Borel construction of M, namely

$$H_T^*(M ; \mathbb{Z}) := H^*(ET \times_T M ; \mathbb{Z})$$

where ET is the total space of the universal principal T-bundle $ET \to BT$. Since T acts on ET, we can consider the diagonal T-action on $ET \times M$, and its orbit space $ET \times_T M$ is called the Borel construction of M. In this paper, we treat the equivariant cohomology ring with $\mathbb{Z}$-coefficients, so we abbreviate $H_T^*(M\,;\mathbb{Z})$ as $H_T^*(M)$. Since a GKM space satisfies the condition that $H_T^*(M)$ is torsion free as a module over $H^*(BT)$, the restriction map

$$\iota^* : H_T^*(M) \to H_T^*(M^T)$$

is injective. Moreover, M^T is a finite set, hence M^T is isolated, so we have

$$H_T^*(M^T) \cong \bigoplus_{p \in M^T} H_T^*(p) \cong \bigoplus_{p \in M^T} H_T^*(BT) \cong \bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n],$$

where $n = \dim T$. Therefore, we shall regard $H_T^*(M)$ as a subring of $\bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n]$ through the map ι^*. Guillemin and Zara defined the subring, denoted by $H_T^*(\mathcal{G}_M)$, of $\bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n]$, by using the combinatorial data of the graph $\mathcal{G}_M$. Then, according to the result of Goresky-Kottwitz-MacPherson in [3], we have

$$H_T^*(M) \otimes \mathbb{Q} \cong H_T^*(\mathcal{G}_M) \otimes \mathbb{Q}.$$

If M is a flag manifold then $H_T^*(M)$ is isomorphic to $H_T^*(\mathcal{G}_M)$ without tensoring with $\mathbb{Q}$ (see [7], for example). Namely, by using data of the GKM graph, we can compute $H_T^*(M)$.

Our purpose is to determine the ring structure of $H_T^*(\mathcal{G}_M)$ for a flag manifold M of different Lie types. In the paper [2], we computed the ring structure of $H_T^*(\mathcal{G}_M)$ for a flag manifold M of classical type directly, namely without using the fact that $H_T^*(\mathcal{G}_M) \cong H_T^*(M)$, and our computation of $H_T^*(\mathcal{G}_M)$ confirms that $H_T^*(M)$ is isomorphic to $H_T^*(\mathcal{G}_M)$. The goal of this paper is to similarly determine the ring structure of $H_T^*(\mathcal{G}_M)$ for a flag manifold of type G_2. The ring structure of $H_T^*(\mathcal{G}_M)$ is given by the following theorem.

Theorem 1.1. *Let $\mathcal{G}_2$ be the labeled graph associated with the flag manifold of type G_2. Then*

$$H_T^*(\mathcal{G}_2) = \mathbb{Z}[\tau_1, \tau_2, \tau_3, t_1, t_2, t_3, f]/I,$$

where $I = (e_1(\tau), e_2(\tau) - e_2(s), 2f - e_3(\tau) - e_3(s), f^2 - fe_3(s))$, and $e_i(\tau)$ (resp. $e_i(s)$) is the ith elementary symmetric polynomial in τ_1, τ_2, τ_3 (resp. $s_1 := t_1 - t_2, s_2 := t_2 - t_3, s_3 := t_3 - t_1$).

To prove Theorem 1.1 we will use the computation of the graph cohomology of type A_2, because the labeled graph of type G_2 contains two labeled subgraphs isomorphic to the labeled graph of type A_2. In fact, the labeled graph of type G_2 can be viewed as the total space of a "GKM fiber bundle"

in the sense of Guillemin, Sabatini and Zara [5] where the type A_2 subgraph is the fiber, although we do not use this perspective in our computation.

This paper is organized as follows. In Section 2, we recall the labeled graph of a flag manifold. The labeled graph is a graph with weight attached to each edge. In Section 3, we give a different description of the Weyl group of type G_2 (the vertex set of the labeled graph) which allows us to describe the labeled graph more concretely. In Section 4, we compute the graph cohomology ring of the labeled graph.
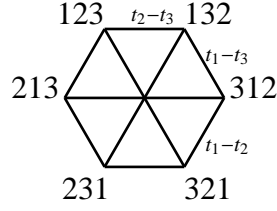
2. The labeled graph $\mathcal{G}_M$

In this section, we recall the definition of the labeled graph $\mathcal{G}_M$ for a flag manifold M. For an n-dimensional torus T, let $\{t_i\}_{i=1}^n$ be a basis of $H^2(BT)$, so that $H^*(BT)$ can be identified with the polynomial ring $\mathbb{Z}[t_1, t_2, \cdots, t_n]$. We take an inner product on $H^2(BT)$ such that the basis $\{t_i\}_{i=1}^n$ is orthonormal. The following is a simplified version of the definition of GKM graph given in [6]. To distinguish our graph from theirs, we call ours a labeled graph.

Definition 2.1. (See 2.2, [4]) Let M be a flag manifold of classical type or exceptional type, namely a flag manifold M is a homogeneous space G/T where G is a compact Lie group and T is a maximal torus of G. Suppose that $\Phi(G)$ be the root system of that type and $W(G)$ be the Weyl group. (We regard $\Phi(G)$ as a subset of $H^2(BT) = \{\sum_{i=1}^n a_i t_i \mid a_i \in \mathbb{Z}\}$.) The labeled graph $\mathcal{G}_M$ has $W(G)$ as a vertex set. Two vertices w and w' in $W(G)$ are connected by an edge e if and only if there is an element α in $\Phi(G)$ such that $w = w'\sigma_\alpha$, where σ_α is the reflection determined by α. The label of the edge e, denoted by $\ell(e)$, is given by $w\alpha$.

Remark 2.2. Guillemin and Zara [6] introduced the notion of a GKM graph which is a graph equipped with an axial function which satisfies a certain compatibility condition. They defined the graph cohomology for a GKM graph, but this definition does not use the compatibility condition of the axial function. One can also see that the graph cohomology is independent of the signs of the labels. Therefore, we omit the axial function in the data in definition 2.1, and will often disregard the signs of the labels on our labeled graph.

Example 2.3 (A_2 type). Let M be the flag manifold of type A_2, namely $M = U(3)/T$ where T is the maximal torus of $U(3)$. Then, the root system $\Phi(A_2)$ is $\{\pm(t_i - t_j) \mid 1 \leq i < j \leq 3\}$ and the Weyl group is the permutation group S_3 on three letters. We use the one-line notation $v = v(1)v(2)v(3)$ for permutations. We denote by $\mathcal{A}_3$ the labeled graph associated with $\Phi(A_2)$. It is shown in Figure 1.

FIGURE 1. $\mathcal{A}_3$

3. LABELED GRAPH OF TYPE G_2

In this section we concretely describe the labeled graph of the flag manifold associated to the compact Lie group of exceptional type G_2. Then the root system of type G_2 is known to be

$$\Phi(G_2) := \{\pm(t_{i_1} - t_{i_2}), \pm(2t_i - t_j - t_k) \mid 1 \leq i_1 < i_2 \leq 3, \{i, j, k\} = [3]\},$$

where $[3] := \{1, 2, 3\}$. Let $s_1 = t_1 - t_2$, $s_2 = t_2 - t_3$ and $s_3 = t_3 - t_1$, then

$$\Phi(G_2) = \{\pm s_k, \pm(s_i - s_j) \mid k \in [3], 1 \leq i < j \leq 3\},$$

so it is easy to see that $\Phi(G_2)$ has $\Phi(A_2)$ as a subset (but s_i's play a role of t_i's). We denote by $\mathcal{G}_2$ the labeled graph associated with $\Phi(G_2)$. The graph $\mathcal{G}_2$ has the Weyl group $W(G_2)$ of type G_2 as the vertex set. Let $\alpha_1 = s_1$ and $\alpha_2 = s_3 - s_1$ be the simple roots, then $W(G_2)$ has a presentation

$$(3.1) \qquad\qquad < \sigma_1, \sigma_2 \mid {\sigma_1}^2 = {\sigma_2}^2 = (\sigma_1 \sigma_2)^6 = 1 >,$$

where σ_i is the reflection defined by α_i for $i = 1, 2$. It is a dihedral group of order 12.

We shall give another description of $W(G_2)$ as a set not as a group, which turns out to be convenient for our purpose, and rewrite the condition about edges and labels. Let

$$\Phi = \{\pm(s_i - s_j) \mid 1 \leq i < j \leq 3\} \subset \Phi(G_2).$$

Let $W(\Phi)$ be the reflection group determined by Φ, namely

$$W(\Phi) =< \sigma_1 \sigma_2 \sigma_1, \sigma_2 >$$

because the reflections $\sigma_{s_i - s_j}$ determined by the roots $s_i - s_j$ in Φ are given by

$$\sigma_{s_3 - s_1} = \sigma_2, \ \sigma_{s_1 - s_2} = \sigma_1 \sigma_2 \sigma_1 \quad \text{and} \quad \sigma_{s_2 - s_3} = (\sigma_1 \sigma_2)^3 \sigma_1$$

It follows from (3.1) that the relations

$$\sigma_2{}^2 = (\sigma_1 \sigma_2 \sigma_1)^2 = (\sigma_1 \sigma_2 \sigma_1 \cdot \sigma_2)^3 = 1$$

hold, so we can identify $W(\Phi)$ with $W(A_3) = S_3$. We choose a group isomorphism ψ between $W(\Phi)$ and S_3 as follows;

(3.2)
$$\psi \ : \ \begin{array}{ccc} W(\Phi) & \cong & S_3 \\ \sigma_{s_i - s_j} & \longmapsto & (i, j) \end{array}$$

where (i, j) is the transposition of i and j. We note that

$$W(G_2) = W(\Phi) \amalg \rho W(\Phi) \quad \text{as a set,}$$

where $\rho := (\sigma_1 \sigma_2)^3$. (Note that ρ is the rotation by angle π.) We record the preceding discussion in a lemma.

Lemma 3.1. *Let Ψ be the map from $W(G_2)$ to $S_3 \times \{\pm\}$ defined as follows; for any w in $W(\Phi)$,*

$$\Psi(w) := (\psi(w), +) \quad and \quad \Psi(\rho w) := (\psi(w), -).$$

Then Ψ is bijective, so that one can identify $W(G_2)$ with $S_3 \times \{\pm\}$ as a set through the map Ψ.

By using the bijection Ψ, we can concretely describe the edge and the label of the graph G_2. The following lemma tells us the way to find the label $w\alpha$ in the Definition 2.1 more concretely.

Lemma 3.2. *For any w_1 and w_2 in $W(G_2)$ connected by an edge e_{w_1, w_2} labeled by $w_1\alpha$ for some α in $\Phi(G_2)$, namely $w_1 = w_2\sigma_\alpha$, one of the following occurs.*
Case 1: both w_1 and w_2 are in $W(\Phi)$. In this case there are distinct integers i and j in [3] such that $\psi(w_1)(i) = \psi(w_2)(j), \psi(w_1)(j) = \psi(w_2)(i)$ and $\ell(e_{w_1, w_2}) = s_{\psi(w_1)(i)} - s_{\psi(w_1)(j)}$.
Case 2: both w_1 and w_2 are in $\rho W(\Phi)$ so that there are unique elements w'_k in $W(\Phi)$ such that $w_k = \rho w'_k$ for $k = 1, 2$. In this case there are distinct integers i and j in [3] such that $\psi(w'_1)(i) = \psi(w'_2)(j), \psi(w'_1)(j) = \psi(w'_2)(i)$ and $\ell(e_{w_1, w_2}) = s_{\psi(w'_1)(i)} - s_{\psi(w'_1)(j)}$.
Case 3: one of w_1 and w_2 is in $W(\Phi)$ and the other is in $\rho W(\Phi)$. Without loss of generality, we may assume $w_1 \in W(\Phi)$. Then there is an element w'_2 in $W(\Phi)$ such that $w_2 = \rho w'_2$. In this case there are distinct integers i and j in [3] such that $\psi(w_1)(i) = \psi(w'_2)(j), \psi(w_1)(j) = \psi(w'_2)(i)$ and $\ell(e_{w_1, w_2}) = s_{\psi(w_1)(k)}$, where $k \in [3] \setminus \{i, j\}$.

Proof. From the definition of ψ, we have $\psi(\sigma_{s_i - s_j}) = (i, j)$ for $\{i, j\} \subset [3]$. Since the graph cohomology is independent of the signs of the label, we do not need to be careful of signs of labels of the labeled graph when we consider the graph cohomology ring of the labeled graph. Therefore, in this proof, we sometime disregard signs in front of roots.

First, we prove the lemma for cases 1 and 2. In these cases α is in Φ, so $\alpha = s_i - s_j$ for some distinct $i, \ j$ in [3].

Case 1. By assumption w_1 and w_2 are in $W(\Phi)$. Remember that $w_1 = w_2\sigma_\alpha$ and $\alpha = s_i - s_j$. Since ψ is a group isomorphism, we have

$$\psi(w_1) = \psi(w_2)\psi(\sigma_\alpha) = \psi(w_2)\psi(\sigma_{s_i-s_j}) = \psi(w_2)(i, j).$$

Therefore $\psi(w_1)(i) = \psi(w_2)(j), \psi(w_1)(j) = \psi(w_2)(i)$. In addition, since $\alpha = s_i - s_j$, we have $\ell(e_{w_1,w_2}) = w_1\alpha = w_1(s_i - s_j)$, so in order to show $\ell(e_{w_1,w_2}) = s_{\psi(w_1)(i)} - s_{\psi(w_1)(j)}$, it is enough to show that

$$(3.3) \qquad\qquad w_1(s_i - s_j) = s_{\psi(w_1)(i)} - s_{\psi(w_1)(j)}.$$

To prove this, it is enough to treat the case when $w_1 = \sigma_2$ or $\sigma_1\sigma_2\sigma_1$, because σ_2 and $\sigma_1\sigma_2\sigma_1$ are the generators of $W(\Phi)$. Since $\sigma_2 = \sigma_{s_3-s_1}$ and $\sigma_1\sigma_2\sigma_1 = \sigma_{s_1-s_2}$, we can check (3.3) this easily. In fact, for any two roots β and γ, $\sigma_\beta\gamma$ is given by $\gamma - 2\frac{\beta \cdot \gamma}{\beta \cdot \beta}\beta$ where $\cdot$ is the inner product on $H^2(BT)$ which we defined in section 2. Therefore we have

$$\sigma_{s_3-s_1}(s_i - s_j) = \begin{cases} s_3 - s_1 & \text{for } \{i, j\} = \{3, 1\} \\ s_2 - s_3 & \text{for } \{i, j\} = \{1, 2\} \\ s_1 - s_2 & \text{for } \{i, j\} = \{2, 3\} \end{cases}$$

while

$$s_{\psi(\sigma_{s_3-s_1})(i)} - s_{\psi(\sigma_{s_3-s_1})(j)} = s_{(3,1)(i)} - s_{(3,1)(j)} = \begin{cases} s_3 - s_1 & \text{for } \{i, j\} = \{3, 1\} \\ s_2 - s_3 & \text{for } \{i, j\} = \{1, 2\} \\ s_1 - s_2 & \text{for } \{i, j\} = \{2, 3\} \end{cases}$$

up to sign. Thus, (3.3) holds when $w_1 = \sigma_2 = \sigma_{s_3-s_1}$. A similar argument proves (3.3) for $w_1 = \sigma_1\sigma_2\sigma_1$.

Case 2. By assumption $w_k = \rho w'_k$ for $k = 1, 2$, where w'_1 and w'_2 are in $W(\Phi)$. Remember that $w_1 = w_2\sigma_\alpha$ and $\alpha = s_i - s_j$. Since $\rho w'_1 = \rho w'_2\sigma_\alpha$,

$$\psi(w'_1) = \psi(w'_2)\psi(\sigma_\alpha) = \psi(w'_2)(i, j).$$

Therefore $\psi(w'_1)(i) = \psi(w'_2)(j), \psi(w'_1)(j) = \psi(w'_2)(i)$, and we have

$$\ell(e_{w_1,w_2}) = w_1\alpha = \rho w'_1(s_i - s_j) = \rho(s_{\psi(w'_1)(i)} - s_{\psi(w'_1)(j)}).$$

Here ρ preserves $s_i - s_j$ up to sign because ρ is the rotation by angle π, so this completes the proof for case (2).

Case 3. By assumption w_1 is in $W(\Phi)$ and $w_2 = \rho w'_2$ with $w'_2 \in W(\Phi)$. In this case α is not in Φ, namely $\alpha = s_{k''}$ for some $k'' \in [3]$. We have

$$(3.4) \qquad\qquad w_1 = w_2\sigma_\alpha = \rho w'_2\sigma_\alpha = \rho\sigma_{w'_2(\alpha)}w'_2.$$

This equation means $\sigma_{w'_2(\alpha)} \in \rho W(\Phi)$ because w_1 and w'_2 are in $W(\Phi)$. If $w'_2(\alpha) \in \Phi$, then $\sigma_{w'_2(\alpha)}$ is in $W(\Phi)$ and this is contradiction, so there is some $k' \in [3]$ such that

$$(3.5) \qquad\qquad w'_2(\alpha) = s_{k'}$$

up to sign. The label of the edge which connects w_1 and w_2 is

(3.6) $\qquad w_1(\alpha) = w_2\sigma_\alpha(\alpha) = \rho w_2'\sigma_\alpha(\alpha) = \rho w_2'(\alpha) = \rho s_{k'} = s_{k'}$,

up to sign.

On the other hand, it follows from (3.4) and (3.5) that

(3.7) $\qquad \psi(w_1) = \psi(\rho\sigma_{w_2'(\alpha)}w_2') = \psi(\rho\sigma_{s_{k'}})\psi(w_2').$

Since

(3.8) $\qquad \rho\sigma_{s_{k'}} = \sigma_{s_{i'}-s_{j'}} \quad \text{for}\{i', j'\} = [3] \setminus \{k'\},$

it follows from (3.7) and (3.8) that

$$\begin{aligned} \psi(w_1) &= \psi(\rho\sigma_{s_k'})\psi(w_2') = \psi(\sigma_{s_{i'}-s_{j'}})\psi(w_2') \\ &= (i', j')\psi(w_2') = \psi(w_2')(\psi(w_2')^{-1}(i'), \psi(w_2')^{-1}(j')). \end{aligned}$$

Let $i := \psi(w_2')^{-1}(i')$, $j := \psi(w_2')^{-1}(j')$. Then we have $\phi(w_1)(i) = \phi(w_2')(i, j)(i) = \phi(w_2')(j)$ and $\phi(w_1)(j) = \phi(w_2')(i, j)(j) = \phi(w_2')(i)$. Let $k \in [3] \setminus \{i, j\}$, namely $k = \psi(w_2')^{-1}(k')$, then we have $\psi(w_1)(k) = \phi(w_2')(i, j)(k) = \phi(w_2')(k) = k'$. This together with (3.5) completes the proof of case (3). $\qquad\square$

Using the above lemma, we can redescribe the labeled graph associated with the root system $\Phi(G_2)$, denoted by $\mathcal{G}_2$, as follows;

- The vertex set $V(\mathcal{G}_2)$ is $\{(v, \varepsilon) \mid v \in S_3, \varepsilon = +, -\}$.
- $w_1 = (v_1, \varepsilon_1)$ and $w_2 = (v_2, \varepsilon_2)$ are connected by an edge e_{w_1,w_2} if and only if there are some integers i and j such that $v_1(i) = v_2(j), v_1(j) = v_2(i)$.
- The label of the edge e_{w_1,w_2} is $s_{v_1(i)} - s_{v_1(j)}$ if $\varepsilon_1 = \varepsilon_2$, and $s_{v_1(k)}$ if $\varepsilon_1 \neq \varepsilon_2$ where $k \in [3] \setminus \{i, j\}$.

See Figure 2. Note that in Figure 2, the parallel edges have the same label.

Remark 3.3. Let V^- (resp. V^+) be a subset of $V(\mathcal{G}_2)$ defined to be $\{(v, \varepsilon) \mid v \in S_3, \varepsilon = -(\text{resp. } +)\}$, and $\mathcal{G}_-$ (resp. $\mathcal{G}_+$) be the labeled full subgraph of $\mathcal{G}_2$ with V^- (resp. V^+) as a vertex set. Clearly $\mathcal{G}_-$ and $\mathcal{G}_+$ are isomorphic as a labeled graph to the labeled graph $\mathcal{A}_3$ associated with the root system of type A_2. Guillemin, Sabatini and Zara introduced the notion of a GKM fiber bundle in [5] (in a GKM fiber bundle, the total space, fiber and base space are all labeled graphs). In fact, it can be seen that $\mathcal{G}_2$ is the total space of a GKM fiber bundle, with fibers isomorphic to $\mathcal{A}_2$. In this sense it is natural to expect that the result of type A_2 plays a role when we determine the ring structure of $H_T^*(\mathcal{G}_2)$ below.
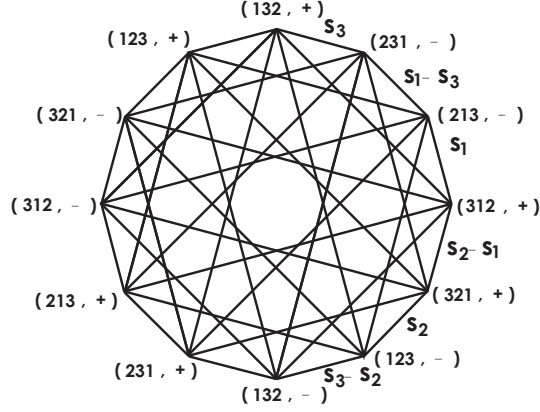
FIGURE 2. The labeled graph $\mathcal{G}_2$

4. GRAPH COHOMOLOGY RING OF $\mathcal{G}_2$

In this section, we state our main result which describes the ring structure of the graph cohomology ring of the labeled graph $\mathcal{G}_2$. We review the definition of the graph cohomology ring of a labeled graph first.

Definition 4.1. Let $\mathcal{G}$ be a labeled graph with a label ℓ taking values in $H^2(BT)$ and let W be the vertex set of $\mathcal{G}$. We identify $\bigoplus_{w \in W} H^*(BT)$ with $\mathrm{Map}(W, H^*(BT))$ where $Map(W, H^*(BT))$ is the set of all maps from W to $H^*(BT)$. Then the graph cohomology $H^*(\mathcal{G})$ of $\mathcal{G}$ is defined to be the set of all $h \in \mathrm{Map}(W, H^*(BT))$ which satisfies the so-called "GKM condition", namely for any two vertices w and w' connected by an edge e, $h(w) - h(w')$ is divisible by $\ell(e)$. The ring structure on $H^*(BT)$ induces a ring structure on $H_T^*(\mathcal{G})$.

To become familiar with a graph cohomology ring, we shall remember the graph cohomology ring of $\mathcal{A}_3$.

Example 4.2. We define elements of $H_T^*(\mathcal{A}_3)$ denoted by τ_i's and t_i's as follows;

$$\tau_i(v) := t_{v(i)} \qquad t_i(v) := t_i \quad \text{for any } v \in S_3, \ i = 1, 2, 3.$$

We regard an element in $\mathrm{Map}(W(\mathcal{A}_3), \mathbb{Z}[t_1, t_2, t_3])$ as a set of six polynomials in t_1, t_2, t_3 such that each polynomial corresponds to some vertex, because $W(\mathcal{A}_3) = S_3$ has six verticies. So the elements τ_i's are described as the following figures.
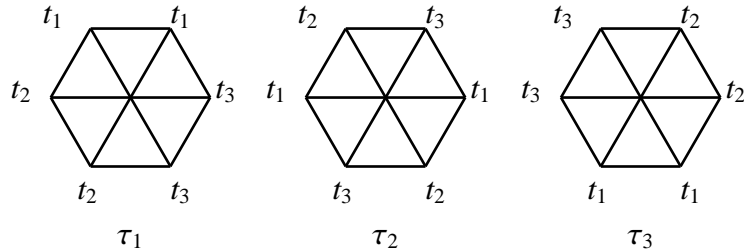
One can easily check that τ_i's are in $H_T^*(\mathcal{A}_3)$. One can also check that the elements t_i's and τ_i's generate $H_T^*(\mathcal{A}_3)$ as a ring, and

$$H_T^*(\mathcal{A}_3) \cong \mathbb{Z}[\tau_1, \tau_2, \tau_3, t_1, t_2, t_3]/J$$

where $J = < e_i(\tau) - e_i(t) \mid i = 1, 2, 3 >$ and $e_i(\tau)$ (resp. $e_i(t)$) is the ith-elementary symmetric polynomial in τ_1, τ_2, τ_3 (resp. t_1, t_2, t_3). (See [2].)

Now we consider elements in $H_T^*(\mathcal{G}_2)$. We set $s_{w(i)} := \varepsilon s_{v(i)}$ for $w = (v, \varepsilon) \in V(\mathcal{G}_2)$. For each $i = 1, 2, 3$, we define elements τ_i, t_i of $\mathrm{Map}(V(\mathcal{G}_2), \mathbb{Z}[t_1, t_2, t_3])$ by

(4.1) $\tau_i(w) := s_{w(i)} = \varepsilon s_{v(i)}, \quad t_i(w) := t_i \quad \text{for} \quad w = (v, \varepsilon) \in V(\mathcal{G}_2)$.

One can check that τ_i's, t_i's are elements of $H_T^*(\mathcal{G}_2)$. We set $s_i := t_i - t_{i+1}$ in $\mathrm{Map}(V(\mathcal{G}_2), \mathbb{Z}[t_1, t_2, t_3])$, namely $s_i(w) = (t_i - t_{i+1})(w) = t_i - t_{i+1} = s_i$, where $t_4 = t_1$. In addition, we define an element f of $\mathrm{Map}(V(\mathcal{G}_2), \mathbb{Z}[t_1, t_2, t_3])$ by

(4.2) $f(w) := \begin{cases} s_1 s_2 s_3 & \text{for} \quad w = (v, +) \\ 0 & \text{for} \quad w = (v, -). \end{cases}$

Clearly, f is also an element of $H_T^*(\mathcal{G}_2)$. Elements τ_1 and f are described in Figures 3 and 4.
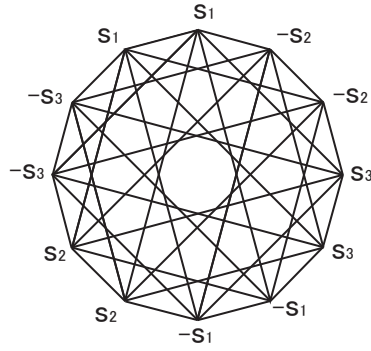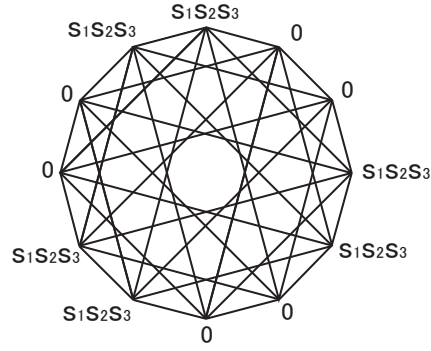
FIGURE 3. τ_1 FIGURE 4. f

Remark 4.3. The restriction of τ_i to the subgraph $\mathcal{G}_+$ (resp. $\mathcal{G}_-$) is τ_i (resp. $-\tau_i$) in $H_T^*(\mathcal{A}_3)$. In other words τ_i in $H_T^*(\mathcal{G}_2)$ is a lift of the τ_i in $H_T^*(\mathcal{A}_3)$. The element f comes from some element in the graph cohomology ring of

the labeled graph of the base space. In fact, the labeled graph of the base space, denoted by $\mathcal{B}$, is described as Figure 5, and Figure 6 describes an element in $H_T^*(\mathcal{B}) \subset \mathrm{Map}(V(\mathcal{B}), \mathbb{Z}[t_1, t_2, t_3])$. The element f in $H_T^*(\mathcal{G}_2)$ is the pullback of the element in Figure 6 by the projection $\mathcal{G}_2 \to \mathcal{B}$.
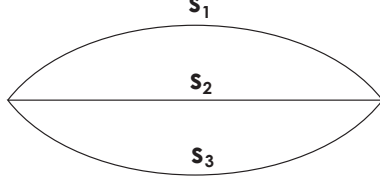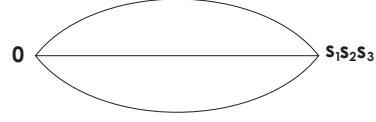

FIGURE 5

FIGURE 6

Guillemin, Sabatini and Zara [5] construct module generators of the graph cohomology of the total space over the graph cohomology of the base space, by using module generators of the graph cohomology of fiber. (They consider the graph cohomology with $\mathbb{R}$ coefficient.)

Theorem 4.4. *Let $\mathcal{G}_2$ be the labeled graph associated with the root system $\Phi(G_2)$ of type G_2. Then*

$$H_T^*(\mathcal{G}_2) = \mathbb{Z}[\tau_1, \tau_2, \tau_3, t_1, t_2, t_3, f]/I,$$

where $I = < e_1(\tau), e_2(\tau) - e_2(s), 2f - e_3(\tau) - e_3(s), f^2 - f e_3(s) >$, and $e_i(\tau)$ (resp. $e_i(s)$) is the ith elementary symmetric polynomial in τ_1, τ_2, τ_3 (resp. s_1, s_2, s_3).

The rest of the paper is devoted to the proof of Theorem 4.4.

Lemma 4.5. *$H_T^*(\mathcal{G}_2)$ is generated by $\tau_1, \tau_2, \tau_3, t_1, t_2, t_3$ and f as a ring.*

Proof. The idea of the proof of the lemma is same as that of Lemma 3.2 in [2].

Claim 4.6. For any homogeneous element h of $H_T^*(\mathcal{G}_2)$, there is a polynomial G in τ_i's and t_i's such that the restrictions of h and G to the subgraph $\mathcal{G}_-$ coincide.

Proof of Claim. We set

$$V_i^- := \{(v, \varepsilon) \in S_3 \times \{\pm\} \mid v(i) = 3, \varepsilon = -\} \quad \text{for} \quad i \in [3].$$

Let $2k$ be the degree of $h \in H_T^*(\mathcal{G}_2)$ and

$$1 \leq q \leq \min\{k + 1, 3\}$$

and assume that

$$h(w_i) = 0 \quad \text{for any } w_i \in V_i^- \text{ whenever } i < q.$$

For any $w = (v, -) \in V_q^-$, there is a unique vertex $w_i = (v_i, -) \in V_i^-$ for each $1 \le i < q$ such that w and w_i are connected by an edge labeled by $s_{w(i)} - s_{w(q)} = s_{w(i)} + s_3$. (Namely $v = v_i(i, q)$.) Then $h(w) - h(w_i) = h(w)$ is divisible by $s_{w(i)} + s_3$ for $i < q$, so there is a homogeneous polynomial $g_w \in \mathbb{Z}[t_1, t_2, t_3]$ of degree $2(k - q + 1)$ such that

$$(4.3) \qquad h(w) = g_w \prod_{i=1}^{q-1} (s_{w(i)} + s_3).$$

On the other hand, since $(\tau_k + s_3)(u) = s_{u(k)} + s_3$ for any $u \in V(\mathcal{G}_2)$,

$$(4.4) \qquad \prod_{i=1}^{q-1} (\tau_i + s_3)(u) = \prod_{i=1}^{q-1} (s_{u(i)} + s_3).$$

In particular, when u is in V_i^- for $i < q$, (4.4) is equal to 0 since $s_{u(i)} = -s_3$. So, it follows from (4.3) and (4.4) that

$$(4.5) \qquad \left(h - g_w \prod_{i=1}^{q-1} (\tau_i + s_3) \right)(u) = 0,$$

whenever $u \in V_i^-$ for $i < q$ or $u = w$. We set

$$(4.6) \qquad h' := h - g_w \prod_{i=1}^{q-1} (\tau_i + s_3).$$

Note that

$$(4.7) \qquad h'(u) = 0 \quad \text{if } u \in V_i^- \text{ for } i < q \text{ or } u = w.$$

Let w' be the other vertex in V_q^-. (Namely, $V_q^- = \{w, w'\}$.) Then w and w' are connected by an edge labeled by $s_1 - s_2$ and there is a unique vertex $w_i' = (v_i', -) \in V_i^-$ for each $1 \le i < q$ such that w' and w_i' are connected by an edge labeled by $s_{w'(i)} - s_{w'(q)} = s_{w'(i)} + s_3$, so $h'(w') - h'(w) = h'(w')$ is divisible by $s_1 - s_2$ and $s_{w'(i)} + s_3$. Therefore, there is a homogeneous polynomial $g_{w'} \in \mathbb{Z}[t_1, t_2, t_3]$ of degree $2(k - q)$ such that

$$(4.8) \qquad h'(w') = (s_1 - s_2) g_{w'} \prod_{i=1}^{q-1} (s_{w'(i)} + s_3).$$

On the other hand, for $j \in [3] \setminus \{q\}$,

$$(4.9) \qquad \left(\tau_j - s_{w(j)} \right)(w') \quad = \quad s_{w'(j)} - s_{w(j)} = \delta(s_1 - s_2)$$

$$(4.10) \qquad \left(\tau_j - s_{w(j)} \right)(w) \quad = \quad s_{w(j)} - s_{w(j)} = 0,$$

where $\delta = \pm 1$ and δ depends on w. Thus, it follows from (4.4), (4.8), (4.9) and (4.10) that

$$\left(h' - (\tau_j - s_{w(j)})(\delta g_{w'}) \prod_{k=1}^{q-1} (\tau_k + s_3) \right)(u) = 0.$$

for $u \in V_i^-$ whenever $i \le q$. Therefore, putting $H = g_w + (\tau_j - s_{w(j)})(\delta g_{w'})$, and subtracting the polynomial $H\Pi_{k=1}^{q-1}(\tau_j + s_3)$ from h, we may assume that

$$h(u) = 0 \quad \text{for any} \quad u \in V_i^- \quad \text{whenever} \quad i < q + 1.$$

The above argument implies that h finally takes zero on all vertices in V^- by subtracting a polynomial in τ_i's and t_i's, and this completes the proof of the claim.

The claim allows us to assume that our homogeneous element h in $H_T^*(\mathcal{G}_2)$ satisfies $h(u) = 0$ for all $u \in V^-$. Any $w = (v, +) \in V^+$ has a unique edge which connects w and some $w_i = (v_i, -) \in V_i^-$ for each $i \in [3]$, and the edge has a label $s_{w(k)}$ where k is determined by $v(k) = v_i(k)$. Namely, $h(w) - h(w_i) = h(w)$ is divisible by $s_{w(k)}$, thus, there is a homogeneous polynomial p_w of $\mathbb{Z}[t_1, t_2, t_3]$ such that

$$h(w) = s_1 s_2 s_3 p_w.$$

In addition the collection of polynomials $\{p_w\}$ satisfies the GKM condition in $\mathcal{G}_+$. In fact, for any vertices w and w' in V^+ connected by an edge with label $s_i - s_j$ for some $i, j \in [3]$, it follows from the definition of $H_T^*(\mathcal{G}_2)$ that $h(w) - h(w') = s_1 s_2 s_3(p_w - p_{w'})$ is divisible by $s_i - s_j$. So $p_w - p_{w'}$ is divisible by the label $s_i - s_j$. Then the same argument as in the proof of the claim above shows that there is a polynomial H' in τ_i's and t_i's such that $H'(w) = p_w$ for $w \in V^+$. Therefore,

$$(h - H'f)(w) = 0 \quad \text{for any} \quad w \in V(\mathcal{G}_2).$$

This completes the proof of the lemma.   □

Remember that the Hilbert series of a graded ring $A^* = \oplus_{j=0}^{\infty} A^j$, where A^j is the degree j part of A^* and of finite rank over $\mathbb{Z}$, is a formal power series defined by

$$F(A^*, x) := \sum_{j=0}^{\infty} (\text{rank}_{\mathbb{Z}} A^j)x^j.$$

Lemma 4.7.

$$F(H_T^*(\mathcal{G}_2), x) = \frac{1}{(1 - x^2)^3}(1 + x^2)(1 + x^2 + x^4)(1 + x^6)$$

Proof. We set $d(k) := \operatorname{rank} H_T^{2k}(\mathcal{G}_2)$ and $r(k) := \operatorname{rank} \mathbb{Z}[t_1, t_2, t_3]^{2k}$, where $\mathbb{Z}[t_1, t_2, t_3]^{2k}$ is the set of homogeneous polynomials of degree $2k$. We note that the degree of t_i is 2. For $h \in H_T^{2k}(\mathcal{G}_2)$, assume that there is some $q \in [3]$ such that

$$h(w) = 0 \text{ for } w \in V_i^- \text{ whenever } i < q.$$

Then, the proof of the claim in Lemma 4.5 shows that there is a polynomial H in τ_i's and t_i's such that

$$(h - H)(w) = 0 \quad \text{for} \quad w \in V_i^- \quad \text{whenever } i \le q,$$

and that the polynomial H is of the form

$$H = \left(g_w + \delta(\tau_j - s_{w(j)})g_{w'}\right) \prod_{k=1}^{q-1} (\tau_k + s_3)$$

where g_w and $g_{w'}$ are some polynomials in t_1, t_2, t_3, and the degree of g_w (resp. $g_{w'}$) is $2(k - q + 1)$ (resp. $2(k - q)$). Therefore the rank of the additive group consisting of all such polynomials H is given by

$$r(k - q + 1) + r(k - q).$$

Similarly, assume that there is some $q \in [3]$ such that

$$h(w) = 0 \text{ for } w \in V^- \text{ or } w \in V_i^+ \text{ whenever } i < q,$$

where $V_i^+ = \{(v, +) \mid v(i) = 3\}$. Then, the proof of Lemma 4.5 shows that there is a polynomial H' in τ_i's, t_i's and f such that

$$(h - H')(w) = 0 \quad \text{for} \quad w \in V^- \quad \text{or} \quad w \in V_i^+ \quad \text{whenever} \quad i \le q,$$

and that the polynomial H' is of the form

$$H' = s_1 s_2 s_3 \left(g_w' + \delta(\tau_j - s_{w(j)})g_{w'}'\right) \prod_{k=1}^{q-1} (\tau_k + s_3)$$

where w and w' are the two vertices of V_q^+ and g_w' and $g_{w'}'$ are some polynomials in t_1, t_2, t_3, and the degree of g_w' (resp. $g_{w'}'$) is $2(k - 2 - q)$ (resp. $2(k - 3 - q)$). Thus, the rank of the the additive group consisting of all such polynomials H' is given by

$$r(k - 2 - q) + r(k - 3 - q).$$

Let G_q^+ (resp. G_q^-) be the the additive group consisting of all polinomials h of degree $2k$ such that

$$h(w) = 0 \text{ for } w \in V^- \text{ or } w \in V_i^+ \text{ whenever } i < q$$

$$(\text{resp. } h(w) = 0 \text{ for } w \in V_i^- \text{whenever } i < q).$$

Then, the rank of G_3^+ is $r(k-5) + r(k-6)$ by the above argument. The rank of G_2^+ is equal to rank G_3^+ plus the rank of additive group consisting of all polynomial H' such that

$$(h - H')(w) = 0 \quad \text{for} \quad w \in V^- \quad \text{or} \quad w \in V_i^+ \quad \text{whenever} \quad i \le 2,$$

so the rank of G_2^+ is rank $G_3^+ + r(k-4) + r(k-5)$. Similarly, the rank of G_1^+ is equal to rank G_2^+ plus $r(k-3) + r(k-4)$, namely

$$(4.11) \qquad\qquad \operatorname{rank} G_1^+ = \sum_{i=4}^{6} (r(i) + r(i-1)).$$

In the same way, we have

$$(4.12) \qquad\qquad \operatorname{rank} G_3^- = \operatorname{rank} G_1^+ + r(k-2) + r(k-3),$$

and

$$(4.13) \qquad \operatorname{rank} G_q^- = \operatorname{rank} G_{q+1}^- + r(k-q+1) + r(k-q) \quad \text{for } q = 1, 2.$$

Therefore, it follows from (4.11), (4.12) and (4.11) that

$$d(k) = \operatorname{rank} G_1^- = \sum_{i=k}^{k-5} \left(r(i) + r(i-1) \right) = r(k) + 2 \sum_{i=k-5}^{k-1} r(i) + r(k-6)$$

where $r(i) = 0$ for $i < 0$. Namely,

$$d(k) = \begin{cases} r(0) & \text{for} \quad k = 0 \\ 2 \sum_{i=0}^{k-1} r(i) + r(k) & \text{for} \quad 1 \le k \le 5 \\ r(k-6) + 2 \sum_{i=k-5}^{k-1} r(i) + r(k) & \text{for} \quad k > 5. \end{cases}$$

Therefore,

$$
\begin{aligned}
F(H_T^*(\mathcal{G}_2), x) &= \sum_{k=0}^{\infty} d(k) x^{2k} \\
&= r(0) + \sum_{k=1}^{5} \left(2 \sum_{i=0}^{k-1} r(i) + r(k) \right) x^{2k} \\
&\quad + \sum_{k=6}^{\infty} \left(r(k-6) + 2 \sum_{i=k-5}^{k-1} r(i) + r(k) \right) x^{2k} \\
&= \sum_{k=0}^{\infty} r(k) x^{2k} + \sum_{k=6}^{\infty} r(k-6) x^{2k} \\
&\quad + 2 \left(\sum_{k=1}^{5} \sum_{i=0}^{k-1} r(i) x^{2k} + \sum_{k=6}^{\infty} \sum_{i=k-5}^{k-1} r(i) x^{2k} \right) \\
&= (1 + x^{12})(r(0) + r(1)x^2 + r(2)x^4 + \cdots) \\
&\quad + 2(x^2 + x^4 + x^6 + x^8 + x^{10})(r(0) + r(1)x^2 + r(2)x^4 + \cdots) \\
&= (1 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + 2x^{10} + x^{12}) F(\mathbb{Z}[t_1, t_2, t_3], x) \\
&= (1 + x^2)(1 + x^2 + x^4)(1 + x^6) F(\mathbb{Z}[t_1, t_2, t_3], x) \\
&= \frac{1}{(1 - x^2)^3} (1 + x^2)(1 + x^2 + x^4)(1 + x^6),
\end{aligned}
$$

proving the lemma.   □

We abbreviate the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \tau_3, t_1, t_2, t_3, f]$ as $\mathbb{Z}[\tau, t, f]$. The canonical map $\mathbb{Z}[\tau, t, f] \to H_T^*(\mathcal{G}_2)$ is a grade preserving homomorphism which is surjective by Lemma 4.5. It easily follows from (4.1) and (4.2) that

$$(4.14) \qquad\qquad e_1(\tau) = 0,$$

$$(4.15) \qquad\qquad e_2(\tau) - e_2(s) = 0,$$

$$(4.16) \qquad 2f - e_3(\tau) - e_3(s) = 0, \qquad \text{and}$$

$$(4.17) \qquad\qquad f^2 - f e_3(s) = 0.$$

Therefore the canonical map above induces a grade preserving epimorphism

$$(4.18) \qquad\qquad \mathbb{Z}[\tau, t, f]/I \to H_T^*(\mathcal{G}_2),$$

where

$$I = < e_1(\tau), e_2(\tau) - e_2(s), 2f - e_3(\tau) - e_3(s), f^2 - f e_3(s) > .$$

We note that $\mathbb{Z}[\tau, t, f]/I$ is a $\mathbb{Z}[t]$-module in a natural way.

Lemma 4.8. $\mathbb{Z}[\tau, t, f]/I$ *is generated by* $\prod_{k=1}^{3} \tau_k^{i_k} f^j$ *for* $0 \le i_k \le 3 - k$ *and* $j = 0, 1$ *as a* $\mathbb{Z}[t]$*-module.*

Proof. Clearly the elements $\prod_{k=1}^{3} \tau_k^{i_k} f^j$ with no restriction on exponents, generate $\mathbb{Z}[\tau, t, f]/I$ as a $\mathbb{Z}[t]$-module. The identity (4.14) means

$$(4.19) \qquad\qquad \tau_3 = -\tau_1 - \tau_2.$$

By using (4.19), we have

$$e_2(\tau) = \tau_1 \tau_2 + \tau_3(\tau_1 + \tau_2) = \tau_1 \tau_2 - (\tau_1 + \tau_2)^2 = -\tau_1^2 - \tau_2^2 - \tau_1 \tau_2.$$

Therefore, $\tau_2^2 = -\tau_1^2 - \tau_1 \tau_2 - e_2(\tau)$, and hence

$$(4.20) \qquad\qquad \tau_2^2 = -\tau_1^2 - \tau_1 \tau_2 - e_2(s)$$

by (4.15). It follows from (4.19) and (4.20) that

$$
\begin{aligned}
e_3(\tau) &= \tau_1 \tau_2 \tau_3 \\
&= -\tau_1 \tau_2 (\tau_1 + \tau_2) \\
&= -\tau_1^2 \tau_2 - \tau_1 \tau_2^2 \\
&= -\tau_1^2 \tau_2 + \tau_1(\tau_1^2 + \tau_1 \tau_2 + e_2(s)) \\
&= \tau_1^3 + \tau_1 e_2(s),
\end{aligned}
$$

Therefore $\tau_1^3 = -\tau_1 e_2(s) + e_3(\tau)$ and hence

$$(4.21) \qquad\qquad \tau_1^3 = -\tau_1 e_2(s) + 2f - e_3(s)$$

by (4.16). Therefore, τ_k^{3-k+1} is written as (4.19), (4.20) and (4.21), satisfying that the exponent of τ_k is less than or equal to $3 - k$ and that of f is 0 or 1. In addition, f^2 is written as (4.17), so we can always assume the exponent of f to be 0 or 1. This completes the proof of the lemma.   □

Now we are in a position to complete the proof of Theorem 4.4.

Proof of Theorem 4.4. If two formal power series $a(x) = \sum_{i=0}^{\infty} a_i x^i$ and $b(x) = \sum_{i=0}^{\infty} b_i x^i$ in x with real coefficients a_i and b_i satisfy $a_i \le b_i$ for every i, then we express this as $a(x) \le b(x)$.

The Hilbert series of the free $\mathbb{Z}[t]$-module generated by $\prod_{k=1}^{3} \tau_k^{i_k} f^j$ is given by $\frac{1}{(1-x^2)^3} x^{2(\sum_{k=1}^{3} i_k + 3j)}$, so it follows from Lemma 4.8 that

$$(4.22) \qquad F(\mathbb{Z}[\tau, t, f]/I, x) \le \frac{1}{(1 - x^2)^3} \sum_{0 \le i_1 \le 2,\ 0 \le i_2 \le 1} x^{2(i_1 + i_2)} \sum_{j=0}^{1} x^{6j}$$

and the equality holds above if and only if $\mathbb{Z}[\tau, t, f]/I$ is free as a $\mathbb{Z}[t]$-module. Here the right hand side in (4.22) above is equal to

$$\frac{1}{(1 - x^2)^3}(1 + x^2)(1 + x^2 + x^4)(1 + x^6)$$

which agrees with $F(H_T^*(\mathcal{G}_2), x)$ by Lemma 4.7. Therefore $F(\mathbb{Z}[\tau, t, f]/I, x) \leq F(H_T^*(\mathcal{G}_2), x)$. On the other hand, the surjectivity of the map (4.18) implies the opposite inequality. Therefore $F(\mathbb{Z}[\tau, t, f]/I, x) = F(H_T^*(\mathcal{G}_2), x)$. This means that the inequality in (4.22) must be an equality and hence $\mathbb{Z}[\tau, t, f]/I$ is free as a $\mathbb{Z}[t]$-module, in particular, as a $\mathbb{Z}$-module. Since the map in (4.18) is surjective and $F(\mathbb{Z}[\tau, t, f]/I, x) = F(H_T^*(\mathcal{G}_2), x)$, we conclude that the map in (4.18) is actually an isomorphism. This proves Theorem 4.4. $\square$

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