Asymptotic behavior of densities for stochastic functional differential equations

AKIHIRO KITAGAWA* AND ATSUSHI TAKEUCHI†‡

Abstract

Consider stochastic functional differential equations depending on whole past histories in a finite time interval, which determine non-Markovian processes. Under the uniformly elliptic condition on the coefficients of the diffusion terms, the solution admits a smooth density with respect to the Lebesgue measure. In the present paper, we shall study the large deviations for the family of the solution process, and the asymptotic behaviors of the density. The Malliavin calculus plays a crucial role in our argument.

Keywords: stochastic functional differential equations, large deviations, Malliavin calculus, density estimate.
MSC 2010: 34K50, 60F10, 60H07, 62G07.

1 Introduction

Stochastic functional differential equations, or stochastic delay differential equations determine non-Markovian processes, because the current states of the process in the equation depend on the past histories of the process. Such kind of equations was initiated by K. Itô and M. Nisio in their pioneering work [7] about 50 years ago. As stated in [14], there are some difficulties to study such equations, because we cannot use any methods in analysis, partial differential equations, and potential theory at all. On the other hand, it seems to be more natural to consider

This version: November 22, 2012.

*Email address: rena.last-genesis@dance.ocn.ne.jp. Postal address: Aikou Educational Institute, Kinuyama 5-1610-1, Matsuyama, Ehime 791-8501, Japan.

†Email address: takeuchi@sci.osaka-cu.ac.jp. Postal address: Department of Mathematics, Osaka City University, Sugimoto 3-3-138, Sumiyoshi-ku, Osaka 558-8585, Japan.

‡Corresponding author

the models determined by the solutions to the stochastic functional differential equations in finance, physics, biology, etc., because such processes include their past histories, and can be recognized to reflect real phenomena in various fields much more exactly.

The Malliavin calculus is well known as a powerful tool to study some properties on the density function by a probabilistic approach. There are a lot of works on the densities for diffusion processes by many authors, from the viewpoint of the Malliavin calculus (cf. [2]). Moreover, it is also applicable to the case of solutions to stochastic functional differential equations, regarding as one of the examples of the Wiener functionals. Kusuoka and Stroock in [9] studied the application of the Malliavin calculus to solutions to stochastic functional differential equations, and obtained the result on the existence of the smooth density for the solution with respect to the Lebesgue measure. On the other hand, it is well known that the Malliavin calculus is very fruitful to study the asymptotic behavior of the density function related to the large deviations theory (cf. Léandre [10, 11, 12, 13] and Nualart [17]). In fact, the Varadhan-type estimate of the density function for the diffusion processes can be also obtained from this viewpoint. Ferrante et al. in [4] discussed such problem in the case of stochastic delay differential equations, where the drift term depends on the whole past histories on the finite time interval, while the diffusion terms depend on the state only for the edges of the finite time interval. Mohammed and Zhang in [16] studied the large deviations for the solution process under a similar situation to [4]. But, the special forms on the diffusion terms play a crucial role throughout their arguments in [4, 16].

In the present paper, we shall study the large deviations on the solution process to the stochastic functional differential equations. Our stochastic functional differential equations are much more general, because they are time inhomogeneous, and not only the drift terms but also the diffusion terms in the equation depend on the whole past histories of the process over a finite interval. Furthermore, as a typical application of the large deviation theory and the Malliavin calculus, we shall study the asymptotic behavior, so-called the Varadhan-type estimate, of the density function for the solution process, which is quite similar to the case of diffusion processes. The effect of the time delay plays a crucial role in the behavior of the density function, and the obtained result can be also regarded as the natural extension of the estimate for diffusion processes, which are the most interesting points in the present paper.

The paper is organized as follows: In Section 2, we shall prepare some notations and introduce our stochastic functional differential equations. Section 3 will be devoted to the brief summary on the Malliavin calculus, and the application it to our equations. We shall consider some estimates, which guarantee the smoothness of the solution process, and the non degeneracy in the Malliavin sense. The existence of the smooth density will be also discussed in Section

3. The negative-order moments of the Malliavin covariance matrix will be studied there, which is important in order to give the estimate of the density function. Sections 4 and 5 are our main goals in the present paper. In Section 4, we shall focus on the large deviation principles on the solution processes. As an application of the result obtained in Section 4, we shall study the asymptotic behavior on the density for the solution process. Moreover, we can also derive the short time asymptotics on the density function, which can be interpreted as the generalization of the Varadhan-type estimate on diffusion processes (cf. [10, 11, 12, 13, 17]).

2 Preliminaries

Let r and T be positive constants, and denote an m-dimensional Brownian motion by $W = \big\{W(t) = \big(W^1(t), \ldots, W^m(t)\big)\,;\, t \in [0,T]\big\}$. Let A_i $(i = 0, 1, \ldots, m)$ be $\mathbb{R}^d$-valued functions on $[0,T] \times C\big([-r,0]\,;\mathbb{R}^d\big)$ such that, for each $t \in [0,T]$, the mapping $A_i(t,\cdot) : C\big([-r,0]\,;\mathbb{R}^d\big) \ni f \longmapsto A_i(t,f) \in \mathbb{R}^d$ is smooth in the Frechét sense, and all Frechét derivatives of any orders greater than 1 are bounded. Under the conditions stated above, the functions A_i $(i = 0, 1, \ldots, d)$ satisfy the linear growth condition and the Lipschitz condition in the functional sense of the form:

$$\sup_{t\in[0,T]} \sum_{i=0}^{m} \big|A_i(t,f)\big| \le C_{1,T}\,\big(1 + \|f\|_\infty\big), \tag{1}$$

$$\sup_{t\in[0,T]} \sum_{i=0}^{m} \big|A_i(t,f) - A_i(t,g)\big| \le C_{2,T}\,\|f-g\|_\infty \tag{2}$$

for $f, g \in C\big([-r,0]\,;\mathbb{R}^d\big)$, where $\|f\|_\infty = \sup_{t\in[-r,0]} |f(t)|$. Denote by $A = \big(A_1, \ldots, A_m\big)$.

Let $0 < \varepsilon \le 1$ be sufficiently small. For a deterministic path $\eta \in C\big([-r,0]\,;\mathbb{R}^d\big)$, we shall consider the $\mathbb{R}^d$-valued process $X^\varepsilon = \{X^\varepsilon(t)\,;\, t \in [-r,T]\}$ given by the stochastic functional differential equation of the form:

$$\begin{cases} X^\varepsilon(t) = \eta(t) & (t \in [-r,0]), \\ \mathrm{d}X^\varepsilon(t) = A_0\big(t, X_t^\varepsilon\big)\,\mathrm{d}t + \varepsilon\, A\big(t, X_t^\varepsilon\big)\,\mathrm{d}W(t) & (t \in (0,T]), \end{cases} \tag{3}$$

where $X_s^\varepsilon = \{X^\varepsilon(s+u)\,;\, u \in [-r,0]\}$ is the segment. Since the current state of the solution depends on its past histories, the process X^ε is non-Markovian clearly. Since the coefficients of the equation (3) satisfy the Lipschitz and the linear growth condition in the functional sense, there exists a unique solution to the equation (3), via the successive approximation $X^{\varepsilon,(n)} =$

$\left\{X^{\varepsilon,(n)}(t)\,;\,t\in[-r,T]\right\}$ $(n\in\mathbb{Z}_+)$ of the solution process X to the equation (3) as follows:

$$\begin{cases} X^{\varepsilon,(0)}(t)=\eta(t) & (t\in[-r,0]),\\ X^{\varepsilon,(0)}(t)=\eta(0) & (t\in(0,T]), \end{cases} \tag{4}$$

$$\begin{cases} X^{\varepsilon,(n)}(t)=\eta(t) & (t\in[-r,0]),\\ \mathrm{d}X^{\varepsilon,(n)}(t)=A_0\big(t,X_t^{\varepsilon,(n-1)}\big)\,\mathrm{d}t+\varepsilon A\big(t,X_t^{\varepsilon,(n-1)}\big)\,\mathrm{d}W(t) & (t\in(0,T]) \end{cases} \tag{5}$$

for $n\in\mathbb{N}$ (cf. Ito-Nisio [7], Mohammed [14, 15]).

Proposition 2.1 *For any $p>1$, it holds that*

$$\sup_{0<\varepsilon\le 1}\mathbb{E}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|^{p}\right]\le C_{3,p,T,\eta}.$$

Proof. Let $p>2$ and $t\in[0,T]$. The Hölder inequality and the Burkholder inequality tell us to see that

$$\begin{aligned}
\mathbb{E}\left[\sup_{\tau\in[-r,t]}\left|X^{\varepsilon}(\tau)\right|^{p}\right] &\le C_{4,p}\,\|\eta\|_\infty^p+C_{4,p}\,\mathbb{E}\left[\sup_{\tau\in[0,t]}\left|X^{\varepsilon}(\tau)\right|^{p}\right]\\
&\le C_{4,p}\,\|\eta\|_\infty^p+C_{5,p}\,\mathbb{E}\left[\sup_{\tau\in[0,t]}\left|\int_0^\tau A_0\big(s,X_s^{\varepsilon}\big)\,\mathrm{d}s\right|^{p}\right]\\
&\qquad+C_{5,p}\,\varepsilon^p\,\mathbb{E}\left[\sup_{\tau\in[0,t]}\left|\int_0^\tau A\big(s,X_s^{\varepsilon}\big)\,\mathrm{d}W(s)\right|^{p}\right]\\
&\le C_{4,p}\,\|\eta\|_\infty^p+C_{5,p}\,T^{p-1}\int_0^t\mathbb{E}\left[\left|A_0\big(s,X_s^{\varepsilon}\big)\right|^{p}\right]\mathrm{d}s\\
&\qquad+C_{6,p}\,\varepsilon^p\,T^{p/2-1}\int_0^t\sum_{i=1}^m\mathbb{E}\left[\left|A_i\big(s,X_s^{\varepsilon}\big)\right|^{p}\right]\mathrm{d}s\\
&\le C_{7,p,T,\eta}+C_{8,p,T}\int_0^t\mathbb{E}\left[\sup_{\tau\in[-r,s]}\left|X^{\varepsilon}(\tau)\right|^{p}\right]\mathrm{d}s
\end{aligned}$$

from the linear growth condition on the coefficients A_i $(i=0,1,\ldots,m)$. Hence, the Gronwall inequality enables us to obtain the assertion for $p>2$.

As for $1<p\le 2$, the Jensen inequality yields us to see that

$$\sup_{0<\varepsilon\le 1}\mathbb{E}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|^{p}\right]\le\left(\sup_{0<\varepsilon\le 1}\mathbb{E}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|^{2p}\right]\right)^{1/2},$$

which implies the assertion by using the consequence stated above. The proof is complete. □

3 Applications of the Malliavin calculus

At the beginning, we shall introduce the outline of the Malliavin calculus on the Wiener space $C_0\big([0,T]\,;\mathbb{R}^m\big)$, briefly, where $C_0\big([0,T]\,;\mathbb{R}^m\big)$ is the set of $\mathbb{R}^m$-valued continuous functions on $[0,T]$ starting from the origin. See Di Nunno et al. [5] and Nualart [17, 18]) for details. Let H be the Cameron-Martin subspace of $C\big([0,T]\,;\mathbb{R}^m\big)$ with the inner product

$$\langle g,h\rangle_H = \int_0^T \dot{g}(t)\cdot\dot{h}(t)\,dt \quad (g,h\in H).$$

Denote by $\mathscr{S}$ the set of $\mathbb{R}$-valued random variables such that a random variable F is represented as the following form:

$$F(W) = f\big(W[h_1],\cdots,W[h_n]\big)$$

for $W\in C_0\big([0,T]\,;\mathbb{R}^m\big)$, where $h_1,\cdots,h_n\in H$, $W[h]=\int_0^T h(s)\cdot dW(s)$ for $h\in H$, and $f\in C_p^\infty(\mathbb{R}^n;\mathbb{R})$. Here, we shall denote by $C_p^\infty(\mathbb{R}^n;\mathbb{R})$ the set of smooth functions on $\mathbb{R}^n$ such that all derivatives of any orders have polynomial growth. For $k\in\mathbb{N}$, the k-th Malliavin-Shigekawa derivative $D^kF=\Big\{D^k_{(u_1,\cdots,u_k)}F\,;\,u_1,\cdots,u_k\in[0,T]\Big\}$ for $F\in\mathscr{S}$ is defined by

$$D^k_{(u_1,\cdots,u_k)}F(W) = \begin{cases} \displaystyle\sum_{j=1}^n \partial_j f\big(W[h_1],\cdots,W[h_n]\big)\int_0^{u_1} h_j(s)\,\mathrm{d}s & (k=1),\\ D_{u_1}\cdots D_{u_k}F(W) & (k\geq 2).\end{cases}$$

We shall consider $D^0F=F$, which helps us to define the operator D^k for $k\in\mathbb{Z}_+$. For $p>1$ and $k\in\mathbb{Z}_+$, let $\mathbb{D}_{k,p}$ be the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,p} = \begin{cases} \big(\mathbb{E}\big[|F|^p\big]\big)^{1/p} & (k=0),\\ \mathbb{E}\big[\,|F|^p\,\big]^{1/p} + \displaystyle\sum_{j=1}^k \mathbb{E}\Big[\,\|D^jF\|^p_{H^{\otimes j}}\,\Big]^{1/p} & (k\in\mathbb{N}).\end{cases}$$

Let $\mathbb{D}_{k,p}(\mathbb{R}^d)$ be the set of $\mathbb{R}^d$-valued random variables with the components of which belong to $\mathbb{D}_{k,p}$, and set $\mathbb{D}_\infty(\mathbb{R}^d)=\bigcap_{p>1}\bigcap_{k\in\mathbb{Z}_+}\mathbb{D}_{k,p}(\mathbb{R}^d)$. For $F\in\mathbb{D}_{1,2}(\mathbb{R}^d)$, the $\mathbb{R}^d\otimes\mathbb{R}^d$-valued random variable V_F given by

$$V_F = \langle DF,DF\rangle_H = \int_0^T \frac{\mathrm{d}}{\mathrm{d}u}D_uF\cdot\frac{\mathrm{d}}{\mathrm{d}u}D_uF\,\mathrm{d}u$$

is well defined, which is called the Malliavin covariance matrix for F.

Before studying the application of the Malliavin calculus to the solution process X to the equation (3), we shall prepare two basic and well-known facts.

Lemma 3.1 (cf. Kusuoka-Stroock [9], Lemma 2.1) *Let Γ be a real separable Hilbert space, and $\alpha : [0,T]\times\Omega \to \mathbb{R}^m\otimes\Gamma$ be a progressively measurable process such that*

$$\mathbb{E}\left[\int_0^T \|\alpha(s)\|^p_{\mathbb{R}^m\otimes\Gamma}\, ds\right] < +\infty$$

for all $p > 1$. Then, for any $p > 2$ and $\tau \in [0,T]$, it holds that

$$\mathbb{E}\left[\sup_{t\in[0,\tau]}\left\|\int_0^t \alpha(s)\, dW(s)\right\|^p_\Gamma\right] \le C_{9,p,\tau}\int_0^\tau \sum_{i=1}^m \mathbb{E}\left[\|\alpha_i(s)\|^p_\Gamma\right] ds.$$

Lemma 3.2 (cf. Nualart [17], Proposition 1.3.8) *Let $\{\beta(t)\,;\,t\in[0,T]\}$ be a $(\mathscr{F}_t)$-adapted, $\mathbb{R}^m\otimes\mathbb{R}^d$-valued process such that $\beta(t)\in\mathbb{D}_{1,2}(\mathbb{R}^m\otimes\mathbb{R}^d)$ for almost all $t\in[0,T]$, and that*

$$\mathbb{E}\left[\int_0^T\int_0^T |D_u\beta(t)|^2\, du\, dt\right] < +\infty.$$

Then, for each $t\in[0,T]$, it holds that $\int_0^t \beta(s)\, dW(s)\in\mathbb{D}_{1,2}(\mathbb{R}^d)$, and that

$$D_u\left(\int_0^t \beta(s)\, dW(s)\right) = \int_0^{u\wedge t}\beta(s)\, du + \int_0^t D_u\beta(s)\, dW(s).$$

Now, we shall return our position to study the application of the Malliavin calculus to the solution process (3).

Proposition 3.3 *Let $n\in\mathbb{Z}_+$ and $0<\varepsilon\le 1$. Then, for each $t\in[-r,T]$, the $\mathbb{R}^d$-valued random variable $X^{\varepsilon,(n)}(t)$ is in $\mathbb{D}_\infty(\mathbb{R}^d)$. Moreover, for each $k\in\mathbb{Z}_+$, it holds that*

$$\mathbb{E}\left[\sup_{t\in[-r,T]}\left\|D^kX^{\varepsilon,(n)}(t)\right\|^p_{H^{\otimes k}\otimes\mathbb{R}^d}\right] \le C_{10,k,p,T,\eta}, \tag{6}$$

$$\mathbb{E}\left[\sup_{t\in[-r,T]}\left\|D^kX^{\varepsilon,(n)}(t) - D^kX^{\varepsilon,(n-1)}(t)\right\|^p_{H^{\otimes k}\otimes\mathbb{R}^d}\right] \le \frac{C_{11,k,p,T,\eta}}{2^{n/2}}. \tag{7}$$

Proof. At the beginning, we shall consider the case $p>2$, inductively on $k\in\mathbb{Z}_+$. As for $k=0$, it is a routine work to check the assertion via the Hölder inequality and the Burkholder inequality, from the Lipschitz condition and the linear growth condition on the coefficients A_i $(i=0,1,\ldots,m)$, similarly to Proposition 2.1. Next, we shall discuss the case $k=1$. Let $n\in\mathbb{N}$, because the assertion of $n=0$ is trivial. Since $DX^{\varepsilon,(n)}=0$ for $t\in[-r,0]$, we have only to prove the assertion for $t\in(0,T]$. The chain rule on the operator D and Lemma 3.2 tell us to see that

$$\begin{aligned} D_uX^{\varepsilon,(n)}(t) = \varepsilon\int_0^u A\big(s,X_s^{\varepsilon,(n-1)}\big)\,\mathbb{I}_{(s\le t)}\,\mathrm{d}s + \int_0^t \nabla A_0\big(s,X_s^{\varepsilon,(n-1)}\big)\, D_uX_s^{\varepsilon,(n-1)}\,\mathrm{d}s \\ + \varepsilon\int_0^t \nabla A\big(s,X_s^{\varepsilon,(n-1)}\big)\, D_uX_s^{\varepsilon,(n-1)}\,\mathrm{d}W(s) \end{aligned} \tag{8}$$

for $u \in [0,T]$ (cf. Ferrante et al. [4], Lemma 6.1), where the symbol ∇ is the Frechét derivative in $C\big([-r,0]\,;\mathbb{R}^d\big)$. Thus, the Hölder inequality and Lemma 3.1 enable us to get the assertions. Finally, we shall discuss the general case $k \in \mathbb{Z}_+$. Suppose that the assertions are right until the case $n-1$. Remark that

$$\begin{aligned}
&D^k_{u_1,\ldots,u_k}\left(\int_0^t A\big(s,X_s^{\varepsilon,(n-1)}\big)\,\mathrm{d}W(s)\right)\\
&= D^{k-1}_{u_1,\ldots,u_{k-1}}\left(\int_0^t D_{u_k}\Big(A\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}W(s)\right) + D^{k-1}_{u_1,\ldots,u_{k-1}}\left(\int_0^{u_k\wedge t} A\big(s,X_s^{\varepsilon,(n-1)}\big)\,\mathrm{d}s\right)\\
&= \cdots\\
&= \int_0^t D^k_{u_1,\ldots,u_k}\Big(A\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}W(s) + \sum_{\sigma\in\mathfrak{S}_k}\int_0^{u_{\sigma(k)}\wedge t} D^{k-1}_{u_{\sigma(1)},\ldots,u_{\sigma(k-1)}}\Big(A\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}s
\end{aligned}$$

from Lemma 3.2, where $\mathfrak{S}_k$ is the set of permutations of $\{1,\ldots,k\}$. Since

$$\begin{aligned}
&D^k_{u_1,\ldots,u_k}\Big(A_i\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\\
&= D^{k-1}_{u_1,\ldots,u_{k-1}}\Big(\nabla A_i\big(s,X_s^{\varepsilon,(n-1)}\big)\,D_{u_k}X_s^{\varepsilon,(n-1)}\Big)\\
&= \sum_{\sigma\in\mathfrak{S}_k}\sum_{j=0}^{k-1}\binom{k-1}{j} D^{k-1-j}_{u_{\sigma(1)},\ldots,u_{\sigma(k-j)}}\Big(\nabla A_i\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,D^{j+1}_{u_{\sigma(k-j+1)},\ldots,u_{\sigma(k-1)},u_k}X_s^{\varepsilon,(n-1)}
\end{aligned}$$

for $i = 0, 1, \ldots, m$, and

$$\begin{aligned}
&D^k_{u_1,\ldots,u_k}X^{\varepsilon,(n)}(t)\\
&= D^k_{u_1,\ldots,u_k}\left(\int_0^t A_0\big(s,X_s^{\varepsilon,(n-1)}\big)\,\mathrm{d}s\right) + D^k_{u_1,\ldots,u_k}\left(\varepsilon\int_0^t A\big(s,X_s^{\varepsilon,(n-1)}\big)\,\mathrm{d}W(s)\right)\\
&= \sum_{\sigma\in\mathfrak{S}_k}\varepsilon\int_0^{u_{\sigma(k)}\wedge t} D^{k-1}_{u_{\sigma(1)},\ldots,u_{\sigma(k-1)}}\Big(A\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}s\\
&\qquad + \int_0^t D^k_{u_1,\ldots,u_k}\Big(A_0\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}s + \varepsilon\int_0^t D^k_{u_1,\ldots,u_k}\Big(A\big(s,X_s^{\varepsilon,(n-1)}\big)\Big)\,\mathrm{d}W(s),
\end{aligned}$$

we can get the assertion by using the Hölder inequality, Lemma 3.1 and the assumption on the case until $k-1$ of the induction.

The case $1 < p \le 2$ is the direct consequence by the Jensen inequality. The proof is complete. □

Proposition 3.4 *For $t \in [-r,T]$, the $\mathbb{R}^d$-valued random variable $X^\varepsilon(t)$ is in $\mathbb{D}_\infty\big(\mathbb{R}^d\big)$. Moreover, for each $u \in [0,T]$, the $\mathbb{R}^d\otimes\mathbb{R}^d$-valued process $\{D_uX^\varepsilon(t)\,;\,t\in[-r,T]\}$ satisfies the equation of* *the form:*

$$\begin{cases} D_u X^\varepsilon(t) = \mathbf{0} & (t \in [-r,0] \text{ or } t < u) \\ D_u X^\varepsilon(t) = \varepsilon \int_0^{u\wedge t} A\big(s, X_s^\varepsilon\big)\, ds + \int_0^t \nabla A_0\big(s, X_s^\varepsilon\big)\, D_u X_s^\varepsilon\, ds & \\ \qquad\qquad + \varepsilon \int_0^t \nabla A\big(s, X_s^\varepsilon\big)\, D_u X_s^\varepsilon\, dW(s) & (t \in [u,T]). \end{cases} \tag{9}$$

Proof. Let $p > 1$ and $k \in \mathbb{Z}_+$ be arbitrary. For each $t \in [-r,T]$, the sequence $\big\{X^{\varepsilon,(n)}(t)\,;\, n \in \mathbb{N}\big\}$ is the Cauchy one in $\mathbb{D}_{k,p}\big(\mathbb{R}^d\big)$, from Proposition 3.3. Hence, we can find the limit, denoted by $\tilde{X}^\varepsilon(t)$, in $\mathbb{D}_{k,p}\big(\mathbb{R}^d\big)$. Then, it is a routine work to see that the process $\big\{\tilde{X}^\varepsilon(t)\,;\, t \in [-r,T]\big\}$ satisfies the equation (3), via the Hölder inequality and the Burkholder inequality, from the conditions on the coefficients A_i $(i = 0, 1, \ldots, m)$, which implies $\tilde{X}^\varepsilon(t) = X^\varepsilon(t)$ for $t \in [-r,T]$ from the uniqueness of the solutions. Thus, we can get $X(t) \in \mathbb{D}_{k,p}\big(\mathbb{R}^d\big)$ for $t \in [-r,T]$. Similarly, we can check that $\big\{D_u X(t)\,;\, u \in [0,T]\big\}$ satisfies the equation (9), by taking the limit in each terms of (8) via the Hölder inequality and Lemma 3.1. □

For $u \in [0,T]$, denote by $\{Z^\varepsilon(t,u)\,;\, t \in [-r,T]\}$ the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process determined by the following equation:

$$\begin{cases} Z^\varepsilon(t,u) = \mathbf{0} & (t \in [-r,0] \text{ or } t < u), \\ Z^\varepsilon(u,u) = I_d & \\ \mathrm{d}Z^\varepsilon(t,u) = \nabla A_0\big(t, X_t^\varepsilon\big)\, Z_t^\varepsilon(\cdot,u)\, \mathrm{d}t + \varepsilon \nabla A\big(t, X_t^\varepsilon\big)\, Z_t^\varepsilon(\cdot,u)\, \mathrm{d}W(t) & (t \in (u,T]), \end{cases} \tag{10}$$

where $Z_t^\varepsilon(\cdot,u) = \big\{Z^\varepsilon(t+\tau,u)\,;\, \tau \in [-r,0]\big\}$.

Corollary 3.5

$$D_u X^\varepsilon(t) = \varepsilon \int_0^{u\wedge t} Z^\varepsilon(t,s)\, A\big(s, X_s^\varepsilon\big)\, ds.$$

Proof. Direct consequence of Proposition 3.4, and the uniqueness of the solution to (9). □

Finally, we shall introduce the well-known criterion on the existence of the smooth density for the probability law of $X^\varepsilon(t)$ with respect to the Lebesgue measure on $\mathbb{R}^d$.

Lemma 3.6 (cf. Kusuoka-Stroock [9]) *Suppose that the uniformly elliptic condition on the coefficients A_i $(i = 1, \ldots, m)$ of the equation (3):*

$$\inf_{\zeta \in \mathbb{S}^{d-1}} \inf_{t \in [0,T]} \inf_{f \in C([-r,0]\,;\,\mathbb{R}^d)} \sum_{i=1}^m \big(\zeta \cdot A_i(t,f)\big)^2 > 0. \tag{11}$$

Then, for each $t \in (0,T]$ and $0 < \varepsilon \le 1$, there exists a smooth density $p^\varepsilon(t,y)$ for the probability law of $X^\varepsilon(t)$ with respect to the Lebesgue measure over $\mathbb{R}^d$.

Proof. Since $X(t) \in \mathbb{D}_\infty\big(\mathbb{R}^d\big)$ from Proposition 3.4, it is sufficiently to study that $\big(\det V^\varepsilon(t)\big)^{-1} \in \bigcap_{p>1} \mathbb{L}^p(\Omega)$ under the uniformly elliptic condition (11). Denote by

$$\tilde{V}^\varepsilon(t) = \int_0^t \sum_{i=1}^m Z^\varepsilon(t,u) A_i\big(u, X_u^\varepsilon\big) A_i\big(u, X_u^\varepsilon\big)^* Z^\varepsilon(t,u)^* \,\mathrm{d}u.$$

Then, $V^\varepsilon(t) = \varepsilon^2 \tilde{V}^\varepsilon(t)$, so we have only to discuss the moment estimate on $\tilde{V}^\varepsilon(t)$. As stated in Lemma 1 of Komatsu-Takeuchi [8], we shall pay attention to the boundedness of

$$\sup_{\zeta \in \mathbb{S}^{d-1}} \mathbb{E}\left[\big(\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)^{-p}\right]$$

for any $p > 1$, which is sufficient to our goal. Since

$$\mathbb{E}\left[\big(\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)^{-p}\right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} \lambda^{p-1}\, \mathbb{E}\left[\exp\big(-\lambda\,\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)\right] \mathrm{d}\lambda,$$

we have to study the decay order of $\sup_{\zeta \in \mathbb{S}^{d-1}} \mathbb{E}\left[\exp\big(-\lambda\,\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)\right]$ as $\lambda \to +\infty$,

Let $\lambda > 1$ be sufficiently large. Remark that

$$\mathbb{E}\left[\big\|Z^\varepsilon(t,u) - I_d\big\|_{\mathbb{R}^d \otimes \mathbb{R}^d}^p\right] \le C_{12,p,T}\,(t-u)^{p/2}$$

for any $p > 1$, from the Burkholder inequality and the Hölder inequality. Let $\xi > 1/2$, $1 < \gamma < 2\xi$ and $0 < \sigma < (\gamma - 1)/2$. Write $t_\xi := t - \lambda^{-\xi}$, and let $\zeta \in \mathbb{S}^{d-1}$. Then, we see that

$$\begin{aligned}
&\mathbb{E}\left[\exp\big(-\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)\right] \\
&\le \mathbb{E}_1\left[\exp\big(-\lambda\,\zeta \cdot \tilde{V}^\varepsilon(t)\,\zeta\big)\right] + \mathbb{P}\left[\int_{t_\xi}^t \big\|Z^\varepsilon(t,u) - I_d\big\|_{\mathbb{R}^d \otimes \mathbb{R}^d}^2 du \ge \lambda^{-\gamma}\right] \\
&\qquad + \mathbb{P}\left[\sup_{s \in [-r,t]} \big|X^\varepsilon(s)\big| \ge \lambda^\sigma\right] \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\mathbb{E}_1[\,\cdot\,] := \mathbb{E}\left[\,\cdot\, : \int_{t_\xi}^t \big\|Z^\varepsilon(t,u) - I_d\big\|_{\mathbb{R}^d \otimes \mathbb{R}^d}^2 \,\mathrm{d}u < \lambda^{-\gamma},\ \sup_{s \in [-r,t]} \big|X^\varepsilon(s)\big| < \lambda^\sigma\right].$$

The Chebyshev inequality yields that

$$I_2 \le \lambda^{\gamma p}\, \mathbb{E}\left[\left(\int_{t_\xi}^t \big\|Z^\varepsilon(t,u) - I_d\big\|_{\mathbb{R}^d \otimes \mathbb{R}^d}^2 \,\mathrm{d}u\right)^p\right] \le C_{13,p,T}\, \lambda^{-(2\xi - \gamma)p}.$$

Similarly, the Chebyshev inequality leads to

$$I_3 \le \lambda^{-\sigma p} \mathbb{E}\left[\sup_{s\in[-r,t]} \left|X^\varepsilon(s)\right|^p\right] \le C_{14,p,T,\eta}\,\lambda^{-\sigma p}$$

from Proposition 2.1. On the other hand, as for I_1, we have

$$\begin{aligned}
I_1 &\le \mathbb{E}_1\left[\exp\left(-\lambda\int_{t_\xi}^t \sum_{i=1}^m \left|\zeta\cdot Z^\varepsilon(t,u)A_i\big(u,X_u^\varepsilon\big)\right|^2 \mathrm{d}u\right)\right] \\
&\le \mathbb{E}_1\Bigg[\exp\left(-\frac{\lambda}{2}\inf_{\zeta\in\mathbb{S}^{d-1}}\int_{t_\xi}^t \sum_{i=1}^m \left|\zeta\cdot A_i\big(u,X_u^\varepsilon\big)\right|^2 \mathrm{d}u\right) \\
&\qquad\qquad \times \exp\left(\lambda\int_{t_\xi}^t \left\|Z^\varepsilon(t,u)-I_d\right\|^2_{\mathbb{R}^d\otimes\mathbb{R}^d} \sum_{i=1}^m \left|A_i\big(u,X_u^\varepsilon\big)\right|^2 \mathrm{d}u\right)\Bigg] \\
&\le \exp(\lambda^{1-\gamma+2\sigma})\,\exp\left(-\frac{\lambda}{2}\inf_{\zeta\in\mathbb{S}^{d-1}}\inf_{u\in[0,T]}\inf_{f\in C([-r,0];\mathbb{R}^d)}\sum_{i=1}^m \left|\zeta\cdot A_i(u,f)\right|^2\right) \\
&\le C_{15}\,\exp(-C_{16}\,\lambda).
\end{aligned}$$

Therefore, we can get

$$\mathbb{E}\left[\exp\left(-\lambda\,\zeta\cdot\tilde{V}^\varepsilon(t)\,\zeta\right)\right] \le C_{17,p,T,\eta}\,\lambda^{-C_{18}\,p},$$

so we have

$$\sup_{\zeta\in\mathbb{S}^{d-1}} \mathbb{E}\left[\left(\zeta\cdot V^\varepsilon(t)\,\zeta\right)^{-p}\right] = \varepsilon^{-2dp}\sup_{\zeta\in\mathbb{S}^{d-1}} \mathbb{E}\left[\left(\zeta\cdot\tilde{V}^\varepsilon(t)\,\zeta\right)^{-p}\right] \le C_{19,p,T}\,\varepsilon^{-2dp}$$

for any $p>1$. The proof is complete. □

Remark 3.7 Consider the case

$$A_i(t,f) = \tilde{A}_i\big(t,f(0)\big) \quad (i=1,\dots,m),$$

where $\tilde{A}_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ with the good conditions on the boundedness and the regularity. Now, our stochastic functional differential equation is as follows:

$$\begin{cases} X^\varepsilon(t) = \eta(t) & (t\in[-r,0]), \\ \mathrm{d}X^\varepsilon(t) = A_0\big(t,X_t^\varepsilon\big)\,\mathrm{d}t + \varepsilon\tilde{A}\big(t,X^\varepsilon(t)\big)\,\mathrm{d}W(t) & (t\in(0,T]), \end{cases} \tag{12}$$

where $\tilde{A} = \big(\tilde{A}_1,\dots,\tilde{A}_m\big)$. Then, we can get the same upper estimate of the inverse of the Malliavin covariance matrix $V^\varepsilon(t)$ for $X^\varepsilon(t)$ in *the hypoelliptic situation*, which means that the linear space generated by the vectors $\tilde{A}_i$ $(i=1,\dots,m)$ and their Lie brackets spans the space $\mathbb{R}^d$ (cf. Takeuchi [19]). □

4 Large deviation principles for X^ε

At the beginning, we shall introduce the well-known fact on the sample-path large deviations for Brownian motions. See also [13]. Recall that H is the Cameron-Martin space of $C_0\big([0,T];\mathbb{R}^m\big)$.

Lemma 4.1 (cf. Dembo-Zeitouni [3], Theorem 5.2.3) *The family* $\left\{\mathbb{P}\circ\left(\varepsilon W\right)^{-1};\, 0<\varepsilon\le 1\right\}$ *of the laws of* εW *over* $C_0\big([0,T];\mathbb{R}^m\big)$ *satisfies the large deviation principle with the good rate function* I*, where*

$$I(f)=\begin{cases}\dfrac{\|f\|_H^2}{2} & (f\in H),\\ +\infty & (f\notin H).\end{cases}$$

For $f\in H$, let $x^f=\left\{x^f(t);\, t\in[-r,T]\right\}$ be the solution to the following functional differential equation:

$$\begin{cases}x^f(t)=\eta(t) & (t\in[-r,0]),\\ \mathrm{d}x^f(t)=A_0\big(t,x_t^f\big)\,\mathrm{d}t+A\big(t,x_t^f\big)\,\dot{f}(t)\,\mathrm{d}t & (t\in(0,T]).\end{cases} \tag{13}$$

Denote by

$$C_\eta\left([-r,T];\mathbb{R}^d\right)=\left\{w\in C\big([-r,T];\mathbb{R}^d\big);\, w(t)=\eta(t)\ (t\in[-r,0])\right\}.$$

Theorem 1 *The family* $\left\{\mathbb{P}\circ\left(X^\varepsilon\right)^{-1};\, 0<\varepsilon\le 1\right\}$ *of the laws of* X^ε *over* $C_\eta\left([-r,T];\mathbb{R}^d\right)$ *satisfies the large deviation principle with the good rate function* $\tilde{I}$*, where*

$$\tilde{I}(g)=\inf\left\{I(f);\ f\in H,\ g=x^f\right\},$$

and I *is the function given in Lemma 4.1.*

Theorem 1 tells us to see, via the contraction principle (cf. Dembo-Zeitouni [3], Theorem 4.2.1).

Corollary 4.2 *For each* $t\in[0,T]$*, the family* $\left\{\mathbb{P}\circ\left(X^\varepsilon(t)\right)^{-1};\, 0<\varepsilon\le 1\right\}$ *of the laws of* $X^\varepsilon(t)$ *over* $\mathbb{R}^d$ *satisfies the large deviation principle with the good rate function* $\overline{I}$*, where*

$$\overline{I}(y)=\inf\left\{\tilde{I}(g);\, g\in C_\eta\left([-r,T];\mathbb{R}^d\right),\, y=g(t)\right\},$$

and $\tilde{I}$ *is the function given in Theorem 1.*

Now, we shall prove Theorem 1, according to Azencott [1] and Léandre [10, 11, 12, 13]. Our strategy stated here is almost parallel to [4, 16].

Proposition 4.3 *For any $a > 0$, the mapping*

$$H_a := \{ f \in H\,;\, \|f\|_H \le a \} \ni f \longmapsto x^f \in C_\eta\big([-r,T]\,;\mathbb{R}^d\big)$$

is continuous.

Proof. Let $f, g \in H_a$. Since

$$x^f(t) = \eta(0) + \int_0^t A_0\big(s, x_s^f\big)\,\mathrm{d}s + \int_0^t A\big(s, x_s^f\big)\,\dot f(s)\mathrm{d}s,$$

we see that

$$\begin{aligned}
\sup_{\tau\in[-r,t]} \big|x^f(\tau)\big| &\le \|\eta\|_\infty + \sup_{\tau\in[0,t]} \big|x^f(\tau)\big| \\
&\le 2\|\eta\|_\infty + \int_0^t \big|A_0\big(s,x_s^f\big)\big|\,\mathrm{d}s + \int_0^t \big\|A\big(s,x_s^f\big)\big\|_{\mathbb{R}^m\otimes\mathbb{R}^d}\,\big|\dot f(s)\big|\,\mathrm{d}s \\
&\le C_{20,T,\eta} + C_{21,T}\int_0^t \big(1+\big|\dot f(s)\big|\big)\left(1+\sup_{\tau\in[-r,s]}\big|x^f(\tau)\big|\right)\mathrm{d}s
\end{aligned}$$

from the linear growth condition on A_0 and the boundedness of A_i $(i = 1,\ldots,m)$, which tells us to see that

$$\sup_{\tau\in[-r,T]} \big|x^f(\tau)\big| \le C_{22,T,\eta,a}.$$

On the other hand, since

$$\begin{aligned}
x^f(t) - x^g(t) &= \int_0^t \Big\{A_0\big(s,x_s^f\big) - A_0\big(s,x_s^g\big)\Big\}\,\mathrm{d}s \\
&\quad + \int_0^t \Big\{A\big(s,x_s^f\big)\,\dot f(s) - A\big(s,x_s^g\big)\,\dot g(s)\Big\}\,\mathrm{d}s
\end{aligned}$$

for $t \in (0,T]$, and the $\mathbb{R}^d$-valued functions A_i $(i = 1,\ldots,m)$ are bounded, we have

$$\begin{aligned}
&\sup_{\tau\in[-r,t]} \big|x^f(\tau) - x^g(\tau)\big| \\
&= \sup_{\tau\in[0,t]} \big|x^f(\tau) - x^g(\tau)\big| \\
&\le \int_0^t \Big|A_0\big(s,x_s^f\big) - A_0\big(s,x_s^g\big)\Big|\,\mathrm{d}s + \int_0^t \Big\|A\big(s,x_s^f\big) - A\big(s,x_s^g\big)\Big\|_{\mathbb{R}^m\otimes\mathbb{R}^d}\,\big|\dot f(s)\big|\,\mathrm{d}s \\
&\quad + \int_0^t \Big\|A\big(s,x_s^g\big)\Big\|_{\mathbb{R}^m\otimes\mathbb{R}^d}\,\big|\dot f(s) - \dot g(s)\big|\,\mathrm{d}s \\
&\le C_{23,T}\int_0^t \sup_{\tau\in[-r,s]}\big|x^f(\tau) - x^g(\tau)\big|\left(1+\sum_{i=1}^m\big|\dot f^i(s)\big|\right)\mathrm{d}s
\end{aligned}$$

$$+C_{24,T,\eta,a}\left\|f-g\right\|_H.$$

The Gronwall inequality tells us to see that

$$\begin{aligned}\sup_{\tau\in[-r,t]}\left|x^f(\tau)-x^g(\tau)\right| &\le C_{24,T,\eta,a}\left\|f-g\right\|_H \exp\left[C_{23,T}\int_0^t\left(1+\sum_{i=1}^m\left|\dot f^i(s)\right|\right)\mathrm{d}s\right]\\ &\le C_{25,T,\eta,a}\left\|f-g\right\|_H,\end{aligned}$$

which completes the proof. □

Proposition 4.4 *Suppose that the $\mathbb{R}^d$-valued functions A_i $(i=1,\dots,m)$ are bounded. Then, for any $f\in H$ and $\rho>0$, there exist $\alpha_\rho>0$ and $\varepsilon_\rho>0$ such that*

$$\begin{aligned}&\mathbb{P}\left[\sup_{\tau\in[-r,T]}\left|X^\varepsilon(\tau)-x^f(\tau)\right|>\rho,\ \sup_{\tau\in[0,T]}\left|\varepsilon W(\tau)-f(\tau)\right|\le\alpha_\rho\right]\\ &\le C_{26,T,f,\rho}\exp\left[-C_{27,T,f}\,\frac{\rho^2}{\varepsilon^2}\right]\end{aligned}$$

for any $0<\varepsilon\le\varepsilon_\rho$.

Proof. Define a new probability measure $\mathrm{d}\tilde{\mathbb{P}}$ by

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T}=\exp\left[\int_0^T\sum_{i=1}^m\frac{\dot f^i(s)}{\varepsilon}\,\mathrm{d}W^i(s)-\frac{\|f\|_H^2}{2\varepsilon^2}\right].$$

The Girsanov theorem tells us to see that the $\mathbb{R}^m$-valued process $\{\tilde W(t):=W(t)-f(t)/\varepsilon\,;t\in[0,T]\}$ is also the m-dimensional Brownian motion under the probability measure $\mathrm{d}\tilde{\mathbb{P}}$. Let $\{X^{\varepsilon,f}(t)\,;t\in[-r,T]\}$ be the $\mathbb{R}^d$-valued process determined by the following equation:

$$\begin{cases}X^{\varepsilon,f}(t)=\eta(t) & (t\in[-r,0]),\\ \mathrm{d}X^{\varepsilon,f}(t)=A_0\big(t,X_t^{\varepsilon,f}\big)\,\mathrm{d}t+A\big(t,X_t^{\varepsilon,f}\big)\,\big\{\varepsilon\,\mathrm{d}\tilde W(t)+\dot f(t)\,\mathrm{d}t\big\} & (t\in(0,T]).\end{cases}\tag{14}$$

Write $M(t):=\int_0^t A\big(s,X_s^{\varepsilon,f}\big)\,\mathrm{d}\tilde W(s)$. Remark that

$$\begin{aligned}\sup_{\tau\in[-r,t]}\left|X^{\varepsilon,f}(\tau)-x^f(\tau)\right| &= \sup_{\tau\in[0,t]}\left|X^{\varepsilon,f}(\tau)-x^f(\tau)\right|\\ &\le\int_0^t\left|A_0\big(s,X_s^{\varepsilon,f}\big)-A_0\big(s,x_s^f\big)\right|\mathrm{d}s\\ &\quad+\int_0^t\left\|A\big(s,X_s^{\varepsilon,f}\big)-A\big(s,x_s^f\big)\right\|_{\mathbb{R}^m\otimes\mathbb{R}^d}\left|\dot f(s)\right|\mathrm{d}s+\sup_{\tau\in[0,t]}\left|\varepsilon M(\tau)\right|\end{aligned}$$

$$
\begin{aligned}
&\le C_{28,T} \int_0^t \sup_{\tau\in[-r,s]} \left|X^{\varepsilon,f}(\tau) - x^f(\tau)\right| \left(1 + \sum_{i=1}^m \left|\dot{f}^i(s)\right|\right) \mathrm{d}s \\
&\quad + \sup_{\tau\in[0,t]} |\varepsilon M(\tau)|.
\end{aligned}
$$

The Gronwall inequality tells us to see that

$$
\begin{aligned}
\sup_{\tau\in[-r,t]} \left|X^{\varepsilon,f}(\tau) - x^f(\tau)\right| &\le \left(\sup_{\tau\in[0,t]} |\varepsilon M(\tau)|\right) \exp\left[C_{28,T} \int_0^t \left(1 + \sum_{i=1}^m \left|\dot{f}^i(s)\right|\right) \mathrm{d}s\right] \\
&\le C_{29,T,f} \left(\sup_{\tau\in[0,t]} |\varepsilon M(\tau)|\right).
\end{aligned}
$$

For each $k = 1, \dots, d$, the martingale representation theorem enables us to see that there exists a 1-dimensional Brownian motion $\left\{B^k(t)\,;\, t \in [0,T]\right\}$ starting at the origin with

$$
M^k(t) = B^k\big(\langle M^k\rangle(t)\big), \quad \langle M^k\rangle(t) = \int_0^t \sum_{i=1}^m \left|A_i^k\big(s, X_s^{\varepsilon,f}\big)\right|^2 \mathrm{d}s
$$

for $k = 1, \dots, d$. Remark that $\langle M^k\rangle(t) \le C_{30,T}$, because of the boundedness of the $\mathbb{R}^d$-valued functions A_i $(i = 1, \dots, m)$. Since

$$
\tilde{\mathbb{P}}\left[\sup_{\tau\in[0,C_{30,T}]} \left|B^k(\tau)\right| > \frac{\rho}{C_{31,T,f}\,\varepsilon}\right] \le \sqrt{2} \exp\left[-\frac{\rho^2}{4C_{30,T}\,C_{31,T,f}^2\,\varepsilon^2}\right]
$$

from the reflection principle on Brownian motions, we have

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{\tau\in[-r,T]} \left|X^\varepsilon(\tau) - x^f(\tau)\right| > \rho,\ \sup_{\tau\in[0,T]} \left|\varepsilon W(\tau) - f(\tau)\right| \le \alpha_\rho\right] \\
&= \tilde{\mathbb{P}}\left[\sup_{\tau\in[-r,T]} \left|X^{\varepsilon,f}(\tau) - x^f(\tau)\right| > \rho,\ \sup_{\tau\in[0,T]} \left|\varepsilon \tilde{W}(\tau)\right| \le \alpha_\rho\right] \\
&\le \tilde{\mathbb{P}}\left[\sup_{\tau\in[0,T]} |M(\tau)| > \frac{\rho}{C_{29,T,f}\,\varepsilon}\right] \\
&\le \tilde{\mathbb{P}}\left[\sup_{\tau\in[0,C_{30,T}]} |B(\tau)| > \frac{\rho}{C_{29,T,f}\,\varepsilon}\right] \\
&\le \tilde{\mathbb{P}}\left[\bigcup_{k=1}^d \left\{\sup_{\tau\in[0,C_{30,T}]} \left|B^k(\tau)\right| > \frac{\rho}{C_{29,T,f}\sqrt{d}\,\varepsilon}\right\}\right] \\
&\le \sqrt{2}\,d \exp\left[-\frac{\rho^2}{4C_{30,T}\,C_{29,T,f}^2\,d\,\varepsilon^2}\right],
\end{aligned}
$$

which completes the proof. □

Proposition 4.5 *It holds that*

$$\lim_{R\to+\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|>R\right]=-\infty.$$

Proof. Let $N>2$ be sufficient large. From the Itô formula, we see that

$$\begin{aligned}\left(1+\left|X^{\varepsilon}(t)\right|^2\right)^N &= \left(1+|\eta(0)|^2\right)^N+\int_0^t N\left(1+\left|X^{\varepsilon}(s)\right|^2\right)^{N-1}2\varepsilon X^{\varepsilon}(s)\cdot A\big(s,X^{\varepsilon}_s\big)\,\mathrm{d}W(s)\\ &\quad+\int_0^t\left\{N\left(1+\left|X^{\varepsilon}(s)\right|^2\right)^{N-1}\left(2X^{\varepsilon}(s)\cdot A_0\big(s,X^{\varepsilon}_s\big)+\varepsilon^2\sum_{i=1}^m\left|A_i\big(s,X^{\varepsilon}_s\big)\right|^2\right)\right.\\ &\quad\left.+2N(N-1)\,\varepsilon^2\left(1+\left|X^{\varepsilon}(s)\right|^2\right)^{N-2}\sum_{i=1}^m\left(X^{\varepsilon}(s)\cdot A_i\big(s,X^{\varepsilon}_s\big)\right)^2\right\}\mathrm{d}s.\end{aligned}$$

Define $\sigma_R=\inf\left\{t>0;\,\left|X^{\varepsilon}(t)\right|>R\right\}$. Then, it holds that

$$\begin{aligned}&\mathbb{E}\left[\left(1+\left|X^{\varepsilon}\big(t\wedge\sigma_R\big)\right|^2\right)^N\right]\\ &\le\left(1+\|\eta\|_{\infty}^2\right)^N\\ &\quad+\mathbb{E}\left[\left|\int_0^{t\wedge\sigma_R}\left\{N\left(1+\left|X^{\varepsilon}(s)\right|^2\right)^{N-1}\left(2X^{\varepsilon}(s)\cdot A_0\big(s,X^{\varepsilon}_s\big)+\varepsilon^2\sum_{i=1}^m\left|A_i\big(s,X^{\varepsilon}_s\big)\right|^2\right)\right.\right.\right.\\ &\quad\left.\left.\left.+2\,N\,(N-1)\,\varepsilon^2\left(1+\left|X^{\varepsilon}(s)\right|^2\right)^{N-2}\sum_{i=1}^m\left(X^{\varepsilon}(s)\cdot A_i\big(s,X^{\varepsilon}_s\big)\right)^2\right\}\mathrm{d}s\right|^2\right]\\ &\le\left(1+\|\eta\|_{\infty}\right)^N+C_{32,T}\,\left(N+\varepsilon^2N+\varepsilon^2N^2\right)\mathbb{E}\left[\int_0^t\left(1+\left|X^{\varepsilon}\big(s\wedge\sigma_R\big)\right|^2\right)^N\mathrm{d}s\right]\end{aligned}$$

from the linear growth condition on the coefficients A_i $(i=0,1,\dots,m)$ of the equation (3). Hence, the Gronwall inequality implies that

$$\mathbb{E}\left[\left(1+\left|X^{\varepsilon}\big(t\wedge\sigma_R\big)\right|^2\right)^N\right]\le\left(1+\|\eta\|_{\infty}\right)^N\exp\left[C_{32,T}\,\left(N+\varepsilon^2N+\varepsilon^2N^2\right)t\right].$$

In particular, taking $N=1/\varepsilon$ yields that

$$\mathbb{E}\left[\left(1+\left|X^{\varepsilon}\big(t\wedge\sigma_R\big)\right|^2\right)^{1/\varepsilon}\right]\le\left(1+\|\eta\|_{\infty}\right)^{1/\varepsilon}\exp\left[C_{33,T}\,\left(\frac{1}{\varepsilon}+1\right)t\right].$$

Therefore, the Chebyshev inequality leads us to see that

$$\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|>R\right]=\mathbb{P}\left[\sigma_R\le T\right]$$

$$
\begin{aligned}
&\le \mathbb{P}\left[\left|X^{\varepsilon}\left(T\wedge\sigma_R\right)\right| \ge R\right]\\
&\le \left(1+R^2\right)^{-1/\varepsilon}\mathbb{E}\left[\left(1+\left|X^{\varepsilon}\left(T\wedge\sigma_R\right)\right|^2\right)^N\right]\\
&\le \left(\frac{1+\|\eta\|_\infty^2}{1+R^2}\right)^{1/\varepsilon}\exp\left[C_{33,T}\left(\frac{1}{\varepsilon}+1\right)T\right],
\end{aligned}
$$

so we have

$$
\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right| > R\right] \le \ln\left(\frac{1+\|\eta\|_\infty^2}{1+R^2}\right)+C_{33,T}\,T,
$$

which completes the proof.

□

Let $R\ge 1$. Define $\sigma_R=\inf\left\{t>0;\,\left|X^{\varepsilon}(t)\right|>R\right\}$, and $X^{\varepsilon,R}(t)=X^{\varepsilon}(t\wedge\sigma_R)$.

Proposition 4.6 *For any $\delta>0$, it holds that*

$$
\lim_{R\to+\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\delta\right]=-\infty.
$$

Proof. Remark that

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\delta\right]\\
&\le \mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\delta,\ \sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|\le R\right]+\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)\right|>R\right]\\
&=\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\delta,\ \sigma_R\ge T\right]+\mathbb{P}\left[\sigma_R\le T\right]\\
&=\mathbb{P}\left[\sigma_R\le T\right]\\
&\le \mathbb{P}\left[\left|X^{\varepsilon}\left(T\wedge\sigma_R\right)\right|\ge R\right]\\
&\le \left(\frac{1+\|\eta\|_\infty^2}{1+R^2}\right)^{1/\varepsilon}\exp\left[C_{33,T}\left(\frac{1}{\varepsilon}+1\right)T\right],
\end{aligned}
$$

as seen in the proof of Proposition 4.5. So, we can get

$$
\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\delta\right]\le\ln\left(\frac{1+\|\eta\|_\infty^2}{1+R^2}\right)+C_{33,T}\,T,
$$

which completes the proof. □

Proof of Theorem 1. We shall prove the assertion in two steps of the form: the case where A_i $(i=1,\ldots,m)$ are bounded, and the general case on A_i $(i=1,\ldots,m)$.

Step 1. Suppose that the coefficients A_i $(i=1,\ldots,m)$ are bounded. Propositions 4.3 and 4.4 are sufficient to our goal (cf. [1, 3]). In fact, the large deviation principle for the family $\{\mathbb{P}\circ (X^\varepsilon)^{-1}; 0<\varepsilon\le 1\}$ comes from the one for $\{\mathbb{P}\circ(\varepsilon W)^{-1}; 0<\varepsilon\le 1\}$ in Lemma 4.1.

Step 2. We shall discuss the general case on A_i $(i=1,\ldots,m)$. Let $R\ge 1$, and F be a closed set in $C_\eta\left([-r,T];\mathbb{R}^d\right)$. Denote by $F_R=F\cap\overline{B(0;R)}$, and by F_R^δ the closed δ-neighborhood of F_R, where $B(0;R)$ is the open ball in $C_\eta\left([-r,T];\mathbb{R}^d\right)$ with radius R centered at $0\in C_\eta\left([-r,T];\mathbb{R}^d\right)$. Then, it holds that

$$
\begin{aligned}
&\mathbb{P}[X^\varepsilon\in F]\\
&\le \mathbb{P}\left[X^\varepsilon\in F,\ \sup_{t\in[-r,T]}\left|X^\varepsilon(t)\right|\le R\right]+\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^\varepsilon(t)\right|>R\right]\\
&=\mathbb{P}\left[X^{\varepsilon,R}\in F_R\right]+\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^\varepsilon(t)\right|>R\right].
\end{aligned}
$$

As seen in Step1, we have already obtained the large deviation principle for $\{\mathbb{P}\circ\left(X^{\varepsilon,R}\right)^{-1}; 0<\varepsilon\le 1\}$ with the good rate function $\tilde{I}_R$, where $I(f)$ is given in Lemma 4.1, and

$$
\tilde{I}_R(g)=\inf\left\{I(f);\ f\in H,\ g=x^f,\ \sup_{t\in[-r,T]}\left|x^f(t)\right|\le R\right\}.
$$

So, we have

$$
\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[X^{\varepsilon,R}\in F_R\right]\le-\inf_{g\in F_R}\tilde{I}_R(g).
$$

Therefore, we can get

$$
\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}[X^\varepsilon\in F]\\
&\le\lim_{R\to+\infty}\left\{\left(-\inf_{g\in F_R}\tilde{I}_R(g)\right)\vee\left(\limsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^\varepsilon(t)\right|>R\right]\right)\right\}\\
&=\lim_{R\to+\infty}\left(-\inf_{g\in F_R}\tilde{I}(g)\right)\\
&\le-\inf_{g\in F}\tilde{I}(g)
\end{aligned}
$$

from Proposition 4.5, which completes the proof on the upper estimate of the large deviation principle.

Next, we shall pay attention to the lower estimate of the large deviation principle. Let G be an open set in $C_\eta\big([-r,T]\,;\mathbb{R}^d\big)$, and take $\tilde{g}$ in $G\cap\overline{B(0\,;R)}$. Then, we can find $\delta>0$ such that $B\big(\tilde{g}\,;\delta\big)\subset G$. Thus, we have

$$
\begin{aligned}
-\tilde{I}\big(\tilde{g}\big) &= -\tilde{I}_R\big(\tilde{g}\big) \\
&\le -\inf_{g\in B(\tilde{g}\,;\delta/2)} \tilde{I}_R(g) \\
&\le \liminf_{\varepsilon\searrow 0} \varepsilon \ln \mathbb{P}\left[X^{\varepsilon,R}\in B\big(\tilde{g}\,;\delta/2\big)\right] \\
&\le \liminf_{\varepsilon\searrow 0} \varepsilon \ln\left\{\mathbb{P}\left[X^{\varepsilon}\in G\right]+\mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\frac{\delta}{2}\right]\right\} \\
&\le \left(\liminf_{\varepsilon\searrow 0} \varepsilon \ln \mathbb{P}\left[X^{\varepsilon}\in G\right]\right)\vee\left(\liminf_{\varepsilon\searrow 0} \varepsilon \ln \mathbb{P}\left[\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|>\frac{\delta}{2}\right]\right).
\end{aligned}
$$

The first equality is right, because of $\tilde{g}\in\overline{B(0\,;R)}$, while the third inequality is the consequence of the large deviation principle for $X^{\varepsilon,R}$ as seen in Step 1. The forth inequality is right, because $X^{\varepsilon}\in B\big(\tilde{g}\,;\delta/2\big)^c$ under $X^{\varepsilon,R}\in B\big(\tilde{g}\,;\delta\big)^c$ and $\sup_{t\in[-r,T]}\left|X^{\varepsilon}(t)-X^{\varepsilon,R}(t)\right|\le\delta/2$. Taking the limit as $R\to+\infty$ leads us to see that

$$
-\tilde{I}\big(\tilde{g}\big)\le \liminf_{\varepsilon\searrow 0} \varepsilon \ln \mathbb{P}\left[X^{\varepsilon}\in G\right]
$$

from Proposition 4.6, which completes the proof on the lower estimate of the large deviation principle. The proof of Theorem 1 is complete. □

5 Density estimates

In this section, we shall consider the estimate of the density $p^{\varepsilon}(t,y)$ for the solution $X^{\varepsilon}(t)$, from the viewpoint of the Malliavin calculus.

Theorem 2 (Upper estimate) *Suppose that the $\mathbb{R}^d$-valued functions A_i $(i=1,\ldots,m)$ satisfy the uniformly elliptic condition* (11). *Then, it holds that*

$$
\limsup_{\varepsilon\searrow 0} \varepsilon^2 \ln p^{\varepsilon}(t,y)\le -\overline{I}(y), \tag{15}
$$

where the function $\overline{I}$ is give in Theorem 1.

Proof. Let $0<\sigma<1$ be sufficiently small, and $\Lambda_\sigma\in C_0^\infty\big(\mathbb{R}^d\,;[0,1]\big)$ such that

$$
\Lambda_\sigma(z)=\begin{cases} 1 & (|z-y|\le\sigma), \\ 0 & (|z-y|>2\sigma). \end{cases}
$$

Take $U = \prod_{j=1}^{d}[a_j, b_j] \subset \mathbb{R}^d$ such that $U \subset \text{Supp}[\Lambda_\sigma]$. Then, the integration by parts formula tells us to see that

$$
\begin{aligned}
&\mathbb{P}\left[X^\varepsilon(t) \in U\right] \\
&= \mathbb{E}\left[\mathbb{I}_U\big(X^\varepsilon(t)\big)\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{X^{\varepsilon,1}(t)} \cdots \int_{-\infty}^{X^{\varepsilon,d}(t)} \mathbb{I}_U\big(y_1,\ldots,y_d\big)\,\mathrm{d}y_1\cdots\mathrm{d}y_d\,\Gamma_{(1,\ldots,d)}\big(X^\varepsilon(t),\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\big)\right] \\
&= \int_U \mathbb{E}\left[\prod_{j=1}^{d}\mathbb{I}_{(y_j,+\infty)}\big(X^{\varepsilon,j}(t)\big)\,\Gamma_{(1,\ldots,d)}\big(X^\varepsilon(t),\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\big)\right]\mathrm{d}y_1\cdots\mathrm{d}y_d,
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_j\big(X^\varepsilon(t),\Lambda_\sigma\big(X^\varepsilon(t)\big)\big) &= \delta\left(\Lambda_\sigma\big(X^\varepsilon(t)\big)\sum_{k=1}^{d}\left[\big(V^\varepsilon(t)\big)^{-1}\right]_{jk} DX^{\varepsilon,k}(t)\right), \\
\Gamma_{(1,\ldots,d)}\big(X^\varepsilon(t),\Lambda_\sigma\big(X^\varepsilon(t)\big)\big) &= \Gamma_d\left(X^\varepsilon(t),\,\Gamma_{(1,\ldots,d-1)}\left(X^\varepsilon(t),\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\right)\right),
\end{aligned}
$$

and δ is the Skorokhod integral operator. Remark that, under the uniformly elliptic condition (11) on the $\mathbb{R}^d$-valued functions A_i $(i = 1,\ldots,m)$,

$$
\begin{aligned}
&\left\|\Gamma_{(1,\ldots,d)}\left(X^\varepsilon(t),\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\right)\right\|_{\mathbb{L}^p(\Omega)} \\
&\leq C_{34,p,T,\eta}\left\|\big(V^\varepsilon(t)\big)^{-1}\right\|_{\mathbb{L}^\alpha(\Omega)}\|X^\varepsilon(t)\|_{\beta,\gamma}\left\|\Lambda_\sigma\big(X^\varepsilon(t)\big)\right\|_{\kappa,\sigma} \\
&\leq C_{35,p,T,\eta}\,\varepsilon^{-2d},
\end{aligned}
$$

where α, γ, $\sigma > 1$ and β, $\kappa \in \mathbb{Z}_+$, by using Proposition 3.4, and the proof of Lemma 3.6. Hence, the density $p^\varepsilon(t,y)$ can be estimated from the above as follows:

$$
\begin{aligned}
p^\varepsilon(t,y) &= \mathbb{E}\left[\prod_{j=1}^{d}\mathbb{I}_{(y_j,+\infty)}\big(X^{\varepsilon,j}(t)\big)\,\Gamma_{(1,\ldots,d)}\big(X^\varepsilon(t),\,\Lambda_\sigma\big(X^\varepsilon(t)\big)\big)\right] \\
&\leq \mathbb{E}\left[\left|\Gamma_{(1,\ldots,d)}\big(X^\varepsilon(t),\Lambda_\sigma\big(X^\varepsilon(t)\big)\big)\right|\mathbb{I}_{Supp[\Lambda_\sigma]}\big(X^\varepsilon(t)\big)\right] \\
&\leq C_{35,p,T,\eta}\,\varepsilon^{-2d}\,\mathbb{P}\left[X^\varepsilon(t) \in \text{Supp}[\Lambda_\sigma]\right]^{1/q},
\end{aligned}
$$

where $q > 1$ such that $1/p + 1/q = 1$. From Theorem 1, we have

$$
\limsup_{\varepsilon \searrow 0}\varepsilon^2 \ln \mathbb{P}\left[X^\varepsilon(t) \in \text{Supp}[\Lambda_\sigma]\right] \leq -\inf_{z \in Supp[\Lambda_\sigma]} \bar{I}(z).
$$

Since the function $\bar{I}$ is lower semi-continuous, taking the limit as $\sigma \searrow 0$ and $q \searrow 1$ enable us to see that

$$
\limsup_{\varepsilon \searrow 0}\varepsilon^2 \ln p^\varepsilon(t,y) \leq -\bar{I}(y),
$$

which is the conclusion of Theorem 2. □

Remark 5.1 As sated in Remark 3.7, a similar problem can be also studied under the hypoelliptic condition, in the case

$$A_i(t,f) = \tilde{A}_i\big(t, f(0)\big) \quad (i=1,\dots,m),$$

where $\tilde{A}_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ with the good conditions on the boundedness and the regularity (cf. [19]). □

Now, we shall study the lower estimate of the density $p^\varepsilon(t,y)$ for the solution process (3). Before doing it, we shall prepare some arguments.

Proposition 5.2 *Let $f\in H$, and suppose the uniformly elliptic condition* (11) *on the functions A_i $(i=1,\dots,m)$. Then, it holds that*

$$\det v^f(t) > 0$$

for each $t\in(0,T]$, where $v^f(t)$ is the Gram matrix for $x^f(t)$.

Proof. Let $u\in[0,T]$, and $\big\{\bar{Z}(t,u)\,;\,t\in[-r,T]\big\}$ be the $\mathbb{R}^d\otimes\mathbb{R}^d$-valued mappings given by the following functional differential equation:

$$\begin{cases} \bar{Z}(t,u) = \mathbf{0} & (t\in[-r,0] \text{ or } t\in(0,u)),\\ \mathrm{d}\bar{Z}(t,u) = \nabla A_0\big(t,x_t^f\big)\,\bar{Z}_t(\cdot,u)\,\mathrm{d}t + \nabla A\big(t,x_t^f\big)\,\bar{Z}_t(\cdot,u)\,\dot{f}(t)\,\mathrm{d}t & (t\in[u,T]), \end{cases} \tag{16}$$

where $\bar{Z}_t(\cdot,u) = \big\{\bar{Z}(t+\tau,u)\,;\,\tau\in[-r,0]\big\}$. From the condition on the coefficients A_i $(i=0,1,\dots,m)$, we see that

$$\begin{aligned}
&\sup_{\tau\in[u,t]}\big\|\bar{Z}(\tau,u)-I_d\big\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\\
&\le \sup_{\tau\in[u,t]}\left\|\int_u^\tau \nabla A_0\big(s,x_s^f\big)\,\bar{Z}_s(\cdot,u)\,\mathrm{d}s\right\|_{\mathbb{R}^d\otimes\mathbb{R}^d} + \sup_{\tau\in[u,t]}\left\|\int_u^\tau \nabla A\big(s,x_s^f\big)\,\bar{Z}_s(\cdot,u)\,\dot{f}(s)\,\mathrm{d}s\right\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\\
&\le \int_u^t\left(\left\|\nabla A_0\big(s,x_s^f\big)\right\|_{C([-r,0];\mathbb{R}^d)\otimes\mathbb{R}^d} + \left\|\nabla A\big(s,x_s^f\big)\right\|\,|_{C([-r,0];\mathbb{R}^d)\otimes\mathbb{R}^d}\,\big|\dot{f}(s)\big|\right)\mathrm{d}s\\
&\quad+\int_u^t\left(\left\|\nabla A_0\big(s,x_s^f\big)\right\|_{C([-r,0];\mathbb{R}^d)\otimes\mathbb{R}^d} + \left\|\nabla A\big(s,x_s^f\big)\right\|_{C([-r,0];\mathbb{R}^d)\otimes\mathbb{R}^d}\,\big|\dot{f}(s)\big|\right)\\
&\qquad\quad\times\sup_{\tau\in[s-r,s]}\|\bar{Z}(\tau,u)-I_d\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\,\mathrm{d}s\\
&\le C_{36,T}\int_u^t\left(1+\sum_{i=1}^m\big|\dot{f}^i(s)\big|\right)\mathrm{d}s + C_{37,T}\int_u^t\left(1+\sum_{i=1}^m\big|\dot{f}^i(s)\big|\right)\sup_{\tau\in[u,s]}\|\bar{Z}(\tau,u)-I_d\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\,\mathrm{d}s.
\end{aligned}$$

Remark that

$$\int_u^t \left(1+\sum_{i=1}^m \left|\dot{f}^i(s)\right|\right) \mathrm{d}s \leq \int_u^T \left(1+\sum_{i=1}^m \left|\dot{f}^i(s)\right|\right) \mathrm{d}s \leq C_{38,f,T}\,(T-u)^{1/2}.$$

Hence, the Gronwall inequality tells us to see that

$$\sup_{\tau\in[u,T]} \|\bar{Z}(\tau,u)-I_d\|_{\mathbb{R}^d\otimes\mathbb{R}^d} \leq C_{39,f,T}\,(T-u)^{1/2}.$$

On the other hand, remark that we have already seen in the proof of Proposition 4.3 that

$$\sup_{\tau\in[-r,t]} \left|x^f(\tau)\right| \leq C_{22,T,\eta,f}.$$

Now, we shall pay attention to the lower estimate of $\det v^f(t)$. Since, for each $u\in[0,T]$, $\left\{D_u x^f(t)\,;\, t\in[-r,T]\right\}$ satisfies the equation

$$\begin{cases} D_u x^f(t) = 0 & (t\in[-r,0]), \\ D_u x^f(t) = \displaystyle\int_0^u A\big(s,x_s^f\big)\,\mathbb{I}_{(s\leq t)}\,\mathrm{d}s + \int_0^t \nabla A_0\big(s,x_s^f\big)\,D_u x_s^f\,\mathrm{d}s & \\ \qquad\qquad + \displaystyle\int_0^t \nabla A\big(s,x_s^f\big)\,D_u x_s^f\,\dot{f}(s)\,\mathrm{d}s & (t\in(0,T]), \end{cases}$$

we have

$$D_u x^f(t) = \int_0^{u\wedge t} \bar{Z}(t,s)\,A\big(s,x_s^f\big)\,\mathrm{d}s,$$

similarly to Corollary 3.5. Hence, the Gram matrix $v^f(t)$ can be expressed as follows:

$$v^f(t) = \int_0^t \bar{Z}(t,u)\,A\big(u,x_u^f\big)\,A\big(u,x_u^f\big)^*\,\bar{Z}(t,u)^*\,\mathrm{d}u.$$

Let $T_\alpha\in[0,T]$ be sufficiently close to T. So, we see that

$$\begin{aligned} \det v^f(t) &= \det\left[\int_0^t \bar{Z}(t,u)\,A\big(u,x_u^f\big)\,A\big(u,x_u^f\big)^*\,\bar{Z}(t,u)^*\,\mathrm{d}u\right] \\ &\geq \left\{\inf_{\zeta\in\mathbb{S}^{d-1}} \int_0^t \sum_{i=1}^m \left(\zeta\cdot\bar{Z}(t,u)\,A_i\big(u,x_u^f\big)\right)^2 \mathrm{d}u\right\}^d \\ &= \left\{\inf_{\zeta\in\mathbb{S}^{d-1}} \int_0^t \sum_{i=1}^m \left(\zeta\cdot\bar{Z}(t,u)\,A_i\big(u,x_u^f\big)\right)^2 \mathrm{d}u\right\}^d \\ &\quad\times \mathbb{I}\left(\sup_{t\in[-r,T]} \left|x^f(t)\right| \leq C_{22,T,\eta,f}\right) \mathbb{I}\left(\sup_{u\in[T_\alpha,T]} \left\|\bar{Z}(T,u)-I_d\right\|_{\mathbb{R}^d\otimes\mathbb{R}^d} \leq C_{39,T,f}\,(T-T_\alpha)^{1/2}\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \left\{ \frac{1}{2} \inf_{\zeta \in \mathbb{S}^{d-1}} \int_0^t \sum_{i=1}^m \left(\zeta \cdot A_i\big(u, x_u^f\big) \right)^2 \mathrm{d}u - \int_{T_\alpha}^T \sum_{i=1}^m \left\| \bar{Z}(T,u) - I_d \right\|_{\mathbb{R}^d \otimes \mathbb{R}^d}^2 \left| A_i\big(u, x_u^f\big) \right|^2 \mathrm{d}u \right\}^d \\
&\quad \times \mathbb{I} \left(\sup_{t \in [-r,T]} \left| x^f(t) \right| \leq C_{22,T,\eta,f} \right) \mathbb{I} \left(\sup_{u \in [T_\alpha, T]} \left\| \bar{Z}(T,u) - I_d \right\|_{\mathbb{R}^d \otimes \mathbb{R}^d} \leq C_{39,T,f} \left(T - T_\alpha \right)^{1/2} \right) \\
&\geq \left\{ \frac{T - T_\alpha}{2} \inf_{\zeta, t, g} \sum_{i=1}^m \left(\zeta \cdot A_i(t,g) \right)^2 - C_{40,T,\eta,f} \left(T - T_\alpha \right)^2 \right\}^d \\
&\quad \times \mathbb{I} \left(\sup_{t \in [-r,T]} \left| x^f(t) \right| \leq C_{22,T,\eta,f} \right) \mathbb{I} \left(\sup_{u \in [T_\alpha, T]} \left\| \bar{Z}(T,u) - I_d \right\|_{\mathbb{R}^d \otimes \mathbb{R}^d} \leq C_{39,T,f} \left(T - T_\alpha \right)^{1/2} \right) \\
&\geq C_{41,T,\eta,f} \left(T - T_\alpha \right)^d \\
&\quad \times \mathbb{I} \left(\sup_{t \in [-r,T]} \left| x^f(t) \right| \leq C_{22,T,\eta,f} \right) \mathbb{I} \left(\sup_{u \in [T_\alpha, T]} \left\| \bar{Z}(T,u) - I_d \right\|_{\mathbb{R}^d \otimes \mathbb{R}^d} \leq C_{39,T,f} \left(T - T_\alpha \right)^{1/2} \right) \\
&= C_{41,T,\eta,f} \left(T - T_\alpha \right)^d,
\end{aligned}
$$

which is strictly positive. Here, we shall remark that there exists the constant $C_{41,T,\eta,f} > 0$ with

$$
\frac{1}{2} \inf_{\zeta, t, g} \sum_{i=1}^m \left(\zeta \cdot A_i(t,g) \right)^2 - C_{40,T,\eta,f} \left(T - T_\alpha \right) \geq C_{41,T,\eta,f},
$$

because the functions A_i $(i = 1, \ldots, m)$ satisfy the uniformly elliptic condition (11), and T_α is sufficiently close to T, which justifies the sixth inequality. □

For $f \in H$, let $\left\{ X^{\varepsilon,f}(t) \,;\, t \in [-r,T] \right\}$ be the $\mathbb{R}^d$-valued process determined by the following equation:

$$
\begin{cases}
X^{\varepsilon,f}(t) = \eta(t) & (t \in [-r,0]), \\
\mathrm{d}X^{\varepsilon,f}(t) = A_0\big(t, X_t^{\varepsilon,f}\big)\,\mathrm{d}t + \varepsilon A\big(t, X_t^{\varepsilon,f}\big)\,\mathrm{d}W(t) + A\big(t, X_t^{\varepsilon,f}\big)\,\dot{f}(t)\,\mathrm{d}t & (t \in (0,T]).
\end{cases}
\tag{17}
$$

Let $\left\{ \tilde{Z}^f(t) \,;\, t \in [-r,T] \right\}$ be the $\mathbb{R}^d$-valued process determined by the following equation:

$$
\begin{cases}
\tilde{Z}^f(t) = 0 & (t \in [-r,0]), \\
\mathrm{d}\tilde{Z}^f(t) = A\big(t, x_t^f\big)\,\mathrm{d}W(t) + \nabla A_0\big(t, x_t^f\big)\,\tilde{Z}_t^f\,\mathrm{d}t + \nabla A\big(t, x_t^f\big)\,\tilde{Z}_t^f\,\dot{f}(t)\,\mathrm{d}t & (t \in (0,T]).
\end{cases}
\tag{18}
$$

Lemma 5.3 *Let $t \in (0,T]$. It holds that*

$$
\lim_{\varepsilon \searrow 0} \left\| Y^{\varepsilon,f}(t) - \tilde{Z}^f(t) \right\|_{k,p} = 0
$$

for any $p > 1$ and $k \in \mathbb{Z}_+$, where $Y^{\varepsilon,f}(t) = \left(X^{\varepsilon,f}(t) - x^f(t) \right) / \varepsilon$.

Proof. We shall prove the statement along the following procedure:

Step 1. For any $p > 1$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\sup_{t\in[-r,T]} \left|X^{\varepsilon,f}(t) - x^f(t)\right|^p\right] = 0.$$

In fact, since

$$X^{\varepsilon,f}(t) - x^f(t) = \int_0^t \left\{A_0\big(s, X_s^{\varepsilon,f}\big) - A_0\big(s, x_s^f\big)\right\} \mathrm{d}s + \int_0^t \left\{A\big(s, X_s^{\varepsilon,f}\big) - A\big(s, x_s^f\big)\right\} \dot{f}(s)\mathrm{d}s + \varepsilon \int_0^t A\big(s, X_s^{\varepsilon,f}\big)\,\mathrm{d}W(s)$$

for $t \in [0,T]$, and the coefficients A_i $(i = 0, 1, \ldots, m)$ satisfy the Lipschitz condition and the linear growth condition, we can get the assertion of Step 1 by using the Hölder inequality, the Burkholder inequality and the Gronwall inequality. for $t \in [0,T]$,

Step 2. For any $p > 1$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\sup_{t\in[-r,T]} \left|Y^{\varepsilon,f}(t) - \tilde{Z}^f(t)\right|^p\right] = 0,$$

which tells us to see that the assertion of Lemma 5.3 holds in the case of $k = 0$.

In fact, we shall remark that

$$\begin{aligned}
&\frac{A_i\big(s, X_s^{\varepsilon,f}\big) - A_i\big(s, x_s^f\big)}{\varepsilon} - \nabla A_i\big(s, x_s^f\big)\,\tilde{Z}_s^f \\
&= \nabla A_i\big(s, x_s^f\big)\,\Big(Y_s^{\varepsilon,f} - \tilde{Z}_s^f\Big) + \frac{1}{2}\Big\langle Y_s^{\varepsilon,f}, \nabla^2 A_i\big(s, \sigma X_s^{\varepsilon,f} + (1-\sigma)x_s^f\big)\,\big(X_s^{\varepsilon,f} - x_s^f\big)\Big\rangle
\end{aligned}$$

from the Taylor theorem for $i = 0, 1, \ldots, m$, where $0 < \sigma < 1$ is the constant. Since

$$\begin{aligned}
Y^{\varepsilon,f}(t) - \tilde{Z}^f(t) &= \int_0^t \left[\frac{A_0\big(s, X_s^{\varepsilon,f}\big) - A_0\big(s, x_s^f\big)}{\varepsilon} - \nabla A_0\big(s, x_s^f\big)\,\tilde{Z}_s^f\right] \mathrm{d}s \\
&\quad + \int_0^t \left[\frac{A\big(s, X_s^{\varepsilon,f}\big) - A\big(s, x_s^f\big)}{\varepsilon} - \nabla A\big(s, x_s^f\big)\,\tilde{Z}_s^f\right] \dot{f}(s)\,\mathrm{d}s \\
&\quad + \int_0^t \left\{A\big(s, X_s^{\varepsilon,f}\big) - A\big(s, x_s^f\big)\right\} \mathrm{d}W(s)
\end{aligned}$$

for $t \in [0,T]$, and the coefficients $A_i(t,\cdot)$ $(i = 0, 1, \ldots, m)$ are in $C^\infty_{1+,b}\big(C([-r,0];\mathbb{R}^d);\mathbb{R}^d\big)$ with respect to the second variable in $C\big([-r,0];\mathbb{R}^d\big)$ for each $t \in [0,T]$, we can get the assertion in Step 2 via the Hölder inequality, the Burkholder inequality and the Gronwall inequality.

Step 3. Let $u \in [0,T]$. Then, for any $p > 1$,

$$\sup_{0<\varepsilon\le 1} \mathbb{E}\left[\sup_{t\in[-r,T]} \left|D_u Y^{\varepsilon,f}(t)\right|^p\right] < +\infty.$$

Remark that

$$D_u Y^{\varepsilon,f}(t) = \int_0^{u\wedge t} \left\{ A\big(s,X_s^{\varepsilon,f}\big) + \frac{A\big(s,X_s^{\varepsilon,f}\big) - A\big(s,x_s^f\big)}{\varepsilon} \right\} \mathrm{d}s$$
$$+ \int_0^t \frac{1}{\varepsilon} \left\{ \nabla A_0\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f} - \nabla A_0\big(s,x_s^f\big) D_u x_s^f \right\} \mathrm{d}s$$
$$+ \int_0^t \frac{1}{\varepsilon} \left\{ \nabla A\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f} - \nabla A\big(s,x_s^f\big) D_u x_s^f \right\} \dot{f}(s)\,\mathrm{d}s$$
$$+ \int_0^t \nabla A\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f}\,\mathrm{d}W(s)$$

for $t \in [u,T]$. Since

$$D_u x^f(t) = \int_0^{u\wedge t} A\big(s,x_s^f\big)\,\mathrm{d}s + \int_0^t \nabla A_0\big(s,x_s^f\big) D_u x_s^f\,\mathrm{d}s + \int_0^t \nabla A\big(s,x_s^f\big) D_u x_s^f\, \dot{f}(s)\,\mathrm{d}s,$$

as seen in Proposition 5.2, we have

$$\sup_{t\in[-r,T]} \left| D_u x^f(t) \right| \le C_{42,T,\eta,f}.$$

Moreover, similarly to Proposition 3.4, we have

$$\mathbb{E}\left[\sup_{t\in[-r,T]} \left| D_u X^{\varepsilon,f}(t) \right|^p \right] \le C_{43,p,T,\eta,f}$$

for any $p > 1$. Then, the assertion in Step 3 can be justified by using the Hölder inequality, the Burkholder inequality and the Gronwall inequality.

Step 4. Let $u \in [0,T]$. Then, for any $p > 1$,

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\left[\sup_{t\in[-r,T]} \left| D_u Y^{\varepsilon,f}(t) - D_u \tilde{Z}^f(t) \right|^p \right] = 0.$$

In fact, since

$$D_u Y^{\varepsilon,f}(t) = \int_0^{u} \left\{ A\big(s,X_s^{\varepsilon,f}\big) + \frac{A\big(s,X_s^{\varepsilon,f}\big) - A\big(s,x_s^f\big)}{\varepsilon} \right\} \mathbb{I}_{(s\le t)}\,\mathrm{d}s$$
$$+ \int_0^t \frac{1}{\varepsilon} \left\{ \nabla A_0\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f} - \nabla A_0\big(s,x_s^f\big) D_u x_s^f \right\} \mathrm{d}s$$
$$+ \int_0^t \frac{1}{\varepsilon} \left\{ \nabla A\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f} - \nabla A\big(s,x_s^f\big) D_u x_s^f \right\} \dot{f}(s)\,\mathrm{d}s$$
$$+ \int_0^t \nabla A\big(s,X_s^{\varepsilon,f}\big) D_u X_s^{\varepsilon,f}\,\mathrm{d}W(s),$$

$$
\begin{aligned}
D_u\tilde{Z}^f(t) = &\int_0^u \left\{A\big(s,x_s^f\big) + \nabla A\big(s,x_s^f\big)\,\tilde{Z}_s^f\right\}\,\mathbb{I}_{(s\le t)}\,\mathrm{d}s \\
&+ \int_0^t \nabla A_0\big(s,x_s^f\big)\,D_u\tilde{Z}_s^f\,\mathrm{d}s + \int_0^t \nabla^2 A_0\big(s,x_s^f\big)\left[D_u\tilde{S}_s^f, \tilde{Z}_s^f\right]\mathrm{d}s \\
&+ \int_0^t \nabla A\big(s,x_s^f\big)\,D_u\tilde{Z}_s^f\,\dot{f}(s)\,\mathrm{d}s + \int_0^t \nabla^2 A\big(s,x_s^f\big)\left[D_u x_s^f, \tilde{Z}_s^f\right]\dot{f}(s)\,\mathrm{d}s \\
&+ \int_0^t \nabla A\big(s,x_s^f\big)\,D_u x_s^f\,\mathrm{d}W(s)
\end{aligned}
$$

for $t \in [u,T]$, and the coefficients $A_i(t,\cdot)$ $(i = 0, 1, \ldots, m)$ are in $C^\infty_{1+,b}\big(C\big([-r,0]\,;\mathbb{R}^d\big)\,;\mathbb{R}^d\big)$ with respect to the second variable in $C\big([-r,0]\,;\mathbb{R}^d\big)$ for each $t \in [0,T]$, the assertion can be obtained via the Hölder inequality, the Burkholder inequality and the Gronwall inequality. Here, $\nabla^2 A_i\big(s,x_s^f\big)[\,\cdot\,,\,\cdot\,]$ $(i = 0, 1, \ldots, m)$ are bilinear mappings on $C\big([-r,0]\,;\mathbb{R}^d\big) \times C\big([-r,0]\,;\mathbb{R}^d\big)$, and $\nabla^2 A\big(s,x_s^f\big) = \Big(\nabla^2 A_1\big(s,x_s^f\big), \ldots, \nabla^2 A_m\big(s,x_s^f\big)\Big)$.

Step 5. Let $k \in \mathbb{N}$ be arbitrary, and $u_1, \ldots, u_k \in [0,T]$. Then, for any $p > 1$,

$$
\lim_{\varepsilon\searrow 0} \mathbb{E}\left[\sup_{t\in[-r,T]} \left|D^k_{(u_1,\ldots,u_k)} Y^{\varepsilon,f}(t) - D^k_{(u_1,\ldots,u_k)} \tilde{Z}^f(t)\right|^p\right] = 0.
$$

We have already proved the case of $k = 1$ in Step 4. Remark that

$$
\begin{aligned}
D^k_{(u_1,\ldots,u_k)}\left(\int_0^t \varphi(s)\,\mathrm{d}W(s)\right) &= \int_0^t D^k_{(u_1,\ldots,u_k)}\big(\varphi(s)\big)\,\mathrm{d}W(s) \\
&\quad + \sum_{j=1}^k D^{k-1}_{(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_k)}\left(\int_0^{u_j\wedge t} \varphi(s)\,\mathrm{d}s\right), \\
D^k_{(u_1,\ldots,u_k)}\left(\int_0^t \psi(s)\,\mathrm{d}s\right) &= \int_0^t D^k_{(u_1,\ldots,u_k)}\big(\psi(s)\big)\,\mathrm{d}s
\end{aligned}
$$

for adapted processes φ and ψ with nice properties. Then, we can get the assertion by induction on $k \in \mathbb{N}$.

Then, the assertion is the direct consequences of Step 2 and Step 5. The proof of Lemma 5.3 is complete. □

Theorem 3 (Lower estimate) *Suppose that the $\mathbb{R}^d$-valued functions A_i $(i = 1, \ldots, m)$ satisfy the uniformly elliptic condition* (11). *Then, it holds that*

$$
\liminf_{\varepsilon\searrow 0} \varepsilon^2 \ln p^\varepsilon(t,y) \ge -\overline{I}(y), \tag{19}
$$

where the function $\overline{I}$ is given in Theorem 1.

Proof. Since the assertion of Theorem 3 is trivial in the case of $\overline{I}(y) = +\infty$, we shall suppose that $\overline{I}(y) < +\infty$. Let $\Phi \in C_0^\infty\big(\mathbb{R}^d\,;\mathbb{R}\big)$ be non negative. For sufficiently small $0 < \sigma < 1$, recall the function Λ_σ as introduced in the proof of Theorem 2: $\Lambda_\sigma \in C_0^\infty\big(\mathbb{R}^d\,;[0,1]\big)$ such that

$$\Lambda_\sigma(z) = \begin{cases} 1 & (|z-y| \leq \sigma), \\ 0 & (|z-y| > 2\sigma). \end{cases}$$

Then, the Girsanov theorem tells us to see that

$$\begin{aligned}
\mathbb{E}\left[\Phi\big(X^\varepsilon(t)\big)\right] &= \mathbb{E}\left[\Phi\big(X^{\varepsilon,f}(t)\big)\exp\left(-\int_0^t \sum_{i=1}^m \frac{\dot f^i(s)}{\varepsilon}\,\mathrm{d}W^i(s) - \frac{1}{2\varepsilon^2}\|f\|_H^2\right)\right] \\
&= \exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\Phi\big(X^{\varepsilon,f}(t)\big)\exp\left(-\int_0^t \sum_{i=1}^m \frac{\dot f^i(s)}{\varepsilon}\,\mathrm{d}W^i(s) + \frac{2\sigma}{\varepsilon^2}\right)\right] \\
&\geq \exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\Phi\big(X^{\varepsilon,f}(t)\big)\,\mathbb{I}\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s) \leq 2\sigma\right)\right] \\
&\geq \exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\Phi\big(X^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s)\right)\right].
\end{aligned}$$

Here, the third inequality comes from the non-negativity in the exponent

$$-\frac{1}{\varepsilon}\int_0^t \dot f(s)\,\mathrm{d}W(s) + \frac{2\sigma}{\varepsilon^2} \geq 0,$$

while the forth inequality holds because of $0 \leq \Lambda_\sigma \leq 1$ and $\Lambda_\sigma \neq 0$ on the complement of $[-2\sigma, 2\sigma]$. Thus, the limiting argument enables us to see that

$$\begin{aligned}
p^\varepsilon(t,y) &\geq \exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\delta_y\big(X^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s)\right)\right] \\
&= \varepsilon^{-d}\exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\delta_0\big(Y^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s)\right)\right]
\end{aligned}$$

where δ_y is the Dirac delta function. Since

$$\mathbb{E}\left[\delta_0\big(Y^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s)\right)\right] \leq 1$$

from Lemme 5.3, we have

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\ln\left(\varepsilon^{-d}\,\mathbb{E}\left[\delta_0\big(Y^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \sum_{i=1}^m \dot f^i(s)\,\mathrm{d}W^i(s)\right)\right]\right) = 0.$$

Moreover, from the definition of the function $\bar{I}(y)$, we can find $f \in H$ with $y = x^f(t)$ such that

$$\frac{\|f\|_H^2}{2} \leq \bar{I}(y) + \sigma.$$

Hence, it holds that

$$\liminf_{\varepsilon \searrow 0} \varepsilon^2 \ln p^\varepsilon(t,y) \geq -\left(\frac{\|f\|_H^2}{2} + 2\sigma\right) \geq -\bar{I}(y) - 3\sigma.$$

Taking the limit as $\sigma \searrow 0$ completes the proof. □

Corollary 5.4 *Suppose that the $\mathbb{R}^d$-valued functions A_i $(i = 1,\ldots,m)$ satisfy the uniformly elliptic condition* (11)*. Then, it holds that*

$$p^\varepsilon(t,y) \sim \exp\left[-\frac{\bar{I}(y)}{\varepsilon^2}\right] \tag{20}$$

as $\varepsilon \searrow 0$, where the function $\bar{I}$ is given in Theorem 1.

Proof. Direct consequences of Theorems 2 and 3. □

Finally, we shall study the asymptotic behavior of the density $p(t,y)$ for $X(t)$ in a short time. Let $0 < r_0 \leq r$ be a constant, and $x \in \mathbb{R}^d$. We shall consider the case

$$\begin{aligned} &\eta(t) = x \quad (t \in [-r,0]), \\ &A_0(t,f) \equiv 0, \qquad A_i(t,f) = \tilde{A}_i\big(\tilde{f}, f(0)\big) \quad (i = 1,\ldots,m), \end{aligned}$$

where $\tilde{f} \in C\big([-r,0];\mathbb{R}^d\big)$ such that $\tilde{f}(t) = f(t)$ $(t \in [-r,-r_0])$ and $\tilde{f}(t) = \tilde{f}(-r_0)$ $(t \in [-r_0,0])$, for $f \in C\big([-r,0];\mathbb{R}^d\big)$. Suppose that the functions $\tilde{A}_i$ $(i = 1,\ldots,m)$ satisfy the uniformly elliptic condition of the form:

$$\inf_{\zeta \in \mathbb{S}^{d-1}} \inf_{f \in C([-r,-r_0];\mathbb{R}^d)} \inf_{y \in \mathbb{R}^d} \sum_{i=1}^m \big(\zeta \cdot \tilde{A}_i\big(f,y\big)\big)^2 > 0. \tag{21}$$

For $0 < \varepsilon \leq 1$, let $X = \big\{X(t)\,;\, t \in [-r,T]\big\}$ and $X^\varepsilon = \big\{X^\varepsilon(t)\,;\, t \in [-r,T]\big\}$ be the $\mathbb{R}^d$-valued processes determined by the equations of the form:

$$\begin{cases} X(t) = x & (t \in [-r,0]), \\ \mathrm{d}X(t) = \tilde{A}\big(\tilde{X}_t, X(t)\big)\,\mathrm{d}W(t) & (t \in (0,T]), \end{cases} \tag{22}$$

$$\begin{cases} X^\varepsilon(t) = x & (t \in [-r,0]), \\ \mathrm{d}X^\varepsilon(t) = \varepsilon\tilde{A}\big(\tilde{X}_t^\varepsilon, X^\varepsilon(t)\big)\,\mathrm{d}W(t) & (t \in (0,T]), \end{cases} \tag{23}$$

where $\tilde{A} = \big(\tilde{A}_1,\ldots,\tilde{A}_m\big)$. Remark that $X = X^\varepsilon\big|_{\varepsilon=1}$. Denote by $p(t,y)$ (or, $p^\varepsilon(t,y)$) the density for the probability law of $X(t)$ ($X^\varepsilon(t)$, respectively), whose existence can be justified under the uniformly elliptic condition (21) on the coefficients $\tilde{A}_i$ $(i = 1,\ldots,m)$. Then, we have

Corollary 5.5 *Suppose that the functions* $\tilde{A}_i$ $(i=1,\ldots,m)$ *satisfy the uniformly elliptic condition* (21)*. Then, it holds that*

$$p(t,y) \sim \exp\left[-\frac{r_0 \bar{I}(y)}{t}\right] \quad (t \searrow 0). \tag{24}$$

Proof. Recall that

$$\begin{aligned} X\left(\varepsilon^2 r_0\right) &= x + \int_0^{\varepsilon^2 r_0} \tilde{A}\left(\tilde{X}_s, X(s)\right) \mathrm{d}W(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{A}\left(\tilde{X}_{\varepsilon^2 s}, X\left(\varepsilon^2 s\right)\right) \mathrm{d}W(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{A}\left(x\, id, X\left(\varepsilon^2 s\right)\right) \mathrm{d}W(s), \end{aligned}$$

where $id \in C\left([-r,-r_0]\,;\mathbb{R}^d\right)$ such that $id(t)=1$ $(t \in [-r,-r_0])$. Here, the second equality holds from the scaling property on the Brownian motion W, while the third equality follows from $\varepsilon^2 s - r_0 \leq 0$. On the other hand, recall that

$$\begin{aligned} X^\varepsilon(r_0) &= x + \varepsilon \int_0^{r_0} \tilde{A}\left(\tilde{X}^\varepsilon_s, X^\varepsilon(s)\right) \mathrm{d}W(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{A}\left(x\, id, X^\varepsilon(s)\right) \mathrm{d}W(s), \end{aligned}$$

because of $s - r_0 \leq 0$. From the uniqueness of the solutions, we have $X\left(\varepsilon^2 r_0\right) = X^\varepsilon(r_0)$ in the sense of the probability law. Hence, we can get

$$p\left(\varepsilon^2 r_0, y\right) = p^\varepsilon(r_0, y).$$

As for the density $p^\varepsilon(r_0,y)$, we have already obtained the asymptotic behavior of the form:

$$p^\varepsilon(r_0,y) \sim \exp\left[-\frac{\bar{I}(y)}{\varepsilon^2}\right]$$

as $\varepsilon \searrow 0$, in Corollary 5.4. Taking $t = \varepsilon^2 r_0$ completes the proof. □

Remark 5.6 In particular, consider the case of

$$A_0(s,f)=0, \quad A_i(s,f) = \bar{A}_i\left(f(0)\right) \ (i=1,\ldots,m), \quad \eta(t) = x \ (t \in [-r,0]),$$

where $\bar{A}_i \in C^\infty_{1+,b}\left(\mathbb{R}^d\,;\mathbb{R}^d\right)$ such that the functions $\bar{A}_i$ $(i=1,\ldots,m)$ satisfy the uniformly elliptic condition of the form:

$$\inf_{\zeta \in \mathbb{S}^{d-1}} \inf_{y \in \mathbb{R}^d} \sum_{i=1}^m \left(\zeta \cdot \bar{A}_i(y)\right)^2 > 0. \tag{25}$$

Then, our equation can be written as follows:

$$\begin{cases} X(t) = x & (t \in [-r,0]), \\ \mathrm{d}X(t) = \bar{A}\big(X(t)\big)\,\mathrm{d}W(t) & (t \in (0,T]), \end{cases} \tag{26}$$

where $\bar{A} = \big(\bar{A}_1, \ldots, \bar{A}_m\big)$. Although our settings include the effect of the time-delay parameter r, the effect of the parameter r in the equation (26) can be ignored. Hence, the solution $\big\{X(t)\,;\, t \in [-r,T]\big\}$ is the diffusion process, so we have only to choose $r = 1$ in the starting point of our study. Moreover, the choice of $r = 1$ tells us to see that Corollary 5.5 is the well-known fact, that is, the Varadhan-type estimate, on the asymptotic behavior of the density function for diffusion processes. Hence, Corollary 5.5 can be also regarded as the generalization of the short-time estimate of the density for diffusion processes. □

Remark 5.7 Ferrante et al. in [4] discussed the large deviation principle for the solution process X^ε and the asymptotic estimate of the density, in the case of

$$A_i(s,f) = \tilde{A}_i\big(s, f(s-r)\big) \quad (i = 1, \ldots, m),$$

where $\tilde{A}_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ with $\tilde{A}_i(t,\cdot) \in C_b^\infty\big(\mathbb{R}^d\,;\mathbb{R}^d\big)$ for each $t \in [0,T]$. Moreover, suppose that the functions $\tilde{A}_i$ $(i = 1, \ldots, m)$ satisfy the uniformly elliptic condition of the form:

$$\inf_{\zeta \in \mathbb{S}^{d-1}} \inf_{t \in [0,T]} \inf_{y \in \mathbb{R}^d} \sum_{i=1}^m \big(\zeta \cdot A_i(t,y)\big)^2 > 0.$$

On the other hand, Mohammed and Zhang in [16] studied the large deviation principle for the solution process X^ε, in the case of

$$A_i(t,f) = \tilde{A}_i\big(t, f(t-r), f(t)\big) \quad (i = 1, \ldots, m),$$

where $\tilde{A}_i : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with $\tilde{A}_i(t,\cdot,\cdot) \in C_{1+,b}^\infty\big(\mathbb{R}^d \times \mathbb{R}^d\,;\mathbb{R}^d\big)$.

Since the special forms of the coefficients on the diffusion terms are quite essential in their arguments [4, 16], our situation cannot be included in their frameworks at all. □

Acknowledgements

This work is partially supported by Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Encouragement of Young Scientists, 23740083. This work is also partially supported from the Research Council of Norway, 219005/F11. This work was largely carried out, while the second author stayed at Center of Mathematics for Applications, University of Oslo from August to September, 2012, on leave from Osaka City University.

References

[1] R. Azencott: Grandes déviations et applications, *in* ”École d’été de Probabilités de Saint-Flour 1978”, Lecture Notes in Mathematics **774**, pp.1-176, Springer-Verlag, 1980.

[2] J. M. Bismut: *Large deviations and the Malliavin calculus*, Birkhäuser, 1984.

[3] A. Dembo and O. Zeitouni: *Large deviations techniques and applications*, 2nd edition, Springer-Verlag, 1998.

[4] M. Ferrante, C. Rovira and M. Sanz-Solé: Stochastic delay equations with hereditary drift: estimates of the density, *Journal of Functional Analysis* **177** (2000), pp.138-177.

[5] G. Di Nunno, B. Oksendal, F. Proske: Malliavin calculus for Lévy processes with applications to finance. Springer-Verlag, 2009.

[6] N. Ikeda and S. Watanabe: *Stochastic differential equations and diffusion processes*, 2nd edition, Kodansha / North-Holland, 1989.

[7] K. Ito and M. Nisio: On stationary solutions of a stochastic differential equation, *Journal of Mathematics of Kyoto University* **4** (1964), pp.1-75.

[8] T. Komatsu and A. Takeuchi: Simplified probabilistic approach to the Hörmander theorem, *Osaka Journal of Mathematics* **38** (2001), pp.681-691.

[9] S. Kusuoka and D. W. Stroock: Applications of the Malliavin calculus I, *in* ”Taniguchi Symposium on Stochastic Analysis, (Katata, 1982)”, pp.271-306, 1984.

[10] R. Léandre: Majoration en temps petit de la densité d’une diffusion dégénérée, *Probability Theory and Related Fields* **74** (1987), pp.289-294.

[11] R. Léandre: Minoration en temps petit de la densité d’une diffusion dégénérée, *Journal of Functional Analysis* **74** (1987), pp.399-414.

[12] R. Léandre: Intégration dans la fibre associée à une diffusion dégénérée, *Probability Theory and Related Fields* **76** (1987), pp.341-358.

[13] R. Léandre: A simple proof for a large deviation theorem, *in* Barcelona Seminar on a Stochastic Analysis, (St. Feliu de Guíols, 1991)”, pp.72-76, 1993.

[14] S. E. A. Mohammed: *Stochastic functional differential equations*, Pitman Advanced Publishing Program, (1984).

[15] S. E. A. Mohammed: Stochastic differential systems with memory: theory, examples and applications, *in* ”Stochastic Analysis and Related Topics VI, (Geilo, 1996)”, Progress in Probability, **42** (1998), Birkhauser.

[16] S. E. A. Mohammed and T. Zhang: Large deviations for stochastic systems with memory, *Discrete and Continuous Dynamical Systems, Series B* **6** (2006), pp.881-893.

[17] D. Nualart: Analysis on Wiener space and anticipating stochastic calculus, *in* ”École d’été de Probabilités de Saint-Flour 1995”, Lecture Notes in Mathematics **1690** (1998), pp.123-220, Springer-Verlag.

[18] D. Nualart: *The Malliavin calculus and related topics*, 2nd edition, Springer-Verlag (2006).

[19] A. Takeuchi: Malliavin calculus for degenerate stochastic functional differential equations, *Acta Applicandae Mathematicae* **97** (2007), pp.281-295.