Morse indices and the number of blow up points of blowing-up solutions for a Liouville equation with singular data

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let $\Gamma = \{p_1, \cdots, p_N\} \subset \Omega$ be the set of prescribed points. Consider the Liouville type equation

$$-\Delta u = \lambda \Pi_{j=1}^N |x - p_j|^{2\alpha_j} V(x) e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where α_j $(j = 1, \cdots, N)$ are positive numbers, $V(x) > 0$ is a given smooth function on $\overline{\Omega}$, and $\lambda > 0$ is a parameter. Let $\{u_n\}$ be a blowing up solution sequence for $\lambda = \lambda_n \downarrow 0$ having the m-points blow up set $S = \{q_1, \cdots, q_m\} \subset \Omega$, i.e.,

$$\lambda_n \Pi_{j=1}^N |x - p_j|^{2\alpha_j} V(x) e^{u_n} dx \rightharpoonup \sum_{i=1}^m b_i \delta_{q_i}$$

in the sense of measures, where $b_i = 8\pi$ if $q_i \notin \Gamma$, $b_i = 8\pi(1 + \alpha_j)$ if $q_i = p_j$ for some $p_j \in \Gamma$. We show that the number of blow up points m is less than or equal to the Morse index of u_n for n sufficiently large, provided $\alpha_j \in (0, +\infty) \setminus \mathbb{N}$ for all $j = 1, \cdots, N$. This is a generalization of the result [14] in which nonsingular case ($\alpha_j = 0$ for all j) was studied.

Keywords: Liouville equation, blow up points, singular data, concentration compactness result, Morse indices.

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $\lambda > 0$ is a parameter. Motivated by some physical problems in selfdual Gauge Field Theories such as

Chern-Simons vortex theories or others (see [12], [15]), some researchers are interested in the analysis of the problem

$$\begin{cases} -\Delta v = \lambda e^v - 4\pi \sum_{j=1}^N \alpha_j \delta_{p_j} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Gamma = \{p_1, \cdots, p_N\} \subset \Omega$ is the set of prescribed singular sources (called "vortices"), δ_p is a Dirac mass supported at p, and $\alpha_j > 0$.

If we introduce the Green's function of $-\Delta$ acting on $H_0^1(\Omega)$:

$$\begin{cases} -\Delta_x G(x,p) = \delta_p & \text{for } x \in \Omega, \\ G(x,p) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

and write $G(x,p) = \frac{1}{2\pi} \log |x-p|^{-1} + H(x,p)$, where $H(x,p)$ is the regular part of G, then the problem (1.1) is equivalent to

$$\begin{cases} -\Delta u = \lambda \Pi_{j=1}^N |x-p_j|^{2\alpha_j} V(x) e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $u = v + 4\pi \sum_{j=1}^N \alpha_j G(x,p_j)$ and $V(x) = e^{-4\pi \sum_{j=1}^N \alpha_j H(x,p_j)}$ is a smooth positive function on $\overline{\Omega}$. By this reason, we are led to consider the problem (1.2) for general smooth positive functions V. In this case, the study of asymptotic behavior of solutions u_n for $\lambda = \lambda_n \to +0$ in (1.2) was done by P. Esposito in [5] (see also [6] [7]), which extends the results of [9], [10] where the regular case ($\alpha_j = 0,\ \forall j$) was considered.

Theorem 1 *(P. Esposito) Let V be a smooth positive function on $\overline{\Omega}$ and set $K(x) = \Pi_{j=1}^N |x-p_j|^{2\alpha_j} V(x)$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\lambda_n \to 0$ and let $\{u_n\}$ be a solution sequence of (1.2) for $\lambda = \lambda_n$ such that*

$$\Sigma_n = \lambda_n \int_\Omega K(x) e^u dx = O(1) \quad \text{as } n \to \infty.$$

Then the following alternative holds:

(i) If $\Sigma_n \to 0$ as $n \to \infty$, then $u_n \to 0$ in $C^{2,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and u_n coincides with the unique minimal solution of (1.2).

(ii) *If* $\Sigma_n \to L$ *for some* $L \neq 0$, *then (up to subsequence) there exists a nonempty finite set* $S = \{q_1, \cdots, q_m\} \subset \Omega$ *(blow up set) such that* $\{u_n\}$ *is uniformly bounded in* $L^\infty_{loc}(\overline{\Omega} \setminus \mathcal{S})$, *and*

$$\lambda_n K(x) e^{u_n} dx \rightharpoonup \sum_{i=1}^{m} b_i \delta_{q_i} \quad \textit{in the sense of measures,} \tag{1.3}$$

$$u_n \to \sum_{i=1}^{m} b_i G(\cdot, q_i) \quad \textit{in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{S}) \tag{1.4}$$

as $n \to \infty$, *where* $b_i = 8\pi$ *if* $q_i \notin \Gamma$, $b_i = 8\pi(1 + \alpha_j)$ *if* $q_i = p_j$ *for some* $p_j \in \Gamma$.

Furthermore, as for the location of blow up points in the case (ii), we have the following:

If $S \cap \Gamma = \phi$, *then* $(q_1, \cdots, q_m)$ *is a critical point for the function*

$$\mathcal{F}(x_1, \cdots, x_m) = \sum_{i=1}^{m} H(x_i, x_i) + \sum_{i,j=1, i\neq j}^{m} G(x_i, x_j) + \frac{1}{4\pi} \sum_{i=1}^{m} \log K(x_i).$$

If $S \cap \Gamma = \{p_{j_1}, \cdots, p_{j_s}\}$ *and* $S \setminus \Gamma = \{q_{i_1}, \cdots, q_{i_k}\}$ *with* $s + k = m$, *then* $(q_{i_1}, \cdots, q_{i_k})$ *is a critical point for the function*

$$\tilde{\mathcal{F}}(x_1, \cdots, x_k) = \mathcal{F}(x_1, \cdots, x_k) + \mathcal{G}(x_1, \cdots, x_k; p_{j_1}, \cdots, p_{j_s}),$$

where

$$\mathcal{G}(x_1, \cdots, x_k; a_1, \cdots, a_s) = \frac{1}{4\pi} \left(\sum_{i=1}^{k} \sum_{j=1}^{s} 8\pi(1 + \alpha_j) G(x_i, a_j) \right).$$

Also, as a vice versa of Theorem 1, Esposito constructed blowing up solutions with a prescribed blow up set S under the additional assumption that $\alpha_j \in (0, +\infty) \setminus \mathbb{N}$ for all $j = 1, \cdots, N$; see [6].

In the following, let $i_M(u)$ denote the Morse index of a solution u of (1.2), i.e., the number of negative eigenvalues of the operator $L_u = -\Delta - \lambda K(x) e^u \cdot$ acting on $H^1_0(\Omega)$.

Now, we state the main result of this note, which is a generalization of [13] [14] in this case.

Theorem 2 *Let $\{u_n\}$ be a solution sequence of (1.2) for $\lambda = \lambda_n$ with $\Sigma_n = O(1)$ as $n \to \infty$ and let $S = \{q_1, \cdots, q_m\}$ be its blow up set (possibly $S = \phi$). Assume $\alpha_j \in (0, +\infty) \setminus \mathbb{N}$ for all $j = 1, \cdots, N$. Then $m \leq i_M(u_n)$ for n sufficiently large.*

As a corollary, we obtain the following assertion.

Corollary 3 *Let $\{u_n\}$ be a solution sequence of (1.2) for $\lambda = \lambda_n$ with $\Sigma_n = O(1)$ as $n \to \infty$. Assume $\alpha_j \in (0, +\infty) \setminus \mathbb{N}$ for all $j = 1, \cdots, N$ and the Morse index $i_M(u_n) = 1$ for any n large. Then the number of blow up points of $\{u_n\}$ is exactly 1.*

Proof. By Theorem 2 and the assumption that $i_M(u_n) = 1$ for n large, we see that the number of blow up points $\sharp S$ is 0 or 1 for the sequence $\{u_n\}$. However, if $\sharp S = 0$, then $\{u_n\}$ is uniformly bounded and $\Sigma_n \to 0$. Thus by Theorem 1, u_n coincides with the minimal solution $\underline{u_n}$ of (1.2) for n large. It is well known that the minimal solution $\underline{u_n}$ is stable and its Morse index is exactly 0. This contradicts to the assumption $i_M(u_n) = 1$, thus we have $\sharp S = 1$. □

2 Proof of Theorem 2

In this section, we prove Theorem 2 along the line of [13], [14]. Analytical tools needed for the study of singular Liouville equations are provided in Tarantello's nice book [12]. In the proof, we need a concentration-compactness alternative result of Bartolucci and Tarantello ([2], [3], see also [12]: Proposition 5.4.32), which we recall here in the following form.

Proposition 4 *Let v_n satisfy*

$$\begin{aligned} -\Delta v_n &= |x-p|^{2\alpha} W_n(x) e^{v_n} \quad \textit{in } B_1(p) \subset \mathbb{R}^2, \\ \int_{B_r(p)} |x-p|^{2\alpha} W_n e^{v_n} dx &\leq C \quad \textit{for some } r \in (0,1], \end{aligned}$$

where $\alpha > 0$ and W_n is a C^1 function on $B_1(p)$ such that

$$0 < b_1 \leq W_n \leq b_2, \quad |\nabla W_n| \leq A \quad \textit{in } B_1(p)$$

for some $b_1, b_2, A > 0$ uniformly in n.

Then there exists $\delta \in (0,1]$ and a subsequence of v_n (denoted by the same symbol), for which only one of the following alternatives hold:

(a) v_n is bounded uniformly in $L^\infty_{loc}(B_\delta(p))$;

(b) $\sup\limits_{\Omega'} v_n \to -\infty$ for every $\Omega' \subset\subset B_\delta(p)$;

(c) there exists $z_n \in B_1(p)$ such that $z_n \to p$ and $v_n(z_n) \to +\infty$, while $\sup_{\Omega'} v_n \to -\infty$ for every $\Omega' \subset\subset B_\delta(p) \setminus \{p\}$ and $|x-p|^{2\alpha} W_n e^{v_n} \rightharpoonup \beta\delta_p$ in the sense of measures in $B_\delta(p)$ with $\beta \geq 4\pi$. Furthermore if $W_n \to W$ in C^0_{loc} for some W, then $\beta \geq 8\pi$.

Let $\{u_n\}$ be a solution sequence to (1.2) for $\lambda = \lambda_n$ with $\Sigma_n = O(1)$ as $n \to \infty$. If $\Sigma_n \to 0$, then $S = \phi$ and we have nothing to prove. Thus we consider the case (ii) of Theorem 1, and we have a blow up set $\mathcal{S} = \{q_1, \cdots, q_m\} \subset \Omega$ for (a subsequence of) $\{u_n\}$.

Let $L_n = -\Delta_x - \lambda_n K(x) e^{u_n(x)} \cdot : H^1_0(\Omega) \to H^{-1}(\Omega)$ be the linearized operator around u_n and let $\lambda_j(L_n, D)$ denote the j-th eigenvalue of L_n acting on $H^1_0(D)$ for a regular subdomain $D \subset \Omega$. Next is the key in the proof of Theorem 2.

Claim: There exist m disjoint open balls $\{B^i\}_{i=1}^m$, each $B^i \subset\subset \Omega$, such that $\lambda_1(L_n, B^i) < 0$ for any $i \in \{1, \cdots, m\}$ and for n large .

Assuming for the moment the validity of Claim, we prove Theorem 2. Indeed, by Claim, there exist m open balls $B^1, \cdots, B^m$ which are disjoint, such that

$$\lambda_1(L_n, B^i) < 0 \quad \text{for } i = 1, \cdots, m.$$

On the other hand, it is well known that

$$\lambda_m(L_n, \Omega) \leq \sum_{i=1}^m \lambda_1(L_n, B^i)$$

holds; see, for example, the Appendix of [13]. Combining these inequalities, we have $\lambda_m(L_n, \Omega) < 0$. Therefore by the definition of the Morse index of u_n, we have $m \leq i_M(u_n)$. This proves Theorem 2. □

In the following, we will prove Claim. Let $S \setminus \Gamma = \{q_{i_1}, \cdots, q_{i_k}\}$. Since $K(x) = \Pi_{j=1}^{N}|x-p_j|^{2\alpha_j}V(x)$ is strictly positive smooth function near any $q \in S \setminus \Gamma$, the argument in [14], which uses a concentration-compactness result of [4] [8], works well around $q \in S \setminus \Gamma$. Thus we can find r disjoint balls $\{B_l'\}_{l=1}^{k}$ with the desired property. We refer the reader to [14] [13].

Next, we consider blow up points in $S \cap \Gamma = \{p_{j_1}, \cdots, p_{j_s}\}$ and, for simplicity, we relabel $S \cap \Gamma = \{p_1, \cdots, p_s\}$. We choose $r > 0$ sufficiently small such that $B_r(p_i) \subset\subset \Omega$, $\{B_r(p_i)\}_{i=1}^{s}$ are disjoint, and p_i is the only blow up point of u_n in $B_r(p_i)$ for all i. Let $x_n^i \in B_r(p_i)$ be a point such that

$$u_n(x_n^i) = \max_{B_r(p_i)} u_n(x) \to +\infty, \quad x_n^i \to p_i \ (i = 1, \cdots, s),$$

as $n \to \infty$.

Now, let us define $\delta_n^i > 0$ and $\tilde{u}_n^i : B_{r/\delta_n^i}(0) \to \mathbb{R}$ so that

$$\begin{aligned}&(\delta_n^i)^{2\alpha_i+2}\lambda_n e^{u_n(p_i)} = 1,\\ &\tilde{u}_n^i(y) = u_n(\delta_n^i y + p_i) - u_n(p_i), \quad y \in B_{r/\delta_n^i}(0)\end{aligned}$$

for $i \in \{1, \cdots, s\}$.

First, we prove

Lemma 5 *$\delta_n^i \to 0$ as $n \to \infty$.*

Proof. Define $v_n(x) = u_n(x) + \log \lambda_n$. Then v_n satisfies

$$-\Delta v_n = |x-p_i|^{2\alpha_i}\hat{K}_i(x)e^{v_n} \quad \text{in } B_r(p_i), \quad v_n = u_n + \log \lambda_n \quad \text{on } \partial B_r(p_i),$$

where $K(x) = |x-p_i|^{2\alpha_i}\hat{K}_i(x)$, $\hat{K}_i(x) = \Pi_{j=1, j\neq i}^{N}|x-p_j|^{2\alpha_j}V(x)$. Note that $\hat{K}_i$ is a smooth, strictly positive function on $B_r(p_i)$. Also, Theorem 1 (1.3), (1.4) implies that

$$|x-p_i|^{2\alpha_i}\hat{K}_i(x)e^{v_n}dx \rightharpoonup 8\pi(1+\alpha_i)\delta_{p_i} \tag{2.1}$$

in the sense of measures on $B_r(p_i)$ and

$$\max_{\partial B_r(p_i)} v_n(x) - \min_{\partial B_r(p_i)} v_n(x) = \max_{\partial B_r(p_i)} u_n(x) - \min_{\partial B_r(p_i)} u_n(x) = O(1) \tag{2.2}$$

as $n \to \infty$. Recall the assumption $\alpha_i \notin \mathbb{N}$ for all i. Therefore, we can apply Proposition 5.6.50 and Corollary 5.4.24 in [12] to v_n to conclude that

$$\sup_{B_\rho(p_i)} \{v_n(x) + (2\alpha_i + 1)\log|x-p_i|\} \leq C$$

for any $\rho < r$, which implies $\left(\frac{|x_n^i - p_i|}{\delta_n^i}\right)^{2(\alpha_i+1)} \le e^C$, and

$$v_n(p_i) = \max_{B_r(p_i)} v_n + O(1) \tag{2.3}$$

as $n \to \infty$. Thus $u_n(p_i) = u_n(x_n^i) + O(1) \to \infty$ for any $i = 1, \cdots, s$ as $n \to \infty$.

Now, we claim that $v_n(p_i) \to +\infty$ as $n \to \infty$ for any $i \in \{1, \cdots, s\}$. Indeed, assume the contrary that there exists $i \in \{1, \cdots, s\}$ and a subsequence (denoted by the same symbol) such that

(i) $v_n(p_i) \to -\infty$, or

(ii) $v_n(p_i) \to C$ for some $C \in \mathbb{R}$.

When (i) happens, we see by (2.3) that

$$\int_{B_r(p_i)} K(x)e^{v_n(x)}dx \le e^{\max_{B_r(p_i)} v_n} \int_{B_r(p_i)} K(x)dx = e^{v_n(p_i)+O(1)} \int_{B_r(p_i)} K(x)dx \to 0$$

as $n \to \infty$. On the other hand, since p_i is the only blow up point of $\{u_n\}$ in $B_r(p_i)$, (2.1) implies

$$\lim_{n\to\infty} \int_{B_r(p_i)} K(x)e^{v_n}dx \ge 8\pi(1 + \alpha_i),$$

which leads to a contradiction.

When (ii) happens, again by (2.3), we see $\max_{B_r(p_i)} v_n = v_n(x_n^i) = O(1)$ as $n \to \infty$. Since $x_n^i \to p_i$ as $n \to \infty$, this case can happen only when the alternative (a) in Proposition 4 occurs: $\{v_n\}$ is bounded in $L^\infty_{loc}(B_r(p_i))$. On the other hand, since $u_n = O(1)$ locally on $B_r(p_i) \setminus \{p_i\}$ by (1.4), $v_n = u_n + \log \lambda_n \to -\infty$ on any compact set in $B_r(p_i) \setminus \{p_i\}$. This again leads to a contradiction and we have proved the claim. Now, since $(\delta_n^i)^{2(1+\alpha_i)} = \frac{1}{e^{v_n(p_i)}}$, we obtain the lemma. □

Incidentally, by (2.1), (2.2) and (2.3), we can apply Theorem 5.6.51 in [12], see also [1], to v_n to obtain the following pointwise estimate

$$\left| v_n(x) - \log \frac{e^{v_n(p_i)}}{\left(1 + \frac{1}{8(\alpha_i+1)^2} c_i e^{v_n(p_i)} |x - p_i|^{2(\alpha_i+1)}\right)^2} \right| \le C \quad \text{for } x \in B_r(p_i),$$

which is equivalent to

$$\left| u_n(x) - \log \frac{e^{u_n(p_i)}}{\left(1 + \frac{\lambda_n}{8(\alpha_i+1)^2} c_i e^{u_n(p_i)} |x - p_i|^{2(\alpha_i+1)}\right)^2} \right| \leq C \quad \text{for } x \in B_r(p_i),$$

where $c_i = \hat{K}_i(p_i)$.

Going back to the proof of Theorem 2, we see that $\tilde{u}_n^i$ satisfies

$$\begin{cases} -\Delta \tilde{u}_n^i = |y|^{2halpha_i} \hat{K}_i(\delta_n^i y + p_i) e^{\tilde{u}_n^i} \quad \text{in } B_{r/\delta_n^i}(0), \\ \hat{K}_i(\delta_n^i y + p_i) \to c_i = \hat{K}_i(p_i) \quad \text{uniformly in } C^0_{loc}(\mathbb{R}^2), \\ \tilde{u}_n^i(0) = 0, \max\limits_{B_{r/\delta_n^i}(0)} \tilde{u}_n^i = u_n(x_n^i) - u_n(p_i) = O(1), \\ \int_{B_{r/\delta_n^i}(0)} |y|^{2\alpha_i} \hat{K}_i(\delta_n^i y + p_i) e^{\tilde{u}_n^i} dy = O(1), \quad (n \to \infty). \end{cases}$$

The third equation comes from (2.3).

At this stage, we can apply Lemma 5.4.21 in [12] to $\tilde{u}_n^i$ to confirm that $\tilde{u}_n^i$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^2)$ and along a subsequence,

$$\tilde{u}_n^i \to U^i(y) \quad \text{in } C^2_{loc}(\mathbb{R}^2) \text{ as } n \to \infty, \tag{2.4}$$

where U^i satisfies

$$\begin{cases} -\Delta U^i = c_i |y|^{2\alpha_i} e^{U^i} \quad \text{in } \mathbb{R}^2, \\ U^i(0) = 0, \\ \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} dy < +\infty. \end{cases}$$

By a classification result of Prajapat and Tarantello [11] and the assumption $\alpha_i \notin \mathbb{N}$, we have

$$U^i(y) = -2 \log \left(1 + \frac{c_i}{8(\alpha_i + 1)^2} |y|^{2(\alpha_i+1)} \right) \quad \text{for } i = 1, \cdots, s.$$

Now, we define

$$\tilde{L}_n^i = -\Delta_y - |y|^{2\alpha_i} \hat{K}_i(\delta_n^i y + p_i) e^{\tilde{u}_n^i(y)} \cdot : H_0^1(B_{r/\delta_n^i}(0)) \to H^{-1}(B_{r/\delta_n^i}(0)).$$

This operator is related to L_n by the formula

$$(\delta_n^i)^2 L_n \Big|_{u_n(x) = \tilde{u}_n^i(y) + u_n(p_i)} = \tilde{L}_n^i,$$

where $x = \delta_n^i y + p_i$ for $x \in B_r(p_i)$ and $y \in B_{r/\delta_n^i}(0)$. Also for a domain $D \subset B_r(p_i)$, we have

$$(\delta_n^i)^2 \lambda_1(L_n, D) = \lambda_1(\tilde{L}_n^i, D_n^i), \quad D_n^i = \frac{D - p_i}{\delta_n^i}, \tag{2.5}$$

where $\lambda_1(\tilde{L}_n^i, D_n^i)$ denotes the first eigenvalue of $\tilde{L}_n^i$ acting on $H_0^1(D_n^i)$.

Now, we show

Lemma 6 *There exist disjoint balls $\{B_{\delta_n^i R}(p_i)\}_{i=1,\cdots,s}$ for some $R > 0$ such that $\lambda_1(L_n, B_{\delta_n^i R}(p_i)) < 0$ for n large and for any $i \in \{1, \cdots, s\}$.*

Proof. For $R > 0$, we define

$$w_R(y) = 2 \log \frac{8 + R^2}{8 + |y|^2} \in H_0^1(B_R(0)).$$

We will prove that $(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} < 0$ for $n \in \mathbb{N}$ and $R > 0$ sufficiently large with $B_R(0) \subset B_{r/\delta_n^i}(0)$. Indeed,

$$\begin{aligned}(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} &= \int_{B_R(0)} |\nabla w_R|^2 dy - \int_{B_R(0)} |y|^{2\alpha_i} \hat{K}_i(\delta_n^i y + p_i) e^{\tilde{u}_n^i(y)} w_R^2(y) dy \\ &=: I_1 - I_2.\end{aligned}$$

We observe that

$$I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8 + |y|^2)^2} dy = 2\pi \int_0^R \frac{16r^2}{(8 + r^2)^2} r dr \le 32\pi (\log R) [1 + o_R(1)],$$

where $o_R(1) \to 0$ as $R \to \infty$. On the other hand, we have

$$\begin{aligned}I_2 &= \int_{B_R(0)} |y|^{2\alpha_i} \hat{K}_i(\delta_n^i y + x_n^i) e^{\tilde{u}_n^i(y)} w_R^2(y) dy \\ &= c_i \int_{B_R(0)} \frac{|y|^{2\alpha_i}}{\left(1 + \frac{c_i}{8(\alpha_i+1)^2}|y|^{2\alpha_i+2}\right)^2} \left\{2 \log \frac{8 + R^2}{8 + |y|^2}\right\}^2 dy + o_n(1) \\ &= 8\pi c_i \int_0^R \frac{r^{2\alpha_i+1}}{\left(1 + \frac{c_i}{8(\alpha_i+1)^2} r^{2\alpha_i+2}\right)^2} \left\{\log(8 + R^2) - \log(8 + r^2)\right\}^2 dr + o_n(1) \\ &= 8\pi c_i \left[\frac{4(\alpha_i + 1)}{c_i} + o_R(1)\right] \left\{\log(8 + R^2)\right\}^2 + o_n(1) \\ &= 32\pi(\alpha_i + 1) \left\{\log(8 + R^2)\right\}^2 [1 + o_R(1)] + o_n(1),\end{aligned}$$

where we have used (2.4) and

$$\int_0^R \frac{r^{2\alpha+1}}{(1+cr^{2\alpha+2})^2}dr = \int_0^\infty \frac{r^{2\alpha+1}}{(1+cr^{2\alpha+2})^2}dr + o_R(1) = \frac{1}{2(\alpha+1)c} + o_R(1)$$

for $\alpha, c > 0$. Thus we obtain

$$(\tilde{L}_n^i w_R, w_R)_{L^2(B_R)} = I_1 - I_2 \leq -32\pi(\alpha_i + 1)\left\{\log(8+R^2)\right\}^2 [1 + o_R(1)] < 0$$

by taking n sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset B_{r/\delta_n^i}(0)$. This implies that the first eigenvalue of the operator $\tilde{L}_n^i$ on B_R is negative: $\lambda_1(\tilde{L}_n^i, B_R) < 0$. By this calculation and (2.5) proves that $\lambda_1(L_n, B_{\delta_n^i R}(p_i)) < 0$ for $i = 1, \cdots, s$. These balls $\{B_i\}_{i=1}^s = \{B_{\delta_n^i R}(p_i)\}_{i=1}^s$ can be disjoint if we choose sufficiently large n, since the blow up set S is finite and $\delta_n^i = o(1)$ as $n \to \infty$. □

Since balls $\{B_i\}_{i=1}^s$ in Lemma 6 can also be made disjoint from balls $\{B_l'\}_{l=1}^k$ (former obtained around points in $S \setminus \Gamma$), we obtain Claim. The proof of Theorem 2 is completed. □

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