LATTICE MULTI-POLYGONS

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Abstract. We discuss generalizations of some results on lattice polygons to certain piecewise linear loops which may have a self-intersection but have vertices in the lattice $\mathbb{Z}^2$. We first prove a formula on the rotation number of a unimodular sequence in $\mathbb{Z}^2$ using toric topology. This formula implies the generalized twelve-point theorem in [11]. We then introduce the notion of lattice multi-polygons which is a generalization of lattice polygons, state the generalized Pick's formula and discuss the classification of Ehrhart polynomials of lattice multi-polygons and also of several natural subfamilies of lattice multi-polygons.

Introduction

Lattice polygons are an elementary but fascinating object. Many interesting results such as Pick's formula are known for them. However, not only the results are interesting, but also there are a variety of proofs to the results and some of them use advanced mathematics such as toric geometry, complex analysis and modular form (see [4, 3, 9, 11] for example). These proofs are unexpected and make the study of lattice polygons more fruitful and intriguing.

Some of the results on lattice polygons are generalized to certain generalized polygons. For instance, Pick's formula [10]

$$A(P) = \sharp P^\circ + \frac{1}{2}B(P) - 1$$

for a lattice polygon P, where $A(P)$ is the area of P and $\sharp P^\circ$ (resp. $B(P)$) is the number of lattice points in the interior (resp. on the boundary) of P, is generalized in several directions and one of the generalizations is to certain piecewise linear loops which may have a self-intersection but have vertices in $\mathbb{Z}^2$ ([5, 8]). As is well known, Pick's formula can be proved using toric geometry when P is convex ([4, 9]) but the proof was not applicable when P is concave. However, once we develope toric geometry from the topological point of view, that is *toric topology*, Pick's formula can be proved along the same line in full generality as is done in [8].

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Another such result on lattice polygons is the twelve-point theorem. It says that if P is a convex lattice polygon which contains the origin in its interior as a unique lattice point, then

$$B(P) + B(P^\vee) = 12,$$

where $P^\vee$ is the lattice polygon dual to P. Several proofs are known to the theorem and one of them again uses toric geometry. B. Poonen and F. Rodriguez-Villegas [11] provided a new proof using modular forms. They also formulate a generalization of the twelve-point theorem and claim that their proof works in the general setting. Generalized polygons considered in the generalization are what is called *legal loops*. A legal loop may be regarded as a unimodular sequence of vectors in $\mathbb{Z}^2$. It is mentioned in [11] that the proof using toric geometry is difficult to generalize, but we will show in this paper that a slight generalization of the proof of [8, Theorem 5.1], which uses toric topology and is on the same line of the proof using toric geometry, implies the generalized twelve-point theorem.

We also introduce the notion of lattice multi-polygons. A *lattice multi-polygon* is a piecewise linear loop with vertices in $\mathbb{Z}^2$ together with a sign function which assigns either $+$ or $-$ to each side and satisfies some mild condition. The piecewise linear loop may have a self-intersection and we think of it as a sequence of points in $\mathbb{Z}^2$. A lattice polygon can naturally be regarded as a lattice multi-polygon. The generalized Pick's formula holds for lattice multi-polygons, so Ehrhart polynomials can be defined for them. The Ehrhart polynomial of a lattice multi-polygon is of degree at most two. The constant term is the rotation number of normal vectors to sides of the multi-polygon and not necessarily 1 unlike ordinary Ehrhart polynomials. The other coefficients have similar geometrical meaning to the ordinary ones but they can be zero or negative unlike the ordinary ones. The family of lattice multi-polygons has some natural subfamilies, e.g. the family of all convex lattice polygons. We discuss the characterization of Ehrhart polynomials of not only all lattice multi-polygons but also of some natural subfamilies.

The structure of the present paper is as follows. In Section 1, we give a formula (Theorem 1.1) which describes the rotation number of a unimodular sequence of vectors in $\mathbb{Z}^2$ around the origin. Here the vectors in the sequence may go back and forth. The proof uses toric topology. In Section 2, we observe that Theorem 1.1 implies the generalized twelve-point theorem. In Section 3, we introduce the notion of lattice multi-polygon and state the generalized Pick's formula for lattice multi-polygons. In Section 4, we discuss the characterization of Ehrhart polynomials of lattice multi-polygons and of several natural subfamilies of lattice multi-polygons.

1. Rotation number of a unimodular sequence

We say that a sequence of vectors $v_1, \dots, v_d$ in $\mathbb{Z}^2$ ($d \geq 3$) is *unimodular* if each successive pair (v_i, v_{i+1}) is a basis of $\mathbb{Z}^2$ for $i = 1, \dots, d$, where $v_{d+1} =$

v_1. The vectors in the sequence are not necessarily in counterclockwise or clockwise order. They may go back and forth. We set

$$\epsilon_i = \det(v_i, v_{i+1}) \quad \text{for } i = 1, \dots, d.$$

Since each successive pair (v_i, v_{i+1}) is a basis of $\mathbb{Z}^2$ for $i = 1, \dots, d$, we have $\epsilon_i = \pm 1$ and

$$(v_i, v_{i+1}) = (v_{i-1}, v_i)\begin{pmatrix} 0 & -\epsilon_{i-1}\epsilon_i \\ 1 & -\epsilon_i a_i \end{pmatrix}$$

with some unique $a_i \in \mathbb{Z}$ for each i. The above identity is equivalent to

$$\epsilon_i v_{i+1} + \epsilon_{i-1} v_{i-1} + a_i v_i = 0. \tag{1.1}$$

Theorem 1.1. *The rotation number of the unimodular sequence $v_1, \dots, v_d$ ($d \geq 3$) around the origin is given by*

$$\frac{1}{12}\Big(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i\Big).$$

Proof. This is proved in [8, Section 5] when $\epsilon_i = 1$ for every i and the argument there works in our general setting with a little modification, which we shall explain.

We identify $\mathbb{Z}^2$ with $H_2(BT)$ where $T = (S^1)^2$ and BT is the classifying space of T. We may think of BT as $(\mathbb{C}P^\infty)^2$. For each i ($i = 1, \dots, d$), we form a cone $\angle v_i v_{i+1}$ in $\mathbb{R}^2$ spanned by v_i and v_{i+1} and attach the sign ϵ_i to the cone. The collection of the cones $\angle v_i v_{i+1}$ with the signs ϵ_i attached form a multi-fan Δ and the same construction as in [8, Section 5] produces a real 4-dimensional closed connected smooth manifold M with an action of T satisfying the following conditions:

1. $H^{odd}(M) = 0$.
2. M admits a unitary (or weakly complex) structure preserved under the T-action and the multi-fan associated to M with this unitary structure is the given Δ.
3. Let M_i ($i = 1, \dots, d$) be the characteristic submanifold of M corresponding to the edge vector v_i, that is, M_i is a real codimension two submanifold of M fixed pointwise under the circle subgroup determined by the v_i. Then M_i does not intersect with M_j unless $j = i-1, i, i+1$ and the intersection numbers of M_i with M_{i-1} and M_{i+1} are ϵ_{i-1} and ϵ_i respectively.

Choose an arbitrary element $v \in \mathbb{R}^2$ not contained in any one-dimensional cone in the multi-fan Δ. Then Theorem 4.2 in [8] says that the Todd genus $T[M]$ of M is given by

$$T[M] = \sum_i \epsilon_i, \tag{1.2}$$

where the sum above runs over all i's such that the cone $\angle v_i v_{i+1}$ contains the vector v. Clearly the right hand side in (1.2) agrees with the rotation number of the sequence $v_1, \dots, v_d$ around the origin. In the sequel, we compute the Todd genus $T[M]$.

Let $ET \to BT$ be the universal principal T-bundle and M_T the quotient of $ET \times M$ by the diagonal T-action. The space M_T is called the Borel construction of M and the equivariant cohomology $H^q_T(M)$ of the T-space M is defined to be $H^q(M_T)$. The first projection from $ET \times M$ onto ET induces a fibration

$$\pi\colon M_T \to ET/T = BT$$

with fiber M. The inclusion map ι of the fiber M to M_T induces a surjective homomorphism $\iota^*\colon H^q_T(M) \to H^q(M)$.

Let $\xi_i \in H^2_T(M)$ be the Poincaré dual to the cycle M_i in the equivariant cohomology. The ξ_i restricts to the ordinary Poincaré dual $x_i \in H^2(M)$ to the cycle M_i through the ι^*. By Lemma 1.5 in [8], we have

$$\pi^*(u) = \sum_{j=1}^{d} \langle u, v_j\rangle \xi_j \quad \text{for any } u \in H^2(BT), \tag{1.3}$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology. Multiplying the both sides of (1.3) by ξ_i and restricting the resulting identity to the ordinary cohomology by ι^*, we obtain

$$0 = \langle u, v_{i-1}\rangle x_{i-1}x_i + \langle u, v_i\rangle x_i^2 + \langle u, v_{i+1}\rangle x_{i+1}x_i \quad \text{for all } u \in H^2(BT) \tag{1.4}$$

because M_i does not intersect with M_j unless $j = i-1, i, i+1$, where $x_{d+1} = x_1$. We evaluate the both sides of (1.4) on the fundamental class $[M]$ of M. Since the intersection numbers of M_i with M_{i-1} and M_{i+1} are respectively ϵ_{i-1} and ϵ_i as mentioned above, the identity (1.4) reduces to

$$0 = \langle u, v_{i-1}\rangle \epsilon_{i-1} + \langle u, v_i\rangle \langle x_i^2, [M]\rangle + \langle u, v_{i+1}\rangle \epsilon_i \quad \text{for all } u \in H^2(BT) \tag{1.5}$$

and further reduces to

$$0 = \epsilon_{i-1}v_{i-1} + \langle x_i^2, [M]\rangle v_i + \epsilon_i v_{i+1} \tag{1.6}$$

because (1.5) holds for any $u \in H^2(BT)$. Comparing (1.6) with (1.1), we conclude that $\langle x_i^2, [M]\rangle = a_i$. Summing up the above argument, we have

$$\langle x_i x_j, [M]\rangle = \begin{cases} \epsilon_{i-1} & \text{if } j = i-1, \\ a_i & \text{if } j = i, \\ \epsilon_i & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

By Theorem 3.1 in [8] the total Chern class $c(M)$ of M with the unitary structure is given by $\prod_{i=1}^d (1 + x_i)$. Therefore

$$c_1(M) = \sum_{i=1}^{d} x_i, \qquad c_2(M) = \sum_{i<j} x_i x_j$$

and hence

$$\begin{aligned} T[M] &= \frac{1}{12}\langle c_1(M)^2 + c_2(M), [M]\rangle \\ &= \frac{1}{12}\langle (\sum_{i=1}^d x_i)^2 + \sum_{i<j} x_i x_j, [M]\rangle \\ &= \frac{1}{12}(\sum_{i=1}^d a_i + 3\sum_{i=1}^d \epsilon_i), \end{aligned}$$

where the first identity is known as Noether's formula when M is an algebraic surface and known to hold even for unitary manifolds, and we used (1.7) at the last identity. This proves the theorem because $T[M]$ agrees with the desired rotation number as remarked at (1.2). □

2. Generalized twelve-point theorem

Let P be a convex lattice polygon whose only interior lattice point is the origin. Then the dual $P^\vee$ to P is also a convex lattice polygon whose only interior lattice point is the origin. Let $B(P)$ denote the total number of the lattice points on the boundary of P. The following fact is well known.

Theorem 2.1 (Twelve-point theorem)**.** $B(P) + B(P^\vee) = 12$.

Several proofs are known for this theorem ([1, 2, 11]). B. Poonen and F. Rodriguez-Villegas give a proof using modular forms in [11]. They also formulate a generalization of the twelve-point theorem and claim that their proof works in the general setting. In this section, we will explain the generalized twelve-point theorem and observe that it follows from Theorem 1.1.

If P is a convex lattice polygon whose only interior lattice point is the origin and $v_1, \dots, v_d$ are the vertices of P arranged counterclockwise, then every v_i is primitive and the triangle with the vertices v_i, v_{i+1} and the origin has no lattice point in the interior for each i, where $v_{d+1} = v_1$ as usual. This observation motivates the following definition, see [11, 1].

Definition. A sequence of vectors $\mathcal{P} = (v_1, \dots, v_d)$, where $v_1, \dots, v_d$ are in $\mathbb{Z}^2$ and $d \geq 3$, is called a *legal loop* if every v_i is primitive and whenever $v_i \neq v_{i+1}$, v_i and v_{i+1} are linearly independent (i.e. $v_i \neq -v_{i+1}$) and the triangle with the vertices v_i, v_{i+1} and the origin has no lattice point in the interior. We say that a legal loop is *reduced* if $v_i \neq v_{i+1}$ for any i. A (non-reduced) legal loop $\mathcal{P}$ naturally determines a reduced legal loop, denoted $\mathcal{P}_{red}$, by dropping all the redundant points. We define the *winding number* of a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ to be the rotation number of the vectors $v_1, \dots, v_d$ around the origin.

Joining successive points in a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ by straight lines forms a lattice polygon which may have a self-intersection. A unimodular sequence $v_1, \dots, v_d$ determines a reduced legal loop. Conversely, a reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$ determines a unimodular sequence by adding all the lattice points on the line segment v_iv_{i+1} (called a *side* of $\mathcal{P}$) connecting v_i and v_{i+1} for every i. To each side v_iv_{i+1} with $v_i \neq v_{i+1}$, we assign the sign of $\det(v_i, v_{i+1})$, denoted $\operatorname{sgn}(v_i, v_{i+1})$.

Definition. Let $|v_iv_{i+1}|$ be the number of lattice points on the side v_iv_{i+1} minus 1, so $|v_iv_{i+1}| = 0$ when $v_i = v_{i+1}$. Then we define

$$B(\mathcal{P}) = \sum_{i=1}^{d} \operatorname{sgn}(v_i, v_{i+1})|v_iv_{i+1}|.$$

Clearly, $B(\mathcal{P}) = B(\mathcal{P}_{red})$.

For a reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$, we set

$$w_i = \frac{(v_i - v_{i-1})}{\det(v_{i-1}, v_i)} \quad \text{for } i = 1, \dots, d, \tag{2.1}$$

where $v_0 = v_d$, and define $\mathcal{P}^\vee = (w_1, \dots, w_d)$. It is not difficult to see that $\mathcal{P}^\vee = (w_1, \dots, w_d)$ is again a legal loop although it may not be reduced (see the proof of Theorem 2.2 below). If a legal loop $\mathcal{P}$ is not reduced, then we define $\mathcal{P}^\vee$ to be $(\mathcal{P}_{red})^\vee$. When the vectors $v_1, \dots, v_d$ are the vertices of a convex lattice polygon P with only the origin as an interior lattice point and are arranged in counterclockwise order, the sequence $w_1, \dots, w_d$ is also in counterclockwise order and the convex hull of $w_1, \dots, w_d$ is the polygon $P^\vee$ dual to P.

Theorem 2.2 (Generalized twelve-point theorem [11])**.** *Let $\mathcal{P}$ be a legal loop and let r be the winding number of $\mathcal{P}$. Then $B(\mathcal{P}) + B(\mathcal{P}^\vee) = 12r$.*

Remark. Kasprzyk and Nill ([7, Corollary 2.7]) point out that the generalized twelve-point theorem can further be generalized to what are called *ℓ-reflexive loops*, where ℓ is a positive integer and a 1-reflexive loop is a unimodular sequence.

Proof. We may assume that $\mathcal{P}$ is reduced. As remarked before, the reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$ determines a unimodular sequence by adding all the lattice points on the side v_iv_{i+1} for every i, and the unimodular sequence determines a reduced legal loop, say $\mathcal{Q}$. Clealry, $B(\mathcal{P}) = B(\mathcal{Q})$ and $(\mathcal{P}^\vee)_{red} = (\mathcal{Q}^\vee)_{red}$. In the sequel, we may assume that the vectors $v_1, \dots, v_d$ in our legal loop $\mathcal{P}$ form a unimodular sequence.

Since the sequence $v_1, \dots, v_d$ is unimodular, $\operatorname{sgn}(v_i, v_{i+1}) = \epsilon_i$ and $|v_iv_{i+1}| = 1$ for any i. Therefore

$$B(\mathcal{P}) = \sum_{i=1}^{d} \operatorname{sgn}(v_i, v_{i+1})|v_iv_{i+1}| = \sum_{i=1}^{d} \epsilon_i. \tag{2.2}$$

On the other hand, it follows from (2.1) and (1.1) that

$$\begin{aligned} w_{i+1} - w_i &= \epsilon_i(v_{i+1} - v_i) - \epsilon_{i-1}(v_i - v_{i-1}) \\ &= \epsilon_i v_{i+1} + \epsilon_{i-1}v_{i-1} - (\epsilon_i + \epsilon_{i-1})v_i \\ &= -(a_i + \epsilon_i + \epsilon_{i-1})v_i \end{aligned} \tag{2.3}$$

and that

$$
\begin{aligned}
\det(w_i, w_{i+1}) &= \epsilon_{i-1}\epsilon_i \det(v_i - v_{i-1}, v_{i+1} - v_i) \\
&= \epsilon_{i-1}\epsilon_i \det(v_i - v_{i-1}, -\epsilon_{i-1}\epsilon_i v_{i-1} - \epsilon_i a_i v_i - v_i) \\
&= \epsilon_{i-1}\epsilon_i \big(\det(v_i, -\epsilon_{i-1}\epsilon_i v_{i-1}) + \det(-v_{i-1}, -\epsilon_i a_i v_i - v_i)\big) \\
&= \epsilon_{i-1} + a_i + \epsilon_i.
\end{aligned} \tag{2.4}
$$

Since v_i is primitive, (2.3) shows that $|w_i w_{i+1}| = |\epsilon_{i-1} + \epsilon_i + a_i|$ and this together with (2.4) shows that

$$\operatorname{sgn}(w_i, w_{i+1})|w_i w_{i+1}| = \epsilon_{i-1} + \epsilon_i + a_i.$$

Therefore

$$B(\mathcal{P}^\vee) = \sum_{i=1}^{d} \operatorname{sgn}(w_i, w_{i+1})|w_i w_{i+1}| = \sum_{i=1}^{d} (\epsilon_{i-1} + \epsilon_i + a_i). \tag{2.5}$$

It follows from (2.2) and (2.5) that

$$
\begin{aligned}
B(\mathcal{P}) + B(\mathcal{P}^\vee) &= \sum_{i=1}^{d} \epsilon_i + \sum_{i=1}^{d} (\epsilon_{i-1} + \epsilon_i + a_i) \\
&= 3\sum_{i=1}^{d} \epsilon_i + \sum_{i=1}^{d} a_i,
\end{aligned}
$$

which is equal to $12r$ by Theorem 1.1, proving the theorem. □

Example 2.3. (a) Let

$$\mathcal{P} = ((1,0), (0,1), (-1,0), (0,-1), (-1,-1)).$$

Then $\mathcal{P}$ is a reduced legal loop whose winding number is 1 and

$$\mathcal{P}^\vee = ((2,1), (-1,1), (-1,-1), (1,-1), (1,0)).$$

On the one hand, $B(\mathcal{P}) = 1 + 1 + 1 - 1 + 1 = 3$. On the other hand, $B(\mathcal{P}^\vee) = 3 + 2 + 2 + 1 + 1 = 9$. Thus we have $B(\mathcal{P}) + B(\mathcal{P}^\vee) = 12$. The left-hand side (resp. right-hand side) of the following picture shows $\mathcal{P}$ (resp. $\mathcal{P}^\vee$) together with signs, where the symbols $\circ$ and $\times$ stand for lattice points in $\mathbb{Z}^2$.

(b) Let

$$\mathcal{Q} = ((1,0), (-1,1), (0,-1), (1,1), (-1,0), (1,-1)).$$

Then $\mathcal{Q}$ is a reduced legal loop whose winding number is 2 and

$$\mathcal{Q}^\vee = ((0,1), (-2,1), (1,-2), (1,2), (-2,-1), (2,-1)).$$

On the one hand, $B(\mathcal{Q}) = 6$. On the other hand, $B(\mathcal{Q}^\vee) = 18$. Hence, $B(\mathcal{Q}) + B(\mathcal{Q}^\vee) = 24$. The left-hand side (resp. right-hand side) of the following picture shows $\mathcal{Q}$ (resp. $\mathcal{Q}^\vee$). Note that the signs on the sides of $\mathcal{Q}$ and $\mathcal{Q}^\vee$ are all $+$.

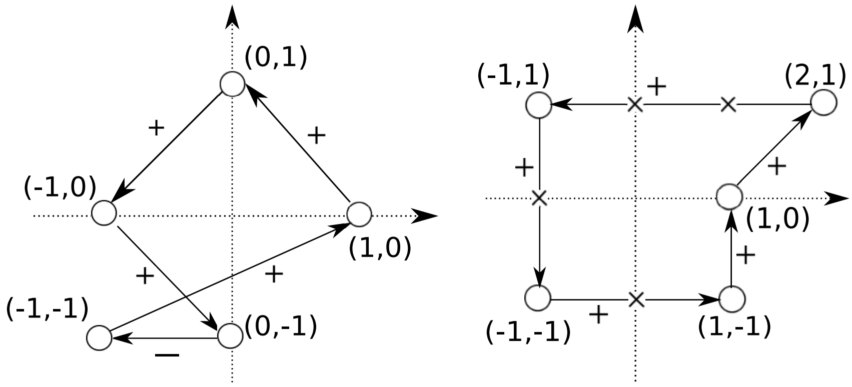


FIGURE 1. lattice points on $\mathcal{P}$ and $\mathcal{P}^\vee$ and sides with signs

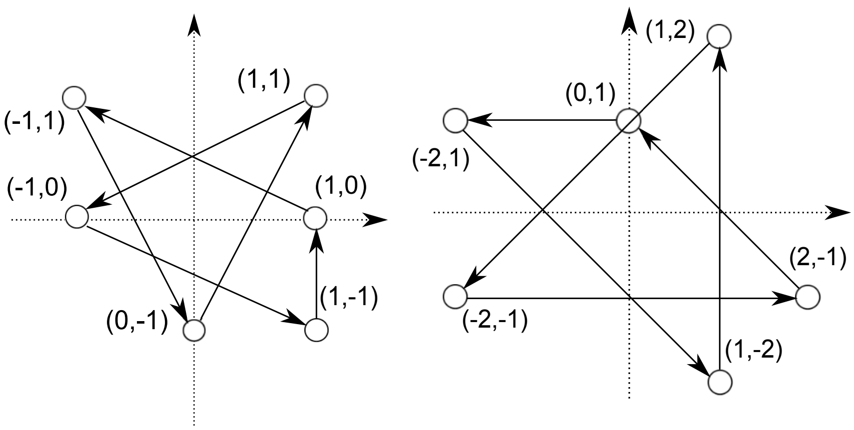


FIGURE 2. lattice points on $\mathcal{Q}$ and $\mathcal{Q}^\vee$

3. Generalized Pick's formula for lattice multi-polygons

In this section, we introduce the notion of lattice multi-polygon and state a generalized Pick's formula for lattice multi-polygons which is essentially proved in [8, Theorem 8.1]. This implies the existence of Ehrhart polynomials for lattice multi-polygons.

We begin with the well-known Pick's formula for lattice polygons ([10]). Let P be a (not necessarily convex) lattice polygon, ∂P the boundary of P and $P^\circ = P \backslash \partial P$. We define

$$A(P) = \text{the area of } P, \quad B(P) = |\partial P \cap \mathbb{Z}^2|, \quad \sharp P^\circ = |P^\circ \cap \mathbb{Z}^2|,$$

where $|X|$ denotes the cardinality of a finite set X. Then Pick's formula says that

$$A(P) = \sharp P^\circ + \frac{1}{2}B(P) - 1. \tag{3.1}$$

We may rewrite (3.1) as

$$\sharp P^\circ = A(P) - \frac{1}{2}B(P) + 1 \quad \text{or} \quad \sharp P = A(P) + \frac{1}{2}B(P) + 1$$

where $\sharp P = |P \cap \mathbb{Z}^2|$.

In [5], the notion of *shaven lattice polygon* is introduced and Pick's formula (3.1) is generalized to shaven lattice polygons. The generalization of Pick's formula discussed in [8] is similar to [5] but a bit more general, which we shall explain.

Let $\mathcal{P} = (v_1, \dots, v_d)$ ($d \geq 3$) be a sequence of points $v_1, \dots, v_d$ in $\mathbb{Z}^2$, where v_i and v_{i+1} are linearly independent for $i = 1, \dots, d$ and $v_{d+1} = v_1$. One may regard $\mathcal{P}$ as an *oriented* piecewise linear loop by connecting all successive points from v_i to v_{i+1} in $\mathcal{P}$ by straight lines as before. To each side v_iv_{i+1}, we assign a sign $+$ or $-$, denoted $\epsilon(v_iv_{i+1})$. In Section 2, we assigned the $\mathrm{sgn}(v_i, v_{i+1})$, which is the sign of $\det(v_i, v_{i+1})$, to v_iv_{i+1} but $\epsilon(v_iv_{i+1})$ may be different from $\mathrm{sgn}(v_i, v_{i+1})$. However we require that the assignment ϵ of signs satisfy the following condition $(\star)$:

($\star$) when there are consecutive three points v_{i-1}, v_i, v_{i+1} in $\mathcal{P}$ lying on a line, we have
  (1) $\epsilon(v_{i-1}v_i) = \epsilon(v_iv_{i+1})$ if v_i is in between v_{i-1} and v_{i+1};
  (2) $\epsilon(v_{i-1}v_i) \neq \epsilon(v_iv_{i+1})$ if v_{i-1} lies on v_iv_{i+1} or v_{i+1} lies on $v_{i-1}v_i$.

A *lattice multi-polygon* is $\mathcal{P}$ equipped with the assignment ϵ satisfying $(\star)$. We need to express a lattice multi-polygon as a pair $(\mathcal{P}, \epsilon)$ to be precise, but we omit ϵ and express a lattice multi-polygon simply as $\mathcal{P}$ in the following. Reduced legal loops introduced in Section 2 are lattice multi-polygons.

Remark. Lattice multi-polygons such that three consecutive points are not on a same line are introduced in [8, Section 8]. But if we require the condition $(\star)$, then the argument developed there works for any lattice multi-polygon. A shaven polygon introduced in [5] is a lattice multi-polygon with $\epsilon = +$ in our terminology, so that v_i is allowed to lie on the line segment $v_{i-1}v_{i+1}$ but v_{i-1} (resp. v_{i+1}) is not allowed to lie on v_iv_{i+1} (resp. $v_{i-1}v_i$) by (2) of $(\star)$, i.e. there is no whisker.

Let $\mathcal{P}$ be a multi-polygon with a sign assignment ϵ. We think of $\mathcal{P}$ as an oriented piecewise linear loop with signs attached to sides. For $i = 1, \dots, d$, let n_i denote a normal vector to each side v_iv_{i+1} such that the 90 degree rotation of $\epsilon(v_iv_{i+1})n_i$ has the same direction as v_iv_{i+1}. The winding number of $\mathcal{P}$ around a point $v \in \mathbb{R}^2 \setminus \mathcal{P}$, denoted $d_{\mathcal{P}}(v)$, is a locally constant function on $\mathbb{R}^2 \setminus \mathcal{P}$, where $\mathbb{R}^2 \setminus \mathcal{P}$ means the set of elements in $\mathbb{R}^2$ which does not belong to any side of $\mathcal{P}$.

Following [8, Section 8], we define

$$\begin{aligned}
A(\mathcal{P}) &:= \int_{v \in \mathbb{R}^2 \setminus \mathcal{P}} d_{\mathcal{P}}(v)\,dv, \\
B(\mathcal{P}) &:= \sum_{i=1}^{d} \epsilon(v_iv_{i+1})|v_iv_{i+1}|, \\
C(\mathcal{P}) &:= \text{ the rotation number of the sequence of } n_1, \dots, n_d.
\end{aligned}$$

Notice that $A(\mathcal{P})$ and $B(\mathcal{P})$ can be 0 or negative. If $\mathcal{P}$ arises from a lattice polygon P, namely $\mathcal{P}$ is a sequence of the vertices of P arranged in counterclockwise order and $\epsilon = +$, then $A(\mathcal{P}) = A(P)$, $B(\mathcal{P}) = B(P)$ and $C(\mathcal{P}) = 1$.

Now, we define $\sharp\mathcal{P}$ in such a way that if $\mathcal{P}$ arises from a lattice polygon P, then $\sharp\mathcal{P} = \sharp P$. Let $\mathcal{P}_+$ be an oriented loop obtained from $\mathcal{P}$ by pushing each side v_iv_{i+1} slightly in the direction of n_i. Since $\mathcal{P}$ satisfies the condition $(\star)$, $\mathcal{P}_+$ misses all lattice points, so the winding numbers $d_{\mathcal{P}_+}(u)$ can be defined for any lattice point u using $\mathcal{P}_+$. Then we define

$$\sharp\mathcal{P} := \sum_{u\in\mathbb{Z}^2} d_{\mathcal{P}_+}(u).$$

As remarked before, lattice multi-polygons treated in [8] are required that three consecutive points v_{i-1}, v_i, v_{i+1} do not lie on a same line. But if the sign assignment ϵ satisfies the condition $(\star)$ above, then the argument developed in [8, Section 8] works and we obtain the following generalized Pick's formula for lattice multi-polygons.

Theorem 3.1 ([8, Theorem 8.1])**.** $\sharp\mathcal{P} = A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P})$.

If we define $\mathcal{P}^\circ$ to be $\mathcal{P}$ with $-\epsilon$ as a sign assignment, then

$$\sharp\mathcal{P}^\circ = A(\mathcal{P}) - \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}) \tag{3.2}$$

and if $\mathcal{P}$ arises from a lattice polygon P, then $\sharp\mathcal{P}^\circ = \sharp P^\circ$.

Given a positive integer m, we dilate $\mathcal{P}$ by m times, denoted $m\mathcal{P}$, in other words, if $\mathcal{P}$ is $(v_1, \dots, v_d)$ with a sign assignment ϵ, then $m\mathcal{P}$ is $(mv_1, \dots, mv_d)$ with $\epsilon(v_iv_{i+1})$ as the sign of the side mv_imv_{i+1} of $m\mathcal{P}$ for each i. Then we have

$$\sharp(m\mathcal{P}) = A(\mathcal{P})m^2 + \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}), \tag{3.3}$$

that is, $\sharp(m\mathcal{P})$ is a polynomial in m of degree at most 2 whose coefficients are as above. Moreover, the equality

$$\sharp(m\mathcal{P}^\circ) = A(\mathcal{P})m^2 - \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}) = (-1)^2\sharp(-m\mathcal{P})$$

holds, so that the reciprocity holds for lattice multi-polygons. We call the polynomial (3.3) the *Ehrhart polynomial* of a lattice multi-polygon $\mathcal{P}$.

Remark. In [6], lattice multi-polytopes $\mathcal{P}$ of dimension n are defined and it is proved that $\sharp(m\mathcal{P})$ is a polynomial in m of degree at most n which satisfies $\sharp(m\mathcal{P}^\circ) = (-1)^n\sharp(-m\mathcal{P})$ whose leading coefficient and constant term have similar geometrical meanings to the 2-dimensional case above.

4. Ehrhart polynomials of lattice multi-polygons

In this section, we will discuss which polynomials appear as the Ehrhart polynomials of lattice multi-polygons. If $am^2 + bm + c$ is the Ehrhart polynomial of a lattice multi-polygon $\mathcal{P}$, then the coefficients (a, b, c) must be in the set

$$\mathcal{A} = \left\{ (a, b, c) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{Z} : a + b \in \mathbb{Z} \right\}$$

because $(a, b, c) = (A(\mathcal{P}), \frac{1}{2}B(\mathcal{P}), C(\mathcal{P}))$ and

$$B(\mathcal{P}) \in \mathbb{Z}, \quad C(\mathcal{P}) \in \mathbb{Z}, \quad A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}) = \sharp\mathcal{P} \in \mathbb{Z}.$$

The following theorem shows that this condition is sufficient.

Theorem 4.1. *A polynomial $am^2 + bm + c$ in m is the Ehrhart polynomial of a lattice multi-polygon if and only if $(a, b, c) \in \mathcal{A}$.*

Proof. It suffices to prove the "if" part. We pick up $(a, b, c) \in \mathcal{A}$. Then one has an expression

$$(a, b, c) = a'(1, 0, 0) + b'\left(\frac{1}{2}, \frac{1}{2}, 0\right) + c'(0, 0, -1) \tag{4.1}$$

with integers a', b', c' because $a' = a - b$, $b' = 2b$ and $c' = -c$. One can easily check that m^2, $\frac{1}{2}m^2 + \frac{1}{2}m$ and -1 are respectively the Ehrhart polynomials of the lattice multi-polygons $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ shown in Figure 3, where the sign of v_iv_{i+1} is given by the sign of $\det(v_i, v_{i+1})$ for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

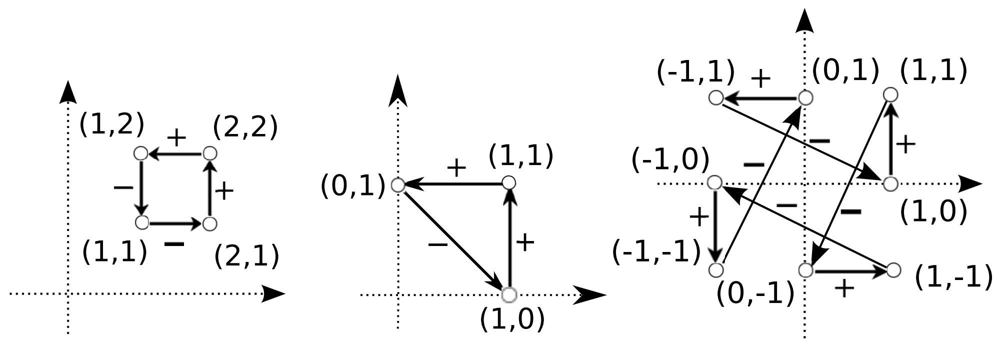


FIGURE 3. lattice multi-polygons $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ from the left

Moreover, reversing both the order of the points and the signs on the sides for $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$, we obtain lattice multi-polygons $\mathcal{P}_1', \mathcal{P}_2'$ and $\mathcal{P}_3'$ whose Ehrhart polynomials are respectively $-m^2$, $-\frac{1}{2}m^2 - \frac{1}{2}m$ and 1. Since all these six multi-polygons have a common lattice point $(1, 1)$, one can produce a multi-polygon by joining them as many as we want at the common point and since Ehrhart polynomials behave additively with respect to the join operation, this together with (4.1) shows the existence of a lattice multi-polygon with the desired Ehrhart polynomial $am^2 + bm + c$. $\square$

In the rest of the paper, we shall consider several natural subfamilies of lattice multi-polygons and discuss the characterization of their Ehrhart polynomials. We note that if a polynomial $am^2 + bm + c$ in m is the Ehrhart polynomial of some lattice multi-polygon, then (a, b, c) must be in the set $\mathcal{A}$.

4.1. **Lattice polygons.** The most natural subfamily of lattice multi-polygons would be the family of convex lattice polygons. Their Ehrhart polynomials are essentially characterized by P. R. Scott as follows.

Theorem 4.2 ([12]). *A polynomial $am^2 + bm + c$ in m with $(a, b, c) \in \mathcal{A}$ is the Ehrhart polynomial of a convex lattice polygon if and only if $c = 1$ and (a, b) satisfies one of the following:*

(1) $a + 1 = b \geq \frac{3}{2}$;
(2) $\frac{a}{2} + 2 \geq b \geq \frac{3}{2}$;
(3) $(a, b) = (\frac{9}{2}, \frac{9}{2})$.

If we do not require the convexity, then the characterization of Ehrhart polynomials becomes simpler than Theorem 4.2.

Proposition 4.3. *A polynomial $am^2 + bm + c$ in m with $(a, b, c) \in \mathcal{A}$ is the Ehrhart polynomial of a (not necessarily convex) lattice polygon if and only if $c = 1$ and $a + 1 \geq b \geq \frac{3}{2}$.*

Proof. If P is a lattice polygon, then we have

$$C(P) = 1, \quad B(P) \geq 3, \quad A(P) - \frac{1}{2}B(P) + 1 = \sharp P^\circ \geq 0$$

and this implies the "only if" part.

On the other hand, let $(a, b, 1) \in \mathcal{A}$ with $a + 1 \geq b \geq \frac{3}{2}$. Thanks to Theorem 4.2, we may assume that $b > \frac{a}{2} + 2$, that is, $4b - 2a - 6 > 2$. Let P be the lattice polygon shown in Figure 4. Then, one has

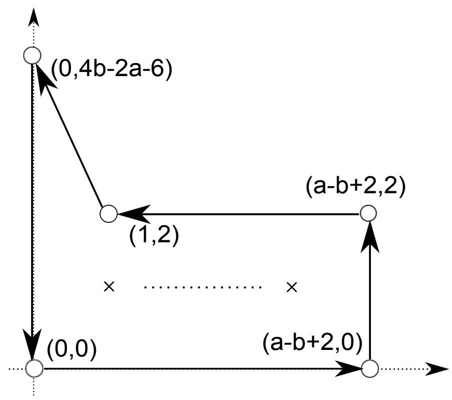


FIGURE 4. a lattice polygon whose Ehrhart polynomial equals $am^2 + bm + 1$

$$A(P) = 2(a - b + 2) + \frac{1}{2}(4b - 2a - 8) = a$$

and

$$B(P) = (a - b + 2) + 2 + (a - b + 1) + 1 + 4b - 2a - 6 = 2b.$$

This shows that $\sharp(mP) = am^2 + bm + 1$. $\square$

4.2. **Unimodular lattice multi-polygons.** We say that a lattice multi-polygon $\mathcal{P} = (v_1, \ldots, v_d)$ is *unimodular* if the sequence $(v_1, \ldots, v_d)$ is unimodular and the sign assignment ϵ is defined by $\epsilon(v_i v_{i+1}) = \det(v_i, v_{i+1})$ for $i = 1, \ldots, d$, where $v_{d+1} = v_1$. When a unimodular lattice multi-polygon $\mathcal{P}$ arises from a convex lattice polygon, $\mathcal{P}$ is essentially the same as so-called a *reflexive polytope* of dimension 2, which is completely classified (16 polygons up to equivalence, see, e.g. [11, Figure 2]) and the Ehrhart polynomials $am^2 + bm + c$ of reflexive polytopes are characterized by the condition that $c = 1$ and $a = b \in \left\{\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}\right\}$.

We can characterize the Ehrhart polynomials of unimodular lattice multi-polygons as follows.

Theorem 4.4. *A polynomial $am^2 + bm + c$ in m with $(a, b, c) \in \mathcal{A}$ is the Ehrhart polynomial of a unimodular lattice multi-polygon if and only if $a = b$.*

Proof. If $\mathcal{P}$ is a unimodular lattice multi-polygon arising from a unimodular sequence $v_1, \ldots, v_d$, then one sees that

$$A(\mathcal{P}) = \frac{1}{2}\sum_{i=1}^{d} \det(v_i, v_{i+1})$$

$$B(\mathcal{P}) = \sum_{i=1}^{d} \det(v_i, v_{i+1})|v_i v_{i+1}| = \sum_{i=1}^{d} \det(v_i, v_{i+1})$$

and this implies the "only if" part.

Conversely, if $(a, b, c) \in \mathcal{A}$ satisfies $a = b$, then one has an expression

$$(a, b, c) = a'\left(\frac{1}{2}, \frac{1}{2}, 0\right) + c'(0, 0, -1)$$

with integers a', c' because $a' = 2a$ and $c' = -c$. We note that the lattice multi-polygons $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_2'$ and $\mathcal{P}_3'$ in the proof of Theorem 4.1 are unimodular lattice multi-polygons. Therefore, joining them as many as we want at the common point $(1, 1)$, we can find a unimodular lattice multi-polygon whose Ehrhart polynomial is equal to $am^2 + bm + c$, as required. $\square$

Example 4.5. The $\mathcal{P}$ and $\mathcal{Q}$ in Example 2.3 are unimodular lattice multi-polygons and we have $\sharp(m\mathcal{P}) = \frac{3}{2}m^2 + \frac{3}{2}m + 1$ and $\sharp(m\mathcal{Q}) = 3m^2 + 3m + 2$.

4.3. **Left-turning (right-turning) lattice multi-polygons.** We say that a lattice multi-polygon $\mathcal{P}$ is *left-turning* (resp. *right-turning*) if $\det(v-u, w-u)$ is always positive (resp. negative) for consecutive three points u, v, w in $\mathcal{P}$ arranged in this order not lying on a same line. In other words, w lies in the left-hand side (resp. right-hand side) with respect to the direction from u to v. For example, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ in Figure 3 and $\mathcal{Q}$ in Example 2.3 (b) are all left-turning.

The following theorem shows that the left-turning (or right-turning) condition does not give any restriction on the Ehrhart polynomials.

Theorem 4.6. *Any polynomial $am^2 + bm + c$ with $(a, b, c) \in \mathcal{A}$ can be the Ehrhart polynomial of a left-turning (or right-turning) lattice multi-polygon.*

Proof. We pick up $(a, b, c) \in \mathcal{A}$. Then one has an expression (4.1). As mentioned above, $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ in Figure 3 are left-turning. Moreover, multi-polygons $\mathcal{P}_4, \mathcal{P}_5$ and $\mathcal{P}_6$ in Figure 5 are also left-turning and their Ehrhart polynomials are respectively $-m^2$, $-\frac{1}{2}m^2 - \frac{1}{2}m$ and 1. Since $\mathcal{P}_4, \mathcal{P}_5$ and $\mathcal{P}_6$ also have the common point $(1, 1)$, we can join $\mathcal{P}_1$ through $\mathcal{P}_6$ as many as we want at the common point so that we can find a left-turning multi-polygon with the desired Ehrhart polynomial.

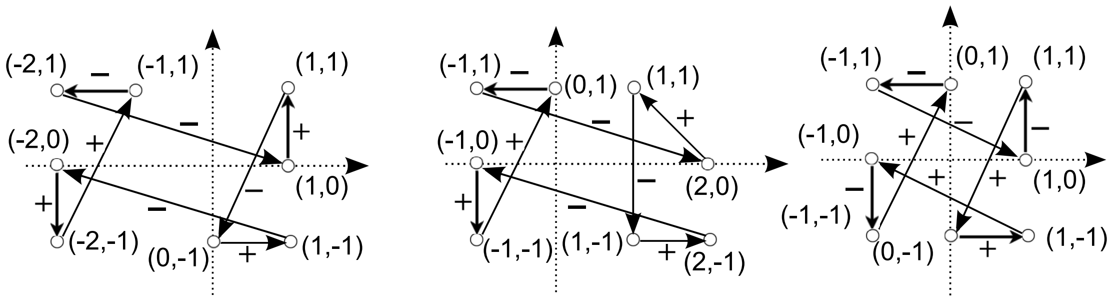


FIGURE 5. lattice multi-polygons $\mathcal{P}_4, \mathcal{P}_5$ and $\mathcal{P}_6$ from the left

Reversing both the order of the points in $\mathcal{P}_j$ and the signs on the sides of $\mathcal{P}_j$ $(j = 1, \dots, 6)$, we obtain the theorem for right-turning lattice multi-polygons. $\square$

Unfortunately, the signs of $\mathcal{P}_4, \mathcal{P}_5$ and $\mathcal{P}_6$ do not always coincide with the sign of $\det(v_i, v_{i+1})$. Thus, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$ are not considered in Section 2 but these lattice multi-polygons are also of interest.

4.4. **Left-turning lattice multi-polygons with all + signs.** We consider left-turning lattice multi-polygons $\mathcal{P}$ and impose one more restriction that the signs on the sides of $\mathcal{P}$ are all $+$. In this case, some interesting phenomena happen. For example, a simple observation shows that

$$B(\mathcal{P}) \geq 2C(\mathcal{P}) + 1 \text{ and } C(\mathcal{P}) \geq 1. \tag{4.2}$$

We note that $C(\mathcal{P}) = 1$ if and only if $\mathcal{P}$ arises from a convex lattice polygon, and those Ehrhart polynomials are characterized by Theorem 4.2. Therefore it suffices to treat the case where $C(\mathcal{P}) \geq 2$.

Theorem 4.7. *A polynomial $am^2 + bm + c$ with $(a, b, c) \in \mathcal{A}$ is the Ehrhart polynomial of a left-turning lattice multi-polygon with all + signs if $b \geq c+1$ and $c \geq 2$.*

Proof. We take an odd number α such that $\beta = a - b + c + \frac{\alpha+3}{2} \geq 2$ and $\alpha + \sum_{j=1}^{2c-4}(-1)^{j-1}(j-1) \geq 1$. Since $a + b \in \mathbb{Z}$, one has $\beta \in \mathbb{Z}$. We also set $\gamma = 2b - 2c - 1$. Since $b \geq c + 1$, one has $\gamma \geq 1$. We then define a left-turning lattice multi-polygon $\mathcal{P} = (v_1, v_2, \dots, v_{2c+2})$ with all $+$ signs on sides by setting

$$v_1 = (0,0),\ v_2 = (-1,\beta),\ v_3 = (-2,2),\ v_4 = (\alpha,0),\ v_{2c+2} = (-\gamma,0),$$

$$\text{and}\quad v_i = (\alpha,0) + \sum_{j=1}^{i-4}((-1)^{j-1}(j-1),(-1)^{j-1}) \ \text{ for } \ i = 5,\dots,2c+1.$$

Figure 6 shows $\mathcal{P}$ with $\alpha = 3$, $\beta = 2$ and $\gamma = 1$.

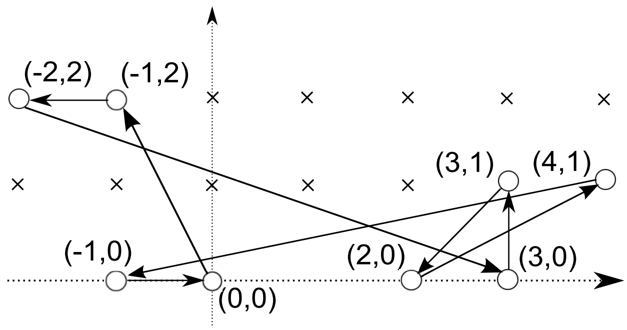


FIGURE 6. a left-turning lattice multi-polygon $\mathcal{P}$ with all $+$ signs

Since $|v_iv_{i+1}| = 1$ for $i = 1,\dots,2c+1$ and $|v_{2c+2}v_1| = \gamma$, we have $B(\mathcal{P}) = 2c+1+\gamma = 2b$. One easily finds that the rotation number of the normal vectors of v_iv_{i+1}'s is equal to c, so $C(\mathcal{P}) = c$. One also easily finds that $\sharp\mathcal{P}^\circ = \beta - 2 + (-1)\cdot\frac{\alpha-1}{2} = a - b + c$. It follows from the formula (3.2), that is $A(\mathcal{P}) = \sharp\mathcal{P}^\circ + \frac{1}{2}B(\mathcal{P}) - C(\mathcal{P})$, that $A(\mathcal{P}) = a$. Therefore, $\sharp(m\mathcal{P}) = am^2 + bm + c$, as desired. $\square$

The condition $b \geq c+1$ in Theorem 4.7 is equivalent to $B(\mathcal{P}) \geq 2C(\mathcal{P})+2$ for a lattice multi-polygon $\mathcal{P}$ because $b = \frac{1}{2}B(\mathcal{P})$ and $c = C(\mathcal{P})$. On the other hand, we have $B(\mathcal{P}) \geq 2C(\mathcal{P}) + 1$ for a left-turning lattice multi-polygon $\mathcal{P}$ with all $+$ signs by (4.2). Therefore, the extreme case where $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$ is not covered by Theorem 4.7 and the following proposition shows that this extreme case is exceptional.

Proposition 4.8. *If $\mathcal{P}$ is a left-turning multi-polygon with all $+$ signs and $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$, then $A(\mathcal{P}) \geq \frac{1}{2}$.*

Proof. Since $A(\mathcal{P}) = \sharp\mathcal{P}^\circ + \frac{1}{2}B(\mathcal{P}) - C(\mathcal{P})$ and $B(\mathcal{P}) = 2C(\mathcal{P})+1$, it suffices to show that $\sharp\mathcal{P}^\circ \geq 0$.

Suppose that $\sharp\mathcal{P}^\circ < 0$. Then there exists a lattice point v such that $d_{\mathcal{P}}(v) < 0$. This implies that there are p sides $l_1,\dots,l_p$ of $\mathcal{P}$ $(p \geq 3)$ winding around v clockwise, see Figure 7. Let q_i (> 0) be the number of sides of $\mathcal{P}$ between l_i and l_{i+1} except l_i and l_{i+1}, where $l_{p+1} = l_1$, and θ_i $(0 < \theta_i < 180)$ the degree of the angle between the sides l_i and l_{i+1}. Then one can verify easily that the counterclockwise winding angle between the normal vectors of l_i and l_{i+1} is $(180 + \theta_i)$ degree and the sum of the angles of the normal vectors of the $(q_i + 2)$ sides, namely from l_i to l_{i+1}, is at most

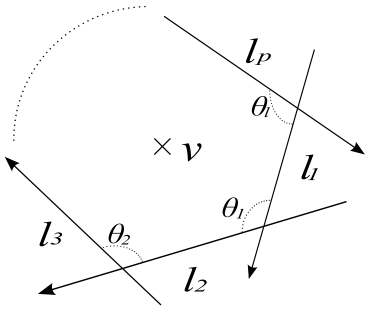


FIGURE 7. a local structure of $\mathcal{P}$ around v with $d_{\mathcal{P}}(v) < 0$

$(360(\frac{q_i-1}{2}) + 180 + \theta_i)$ degree. On the other hand, since all signs are $+$, we have $B(\mathcal{P}) \geq \sum_{i=1}^{p} q_i + p$. In addition, clearly, $\sum_{i=1}^{p} \theta_i = 180(p-2)$. Hence,

$$\begin{aligned} 360 \cdot C(\mathcal{P}) \leq \sum_{i=1}^{p} \left(360 \cdot \frac{q_i - 1}{2} + 180 + \theta_i\right) &= 180 \sum_{i=1}^{p} q_i + 180p - 360 \\ \leq 180 \cdot B(\mathcal{P}) - 360, \end{aligned}$$

which implies that $B(\mathcal{P}) \geq 2C(\mathcal{P}) + 2$, a contradiction. Therefore, there is no lattice point v with $d_{\mathcal{P}}(v) < 0$, which means that $\sharp\mathcal{P}^\circ \geq 0$. $\square$

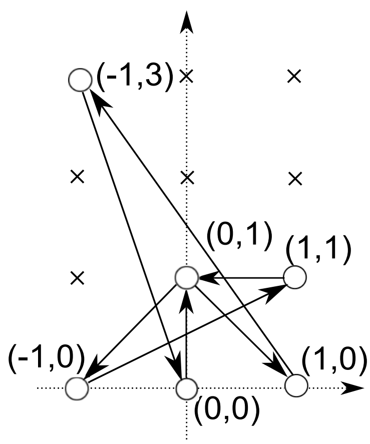
We propose the following conjecture.

Conjecture 4.9. *If $\mathcal{P}$ is a left-turning multi-polygon with all $+$ signs and $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$, then*

$$A(\mathcal{P}) \geq \left\lfloor \frac{C(\mathcal{P})}{2} \right\rfloor + \frac{1}{2}.$$

If the above conjecture is affirmative, then we can characterize the Ehrhart polynomials of left-turning lattice multi-polygons with all $+$ signs as is shown in the following example.

Example 4.10. The Ehrhart polynomial of the left-turning lattice multi-polygon $\mathcal{P}$ with all $+$ signs depicted below is $\frac{3}{2}m^2 + \frac{7}{2}m + 3$. A similar



construction shows that for any integer $c \geq 2$, there is a left-turning lattice multi-polygon $\mathcal{P}$ with all + signs which satisfies $A(\mathcal{P}) \geq \left\lfloor \frac{C(\mathcal{P})}{2} \right\rfloor + \frac{1}{2}$, $B(\mathcal{P}) = 2C(\mathcal{P}) + 1$ and $C(\mathcal{P}) = c$.

4.5. **Lattice multi-polygons with all + signs.** Finally, we consider lattice multi-polygons $\mathcal{P}$ with all + signs, namely, we do not assume that $\mathcal{P}$ is either left-turning or right-turning. However, this case is similar to the previous one (left-turning lattice multi-polygons with all + signs). For example, when $C(\mathcal{P}) \neq 0$, then we still have $B(\mathcal{P}) \geq 2|C(\mathcal{P})| + 1$. Thus, as a corollary of Theorem 4.7, we obtain the following.

Corollary 4.11. *A polynomial $am^2 + bm + c$ in m with $(a, b, c) \in \mathcal{A}$ is the Ehrhart polynomial of a lattice multi-polygon with all + signs if $b \geq |c| + 1$ and $|c| \geq 2$.*

Proof. When $c \geq 2$, the assertion follows directly from Theorem 4.7. When $c \leq -2$, we reverse the order of $v_1, \ldots, v_{2|c|+2}$ in $\mathcal{P}$ described in the proof of Theorem 4.7 and reset $\beta = -(a + b + c) + \frac{\alpha+3}{2}$ and $\gamma = 2b + 2c - 1$, where α is an odd number such that $\beta \geq 2$ and $\alpha + \sum_{j=1}^{2|c|-4}(-1)^{j-1}(j-1) \geq 1$. It gives a (right-turning) lattice multi-polygon $\mathcal{P}'$ with all + signs. One finds that

$$\sharp\mathcal{P}' = -(\beta - 2) + \frac{\alpha - 1}{2} = a + b + c,$$
$$B(\mathcal{P}') = -2c + 1 + \gamma = 2b \quad \text{and} \quad C(\mathcal{P}') = c.$$

It follows from $\sharp\mathcal{P}' = A(\mathcal{P}') + \frac{1}{2}B(\mathcal{P}') + C(\mathcal{P}')$ that $A(\mathcal{P}') = a$. Therefore, the Ehrhart polynomial of $\mathcal{P}'$ equals $am^2 + bm + c$, as desired. □

The condition $b \geq |c| + 1$ is equivalent to $B(\mathcal{P}) \geq 2|C(\mathcal{P})| + 2$. Thus the case where $B(\mathcal{P}) = 2|C(\mathcal{P})| + 1$ is not covered by Corollary 4.11, but $\mathcal{P}$ must be left-turning or right-turning according as $C(\mathcal{P}) > 0$ or $C(\mathcal{P}) < 0$ when $B(\mathcal{P}) = 2|C(\mathcal{P})| + 1$. Hence, we can say that when we discuss the Ehrhart polynomials of lattice multi-polygons with all + signs, it suffices to consider those of left-turning or right-turning ones when $C(\mathcal{P}) \notin \{-1, 0, 1\}$.

On the other hand, on the remaining cases $C(\mathcal{P}) = 0$ or $C(\mathcal{P}) = \pm 1$, which are exceptional, we can characterize the Ehrhart polynomials completely as follows.

Theorem 4.12. *Let $(a, b, c) \in \mathcal{A}$.*

(a) *When $c = 0$, a polynomial $am^2 + bm + c$ in m is the Ehrhart polynomial of a lattice multi-polygon with all + signs if and only if $b \geq 2$.*
(b) *When $c = 1$, a polynomial $am^2 + bm + c$ in m is the Ehrhart polynomial of a lattice multi-polygon with all + signs if and only if either $b \geq \frac{5}{2}$ or $\frac{3}{2} \leq b \leq 2$ and $a - b + 1 \geq 0$.*
(c) *When $c = -1$, a polynomial $am^2 + bm + c$ in m is the Ehrhart polynomial of a lattice multi-polygon with all + signs if and only if either $b \geq \frac{5}{2}$ or $\frac{3}{2} \leq b \leq 2$ and $a + b - 1 \leq 0$.*

Proof. (a) When $\mathcal{P}$ is a lattice multi-polygon with all + signs, it is clear that $B(\mathcal{P}) \geq 3$. Since $C(\mathcal{P})$ must be either 1 or -1 when $B(\mathcal{P}) = 3$, we have $B(\mathcal{P}) \geq 4$ when $C(\mathcal{P}) = 0$, which means the "only if" part of (a).

Conversely, we pick up $(a, b, 0) \in \mathcal{A}$ with $b \geq 2$. If $a + b \geq 2$ (resp. $a + b \leq 2$), then let $\mathcal{P} = (v_1, \dots, v_4)$ be a lattice multi-polygon shown in the left-hand side (resp. right-hand side) of Figure 8. Then we have

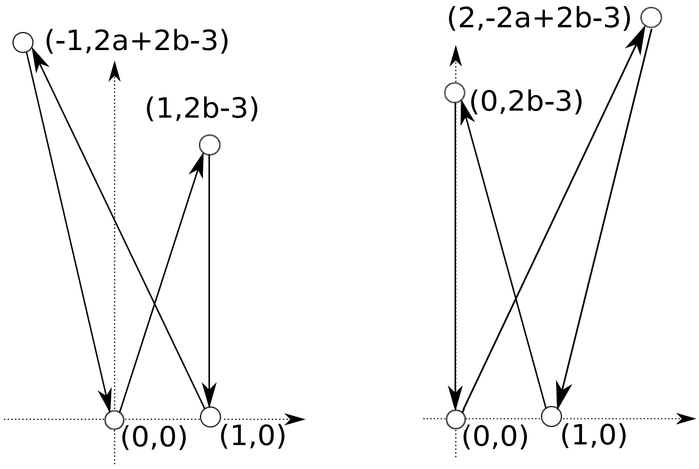


FIGURE 8. lattice multi-polygons with all + signs whose Ehrhart polynomials equal $am^2 + bm$

$B(\mathcal{P}) = 3 + (2b - 3) = 2b$ and $C(\mathcal{P}) = 0$. One also has that

$$A(\mathcal{P}) = \frac{1}{2}(2a + 2b - 3) - \frac{1}{2}(2b - 3) = a, \qquad \text{if } a + b \geq 2,$$
$$A(\mathcal{P}) = \frac{1}{2}(2b - 3) - \frac{1}{2}(-2a + 2b - 3) = a, \quad \text{if } a + b \leq 2,$$

as required.

(b) Let $\mathcal{P}$ be a lattice multi-polygon with all + signs and $C(\mathcal{P}) = 1$. As mentioned above, we have $B(\mathcal{P}) \geq 3$ and $\mathcal{P}$ must be left-turning when $B(\mathcal{P}) = 3$, which implies that $\mathcal{P}$ must arise from a convex lattice polygon. When $B(\mathcal{P}) = 4$, since all its signs are + and $C(\mathcal{P}) = 1$, $\mathcal{P}$ can turn right at most once. Hence $\mathcal{P}$ cannot have a self-intersection, which means that $\mathcal{P}$ must arise from a (not necessarily convex) lattice polygon, and thus $\sharp\mathcal{P}^\circ \geq 0$. This implies the "only if" part of (b) because $\sharp\mathcal{P}^\circ = A(\mathcal{P}) - \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P})$ (cf. (3.2)).

On the other hand, we pick up $(a, b, 1) \in \mathcal{A}$. When $a + 1 \geq b$, Proposition 4.3 guarantees the existence of a lattice multi-polygon $\mathcal{P}$ with all + signs and $\sharp(m\mathcal{P}) = am^2 + bm + 1$. Thus we may consider the case where $b \geq \frac{5}{2}$ and $a + 1 < b$.

Let $\mathcal{P} = (v_1, \dots, v_5)$ be a lattice multi-polygon shown in Figure 9. Then we have $B(\mathcal{P}) = 4 + (2b - 4) = 2b$ and $C(\mathcal{P}) = 1$. One also has that $\sharp\mathcal{P}^\circ = -(-a+b-1) = a-b+1$. Hence it follows from $A(\mathcal{P}) = \sharp\mathcal{P}^\circ + \frac{1}{2}B(\mathcal{P}) - C(\mathcal{P})$ that $A(\mathcal{P}) = a$, as required.

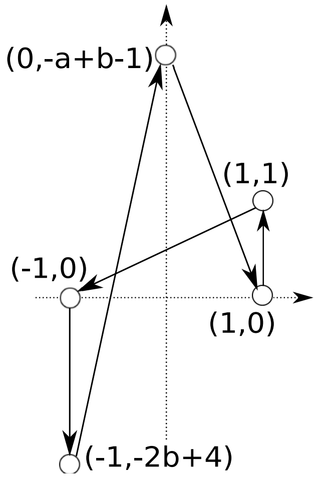


FIGURE 9. a lattice multi-polygon with all + signs whose Ehrhart polynomial equals $am^2 + bm + 1$

(c) Let $\mathcal{P}$ be a lattice multi-poygon with all + signs and $C(\mathcal{P}) = -1$. Similarly to the case where $C(\mathcal{P}) = 1$, we have $B(\mathcal{P}) \geq 3$ and $\mathcal{P}$ must arise from a lattice polygon when $B(\mathcal{P}) = 3$ or 4, where the vertices of a lattice polygon are arranged in clockwise order and $\epsilon = +$. Thus, when $B(\mathcal{P}) = 3$ or 4, we have $\sharp\mathcal{P} \leq 0$, which is equivalent to $A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}) \leq 0$. This implies the "only if" part of (c).

On the other hand, we pick up $(a, b, -1) \in \mathcal{A}$. When $2b = 3$ (resp. $2b = 4$) and $a + b - 1 \leq 0$, we define $\mathcal{P}^{(3)} = (v_1^{(3)}, v_2^{(3)}, v_3^{(3)})$ (resp. $\mathcal{P}^{(4)} = (v_1^{(4)}, v_2^{(4)}, v_3^{(4)}, v_4^{(4)})$) with all + signs by setting

$$v_1^{(3)} = (1, -1),\ v_2^{(3)} = (0, 0),\ v_3^{(3)} = (-2a - 1, 1),$$
$$(\text{resp. } v_1^{(4)} = (1, -1),\ v_2^{(4)} = (0, -1),\ v_3^{(4)} = (0, 0),\ v_4^{(4)} = (-2a - 2, 1)\ ),$$

as shown in Figure 10.

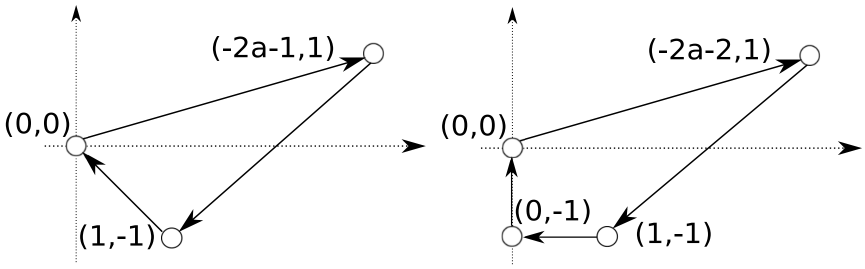


FIGURE 10. lattice multi-polygons $\mathcal{P}^{(3)}$ and $\mathcal{P}^{(4)}$ with all + signs whose Ehrhart polynomials equal $am^2 + bm - 1$

Then we have $B(\mathcal{P}^{(i)}) = i$ and $C(\mathcal{P}^{(i)}) = -1$ for $i = 3, 4$. One also has that

$$A(\mathcal{P}^{(3)}) = \frac{1}{2}\det(v_3^{(3)}, v_1^{(3)}) = a$$

and

$$A(\mathcal{P}^{(4)}) = \frac{1}{2}(\det(v_1^{(4)}, v_2^{(4)}) + \det(v_4^{(4)}, v_1^{(4)})) = a.$$

When $b \geq \frac{5}{2}$, similarly to (b), we define $\mathcal{P}' = (v_1', \dots, v_5')$ with all $+$ signs on sides by setting

$$v_1' = (0, a+b-1),\ v_2' = (-1, -2b+4),\ v_3' = (-1, 0),$$
$$v_4' = (1, 1),\ v_5' = (1, 0).$$

Then we have $B(\mathcal{P}') = 2b$ and $C(\mathcal{P}') = -1$. One also has that $\sharp\mathcal{P}' = a+b-1$. Thus it follows from $\sharp\mathcal{P}' = A(\mathcal{P}') + \frac{1}{2}B(\mathcal{P}') + C(\mathcal{P}')$ that $A(\mathcal{P}') = a$, as required. □

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