Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent

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Abstract. In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p) \quad \begin{cases} -\Delta u + u = 0 & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number.

We study the asymptotic behavior of least energy solutions to (E_p) when the nonlinear exponent p gets large. Following the arguments of X. Ren and J.C. Wei [10], [11], we show that the least energy solutions remain bounded uniformly in p, and it develops one peak on the boundary, the location of which is controlled by the Green function associated to the linear problem.

Keywords: least energy solution, nonlinear Neumann boundary condition, large exponent, concentration.

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1. Introduction.

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number. Let $H^1(\Omega)$ be the usual Sobolev space with the norm $\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + u^2)\,dx$. Since the trace Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$C_p^2 = \inf\left\{ \int_\Omega \left(|\nabla u|^2 + u^2\right) dx \mid u \in H^1(\Omega), \int_{\partial\Omega} |u|^{p+1} ds_x = 1 \right\}. \tag{1.2}$$

Standard variational method implies that C_p^2 is achieved by a positive function $\overline{u}_p \in H^1(\Omega)$ and then $u_p = C_p^{2/(p-1)}\overline{u}_p$ solves (1.1). We call u_p a least energy solution to the problem (1.1).

In this paper, we prove the followings:

Theorem 1 *Let u_p be a least energy solution to (E_p). Then it holds*

$$1 \le \liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \le \limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \le \sqrt{e}.$$

To state further results, we set

$$v_p = u_p / (\int_{\partial\Omega} u_p^p ds_x). \tag{1.3}$$

Theorem 2 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any sequence v_{p_n} of v_p defined in (1.3) with $p_n \to \infty$, there exists a subsequence (still denoted by v_{p_n}) and a point $x_0 \in \partial\Omega$ such that the following statements hold true.*

(1)

$$f_n = \frac{u_{p_n}^{p_n}}{\int_{\partial\Omega} u_{p_n}^{p_n} ds_x} \overset{*}{\rightharpoonup} \delta_{x_0}$$

in the sense of Radon measures on $\partial\Omega$.

(2) *$v_{p_n} \to G(\cdot, x_0)$ in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$, $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \le t < \infty$, where $G(x,y)$ denotes the Green function of $-\Delta$ for the following Neumann problem:*

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = 0 & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu_x}(x,y) = \delta_y(x) & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

(3) *x_0 satisfies*

$$\nabla_{\tau(x_0)} R(x_0) = \vec{0},$$

where $\tau(x_0)$ denotes a tangent vector at the point $x_0 \in \partial\Omega$ and R is the Robin function defined by $R(x) = H(x, x)$, where

$$H(x, y) := G(x, y) - \frac{1}{\pi} \log |x - y|^{-1}$$

denotes the regular part of G.

Concerning related results, X. Ren and J.C. Wei [10], [11] first studied the asymptotic behavior of least energy solutions to the semilinear problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

as $p \to \infty$, where Ω is a bounded smooth domain in $\mathbb{R}^2$. They proved that the least energy solutions remain bounded and bounded away from zero in L^∞-norm uniformly in p. As for the shape of solutions, they showed that the least energy solutions must develop one "peak" in the interior of Ω, which must be a critical point of the Robin function associated with the Green function subject to the Dirichlet boundary condition. Later, Adimurthi and Grossi [1] improved their results by showing that, after some scaling, the limit profile of solutions is governed by the Liouville equation

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^U dx < \infty,$$

and obtained that $\lim_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} = \sqrt{e}$ for least energy solutions u_p. Actual existence of concentrating solutions to (1.1) is recently obtained by H. Castro [4] by a variational reduction procedure, along the line of [7] and [6]. Also in our case, we may conjecture that the limit problem of (1.1) is

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}^2_+, \\ \int_{\partial\mathbb{R}^2_+} e^U ds < \infty, \end{cases}$$

and $\lim_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} = \sqrt{e}$ holds true at least for least energy solutions u_p. Verification of these conjectures remains as the future work.

2. Some estimates for C_p^2.

In this section, we provide some estimates for C_p^2 in (1.2) as $p \to \infty$.

Lemma 3 *For any $s \geq 2$, there exists $\tilde{D}_s > 0$ such that for any $u \in H^1(\Omega)$,*

$$\|u\|_{L^s(\partial\Omega)} \leq \tilde{D}_s s^{\frac{1}{2}} \|u\|_{H^1(\Omega)}$$

holds true. Furthermore, we have

$$\lim_{s\to\infty} \tilde{D}_s = (2\pi e)^{-\frac{1}{2}}.$$

Proof. Let $u \in H^1(\Omega)$. By Trudinger-Moser trace inequality, see [5] and the references therein, we have

$$\int_{\partial\Omega} \exp\left(\frac{\pi|u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right) ds_x \leq C(\Omega)$$

for any $u \in H^1(\Omega)$, where $u_{\partial\Omega} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u ds_x$. Thus, by an elementary inequality $\frac{x^s}{\Gamma(s+1)} \leq e^x$ for any $x \geq 0$ and $s \geq 0$, where $\Gamma(s)$ is the Gamma function, we see

$$\begin{aligned}
&\frac{1}{\Gamma((s/2)+1)} \int_{\partial\Omega} |u - u_{\partial\Omega}|^s ds_x \\
&= \frac{1}{\Gamma((s/2)+1)} \int_{\partial\Omega} \left(\pi\frac{|u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right)^{s/2} ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s \\
&\leq \int_{\partial\Omega} \exp\left(\pi\frac{|u(x) - u_{\partial\Omega}|^2}{\|\nabla u\|_{L^2(\Omega)}^2}\right) ds_x \pi^{-s/2} \|\nabla u\|_{L^2(\Omega)}^s \\
&\leq C(\Omega)\pi^{-s/2}\|\nabla u\|_{L^2(\Omega)}^s.
\end{aligned}$$

Set

$$D_s := (\Gamma(s/2+1))^{1/s} C(\Omega)^{1/s} \pi^{-1/2} s^{-1/2}.$$

Then we have

$$\|u - u_{\partial\Omega}\|_{L^s(\partial\Omega)} \leq D_s s^{1/2} \|\nabla u\|_{L^2(\Omega)}.$$

Stirling's formula says that $(\Gamma(\frac{s}{2}+1))^{\frac{1}{s}} \sim (\frac{s}{2e})^{1/2}$ as $s \to \infty$, so we have

$$\lim_{s\to\infty} D_s = \left(\frac{1}{2\pi e}\right)^{1/2}.$$

On the other hand, by the embedding $\|u\|_{L^2(\partial\Omega)} \leq C(\Omega)\|u\|_{H^1(\Omega)}$ for any $u \in H^1(\Omega)$, we see

$$|u_{\partial\Omega}| \leq \frac{1}{|\partial\Omega|^{1/2}}\left(\int_{\partial\Omega}|u|^2 ds_x\right)^{1/2} \leq \frac{C(\Omega)}{|\partial\Omega|^{1/2}}\|u\|_{H^1(\Omega)}.$$

Thus,

$$\begin{aligned}
\|u\|_{L^s(\partial\Omega)} &\leq \|u-u_{\partial\Omega}\|_{L^s(\partial\Omega)} + \|u_{\partial\Omega}\|_{L^s(\partial\Omega)} \\
&\leq \|u-u_{\partial\Omega}\|_{L^s(\partial\Omega)} + |u_{\partial\Omega}||\partial\Omega|^{1/s} \\
&\leq s^{1/2}\|u\|_{H^1(\Omega)}\left(D(s) + \frac{C(\Omega)|\partial\Omega|^{1/s-1/2}}{s^{1/2}}\right).
\end{aligned}$$

Put

$$\tilde{D}(s) = D(s) + \frac{C(\Omega)|\partial\Omega|^{1/s-1/2}}{s^{1/2}}.$$

Then, we have $\lim_{s\to\infty}\tilde{D}(s) = \lim_{s\to\infty} D(s) = \frac{1}{\sqrt{2\pi e}}$ and

$$\|u\|_{L^s(\partial\Omega)} \leq \tilde{D}_s s^{\frac{1}{2}}\|u\|_{H^1(\Omega)}$$

holds. □

Lemma 4 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. Then we have*

$$\lim_{p\to\infty} pC_p^2 = 2\pi e.$$

Proof. For the estimate from below, we use Lemma 3. By Lemma 3, we have

$$\|u\|^2_{L^{p+1}(\partial\Omega)} \leq \tilde{D}^2_{p+1}(p+1)\|u\|^2_{H^1(\Omega)}$$

for any $u \in H^1(\Omega)$, which leads to $\tilde{D}^{-2}_{p+1}\left(\frac{p}{p+1}\right) \leq pC_p^2$. Thus, we have $2\pi e \leq \liminf_{p\to\infty} pC_p^2$, since $\lim_{p\to\infty}\tilde{D}_{p+1} = (2\pi e)^{-1/2}$.

For the estimate from above, we use the Moser function. Let $0 < l < L$. First, we assume $\Omega\cap B_L(0) = \Omega\cap B_L^+$ where $B_L^+ = B_L(0)\cap\{y=(y_1,y_2) \mid y_2 > 0\}$. Define

$$m_l(y) = \frac{1}{\sqrt{\pi}}\begin{cases} (\log L/l)^{1/2}, & 0 \leq |y| \leq l, y \in B_L^+, \\ \frac{(\log L/|y|)}{(\log L/l)^{1/2}}, & l \leq |y| \leq L, y \in B_L^+, \\ 0, & L \leq |y|, y \in B_L^+. \end{cases}$$

Then $\|\nabla m_l\|_{L^2(B_L^+)} = 1$ and since $m_l \equiv 0$ on $\partial B_L^+ \cap \{y_2 > 0\}$, we have

$$\|m_l\|_{L^{p+1}(\partial B_L^+)}^{p+1} = 2\int_0^l |m_l(y_1)|^{p+1} dy_1 + 2\int_l^L |m_l(y_1)|^{p+1} dy_1$$
$$\geq 2\int_0^l \left(\frac{1}{\sqrt{\pi}}\sqrt{\log(L/l)}\right)^{p+1} dy_1 = 2l\left(\sqrt{\frac{1}{\pi}\log(L/l)}\right)^{p+1}.$$

Thus $\|m_l\|_{L^{p+1}(\partial B_L^+)}^2 \geq (2l)^{\frac{2}{p+1}} \frac{1}{\pi}\log(L/l)$. Also,

$$\|m_l\|_{L^2(B_L^+)}^2 = \int_0^\pi \int_0^L |m_l|^2 r dr d\theta$$
$$= \int_0^\pi \int_0^l |m_l|^2 r dr d\theta + \int_0^\pi \int_l^L |m_l|^2 r dr d\theta$$
$$=: I_1 + I_2.$$

We calculate

$$I_1 = \frac{l^2}{2}\log(L/l),$$
$$I_2 = \frac{1}{\log(L/l)}\int_l^L (\log L/r)^2 r dr$$
$$= -\frac{l^2}{2} - \frac{l^2}{2}\log(L/l) + \frac{1}{\log(L/l)}\frac{L^2 - l^2}{4}.$$

Thus we have $\|m_l\|_{L^2(B_L^+)}^2 = -\frac{l^2}{2} + \frac{1}{\log(L/l)}\frac{L^2-l^2}{4}$.

Now, put $l = Le^{-\frac{p+1}{2}}$ and extend m_l by 0 outside B_L^+ and consider it as a function in $H^1(\Omega)$. Then

$$pC_p^2 \leq p\frac{\|m_l\|_{H^1(B_L^+)}^2}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} = \frac{p}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} + \frac{p\|m_l\|_{L^2(B_L^+)}^2}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2}.$$

We estimate

$$\frac{p}{\|m_l\|_{L^{p+1}(\partial B_L^+)}^2} \leq \frac{p}{(2l)^{\frac{2}{p+1}}\frac{1}{\pi}\log(L/l)} = \left(\frac{p}{p+1}\right) 2\pi e \frac{1}{(2L)^{\frac{2}{p+1}}} \to 2\pi e,$$

and

$$\frac{p\|m_l\|^2_{L^2(B_L^+)}}{\|m_l\|^2_{L^{p+1}(\partial B_L^+)}} \le \frac{p\left(-\frac{l^2}{2}+\frac{1}{\log(L/l)}\frac{L^2-l^2}{4}\right)}{(2l)^{\frac{2}{p+1}}\frac{1}{\pi}\log(L/l)}$$
$$= \frac{2\pi e}{(2L)^{\frac{2}{p+1}}}\left(\frac{p}{p+1}\right)\left\{-\frac{L^2}{2}e^{-(p+1)}+\frac{2}{p+1}\frac{L^2(1-e^{-(p+1)})}{4}\right\} \to 0$$

as $p \to \infty$. Therefore, we have obtained $\limsup_{p\to\infty} pC_p^2 \le 2\pi e$ in this case.

In the general case, we introduce a diffeomorphism which flattens the boundary $\partial\Omega$, see Ni and Takagi [9]. We may assume $0 \in \partial\Omega$ and in a neighborhood U of 0, the boundary $\partial\Omega$ can be written by the graph of function ψ: $\partial\Omega \cap U = \{x = (x_1, x_2) \mid x_2 = \psi(x_1)\}$, with $\psi(0) = 0$ and $\frac{\partial\psi}{\partial x_1}(0) = 0$. Define $x = \Phi(y) = (\Phi_1(y), \Phi_2(y))$ for $y = (y_1, y_2)$, where

$$x_1 = \Phi_1(y) = y_1 - y_2\frac{\partial\psi}{\partial x_1}(y_1), \quad x_2 = \Phi_2(y) = y_2 + \psi(y_1),$$

and put $D_L = \Phi(B_L^+)$. Note that $\partial D_L \cap \partial\Omega = \Phi(\partial B_L^+ \cap \{(y_1, 0)\})$. Since $D\Phi(0) = Id$, we obtain there exists $\Psi = \Phi^{-1}$ in a neighborhood of 0. Finally, define $\tilde{m}_l \in H^1(\Omega)$ as $\tilde{m}_l(x) = m_l(\Psi(x))$ for $x \in U \cap \Omega$. Then, Lemma A.1 in [9] implies the estimates

$$\|\nabla\tilde{m}_l\|^2_{L^2(D_L)} = \|\nabla m_l\|^2_{L^2(B_L^+)} + O(\frac{1}{p}),$$
$$\|\tilde{m}_l\|^2_{L^2(D_L)} \le (1 + O(L))\|m_l\|^2_{L^2(B_L^+)},$$
$$\|\tilde{m}_l\|^2_{L^{p+1}(\partial D_L\cap\partial\Omega)} \ge \|m_l\|^2_{L^{p+1}(\partial B_L^+\cap\{(y_1,0)\})}.$$

The last inequality comes from that, if we put $I = \{(y_1, 0) \mid -L \le y_1 \le L\} \subset \partial B_L^+$ and $J = \Phi(I) \subset \partial\Omega$, then $ds_x = \sqrt{1 + (\psi'(x_1))^2}dx_1$ and $J = \{(x_1, x_2) \mid x_1 = y_1, x_2 = \psi(y_1)\}$. Thus

$$\int_J |\tilde{m}_l(x)|^{p+1} ds_x = \int_I |m_l(y)|^{p+1}\sqrt{1 + (\psi'(y_1))^2}dy_1 \ge \int_I |m_l(y)|^{p+1}dy_1.$$

By testing C_p^2 with $\tilde{m}_l$, again we obtain $\limsup_{p\to\infty} pC_p^2 \le 2\pi e$. □

Corollary 5 *Let u_p be a least energy solution to (E_p). Then we have*

$$\lim_{p\to\infty} p\int_{\partial\Omega} u_p^{p+1} ds_x = 2\pi e, \quad \lim_{p\to\infty} p\int_\Omega \left(|\nabla u_p|^2 + u_p^2\right) dx = 2\pi e.$$

Proof. Since u_p satisfies

$$\int_{\Omega}\left(|\nabla u_p|^2+u_p^2\right)dx=\int_{\partial\Omega}u_p^{p+1}ds_x$$

and

$$pC_p^2=p\frac{\int_{\Omega}\left(|\nabla u_p|^2+u_p^2\right)dx}{\left(\int_{\partial\Omega}u_p^{p+1}ds_x\right)^{\frac{2}{p+1}}}=\left(p\int_{\partial\Omega}u_p^{p+1}ds_x\right)^{\frac{p-1}{p+1}}p^{\frac{2}{p+1}},$$

the results follow from Lemma 4. □

3. Proof of Theorem 1.

The uniform estimate of $\|u\|_{L^\infty(\partial\Omega)}$ from below holds true for any solution u of (E_p), as in [10].

Lemma 6 *There exists $C_1>0$ independent of p such that*

$$\|u\|_{L^\infty(\partial\Omega)}\geq C_1$$

holds true for any solution u to (E_p).

Proof. Let $\lambda_1>0$ be the first eigenvalue of the eigenvalue problem

$$\begin{cases}-\Delta\varphi+\varphi=0 & \text{in }\Omega,\\ \frac{\partial\varphi}{\partial\nu}=\lambda\varphi & \text{on }\partial\Omega\end{cases}$$

and let φ_1 be the corresponding eigenfunction. It is known that λ_1 is simple, isolated, and φ_1 can be chosen positive on $\overline{\Omega}$. (see, [12]). Then by integration by parts, we have

$$\begin{aligned}0&=\int_{\Omega}\{(-\Delta u+u)\,\varphi_1-(-\Delta\varphi_1+\varphi_1)\,u\}\,dx=\int_{\partial\Omega}\left(\frac{\partial\varphi_1}{\partial\nu}u-\frac{\partial u}{\partial\nu}\varphi_1\right)ds_x\\&=\int_{\partial\Omega}\varphi_1u(\lambda_1-u^{p-1})ds_x.\end{aligned}$$

Since $\varphi_1u>0$ on $\partial\Omega$, this implies $\|u\|_{L^\infty(\partial\Omega)}^{p-1}\geq\lambda_1$. □

Lemma 7 *Let u_p be a least energy solution to (E_p). Then it holds*

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \le \sqrt{e}.$$

Proof. We follow the argument of [11], which in turn originates from [8], and use Moser's iteration procedure. Let u be a solution to (E_p). For $s \ge 1$, multiplying $u^{2s-1} \in H^1(\Omega)$ to the equation of (E_p) and integrating, we get

$$\left(\frac{2s-1}{s^2}\right)^2 \int_\Omega |\nabla(u^s)|^2 dx + \int_\Omega u^{2s} dx = \int_{\partial\Omega} u^{2s-1+p} ds_x.$$

Since $\frac{2s-1}{s^2} \le 1$ for $s \ge 1$, we have

$$\left(\frac{2s-1}{s^2}\right) \|u^s\|^2_{H^1(\Omega)} \le \int_{\partial\Omega} u^{2s-1+p} ds_x. \tag{3.1}$$

Also by Lemma 3 applied to $u^s \in H^1(\Omega)$, we have

$$\left(\int_{\partial\Omega} u^{\nu s} ds_x\right)^{1/\nu} \le \tilde{D}_\nu \nu^{\frac{1}{2}} \|u^s\|_{H^1(\Omega)}$$

for any $\nu \ge 2$. Thus by (3.1), we see

$$\left(\int_{\partial\Omega} u^{\nu s} ds_x\right)^{1/\nu} \le \tilde{D}_\nu \nu^{\frac{1}{2}} \left(\frac{s^2}{2s-1}\right)^{1/2} \left(\int_{\partial\Omega} u^{2s-1+p} ds_x\right)^{1/2}.$$

Since $\tilde{D}^2_\nu \left(\frac{s}{2s-1}\right) \le C_1$ for some $C_1 > 0$ independent of $s \ge 1$ and $\nu \ge 2$, we obtain

$$\left(\int_{\partial\Omega} u^{\nu s} ds_x\right)^{2/\nu} \le C_1 \nu s \int_{\partial\Omega} u^{2s-1+p} ds_x. \tag{3.2}$$

Once the iteration scheme (3.2) is obtained, the rest of the argument is exactly the same as one in [11]. Indeed, by Lemma 3, we have

$$\left(\int_{\partial\Omega} u^{\nu} ds_x\right)^{1/\nu} \le (2\pi e)^{-\frac{1}{2}} (1 + o(1)) \nu^{1/2} \|u\|_{H^1(\Omega)}, \tag{3.3}$$

here $o(1) \to 0$ as $\nu \to \infty$. Now, we fix $\alpha > 0$ and $\varepsilon > 0$ which will be chosen small later and put $\nu = (1+\alpha)(p+1) > 2$ in (3.3). By Corollary 5,

$p^{1/2}(2\pi e)^{-1/2}\|u_p\|_{H^1(\Omega)} \to 1$ as $p \to \infty$ for a least energy solution u_p. Thus by (3.3), we see there exists $p_0 > 1$ such that

$$\int_{\partial\Omega} u_p^\nu ds_x \leq (1+\alpha+\varepsilon)^{\nu/2} =: M_0$$

for $p > p_0$. Define $\{s_j\}_{j=0,1,2\cdots}$ and $\{M_j\}_{j=0,1,2\cdots}$ such that

$$\begin{cases} p-1+2s_0 = \nu, \\ p-1+2s_{j+1} = \nu s_j, \ (j=0,1,2,\cdots), \end{cases}$$

and

$$\begin{cases} M_0 = (1+\alpha+\varepsilon)^{\nu/2}, \\ M_{j+1} = (C_1 \nu s_j M_j)^{\nu/2}, \ (j=0,1,2,\cdots). \end{cases}$$

We easily see that $s_0 = \frac{\alpha(p+1)}{2} > 0$, s_j is increasing in j, $s_j \to +\infty$ as $j \to \infty$, and actually,

$$s_j = \left(\frac{\nu}{2}\right)^j (s_0 - x) + x \quad \text{where} \quad x = \frac{p-1}{\nu-2} > 0.$$

At this moment, we can follow exactly the same argument in [11] to obtain the estimates

$$\|u_p\|_{L^{\nu s_{j-1}}(\partial\Omega)} \leq M_j^{\frac{1}{\nu s_{j-1}}} \leq \exp(m(\alpha,p,\varepsilon)),$$

where $m(\alpha,p,\varepsilon)$ is a constant depending on α, p and ε, satisfying

$$\lim_{p\to\infty} m(\alpha,p,\varepsilon) = \frac{1+\alpha}{2\alpha}\log(1+\alpha+\varepsilon).$$

Letting $j \to \infty$, $p \to \infty$ first, we get

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq (1+\alpha+\varepsilon)^{\frac{1+\alpha}{2\alpha}},$$

and then letting $\alpha \to +0$, $\varepsilon \to +0$, we obtain

$$\limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leq \sqrt{e}$$

as desired. □

By Theorem 1 and Hölder's inequality, we also obtain

Corollary 8 *There exists* $C_1, C_2 > 0$ *such that*

$$C_1 \leq p \int_{\partial\Omega} u_p^p ds_x \leq C_2$$

holds.

4. Proof of Theorem 2.

In this section, we prove Theorem 2. First, we recall an L^1 estimate from [6], which is a variant of the one by Brezis and Merle [2].

Lemma 9 *Let* u *be a solution to*

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases}$$

with $h \in L^1(\partial\Omega)$*, where* Ω *is a smooth bounded domain in* $\mathbb{R}^2$*. For any* $\varepsilon \in (0, \pi)$*, there exists a constant* $C > 0$ *depending only on* ε *and* Ω*, independent of* u *and* h*, such that*

$$\int_{\partial\Omega} \exp\left(\frac{(\pi - \varepsilon)|u(x)|}{\|h\|_{L^1(\partial\Omega)}} \right) ds_x \leq C \tag{4.1}$$

holds true.

Also we need an elliptic L^1 estimate by Brezis and Strauss [3] for weak solutions with the L^1 Neumann data.

Lemma 10 *Let* u *be a weak solution of*

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$ *and* $g \in L^1(\partial\Omega)$*, where* Ω *is a smooth bounded domain in* $\mathbb{R}^N$*,* $N \geq 2$*. Then we have* $u \in W^{1,q}(\Omega)$ *for all* $1 \leq q < \frac{N}{N-1}$ *and*

$$\|u\|_{W^{1,q}(\Omega)} \leq C_q \left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \right)$$

holds.

For the proof, see [3]:Lemma 23.

Now, following [10], [11], we define the notion of δ-regular points. Put $u_n = u_{p_n}$ for any subsequence of u_p. Since u_n satisfies

$$\int_{\partial\Omega} \frac{u_n^{p_n}}{\int_{\partial\Omega} u_n^{p_n} ds_x} ds_x = 1,$$

we can select a subsequence $p_n \to \infty$ (without changing the notation) and a Radon measure $\mu \geq 0$ on $\partial\Omega$ such that

$$f_n := \frac{u_n^{p_n}}{\int_{\partial\Omega} u_n^{p_n} ds_x} \overset{*}{\rightharpoonup} \mu$$

weakly in the sense of Radon measures on $\partial\Omega$, i.e.,

$$\int_{\partial\Omega} f_n \varphi \, ds_x \to \int_{\partial\Omega} \varphi \, d\mu$$

for all $\varphi \in C(\partial\Omega)$. As in [11], we define

$$L_0 = \frac{1}{2\sqrt{e}} \limsup_{p\to\infty} \left(p \int_{\partial\Omega} u_p^p ds_x \right). \tag{4.2}$$

By Corollary 5 and Hölder's inequality, we have

$$L_0 \leq \pi\sqrt{e}.$$

For some $\delta > 0$ fixed, we call a point $x_0 \in \partial\Omega$ a *δ-regular point* if there is a function $\varphi \in C(\partial\Omega)$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ in a neighborhood of x_0 such that

$$\int_{\partial\Omega} \varphi \, d\mu < \frac{\pi}{L_0 + 2\delta}$$

holds. Define $S = \{x_0 \in \partial\Omega \mid x_0 \text{ is not a } \delta\text{-regular point for any } \delta > 0.\}$. Then,

$$\mu(\{x_0\}) \geq \frac{\pi}{L_0 + 2\delta} \tag{4.3}$$

for all $x_0 \in S$ and for any $\delta > 0$.

Here, following the argument in [11], we prove a key lemma in the proof of Theorem 2.

Lemma 11 *Let $x_0 \in \partial\Omega$ be a δ-regular point for some $\delta > 0$. Then $v_n = \frac{u_n}{\int_{\partial\Omega} u_n^{p_n} ds_x}$ is bounded in $L^\infty(B_{R_0}(x_0) \cap \Omega)$ for some $R_0 > 0$.*

Proof. Let $x_0 \in \partial\Omega$ be a δ-regular point. Then by definition, there exists $R > 0$ such that

$$\int_{\partial\Omega \cap B_R(x_0)} f_n ds_x < \frac{\pi}{L_0 + \delta}$$

holds for all n large. Put $a_n = \chi_{B_R(x_0)} f_n$ and $b_n = (1 - \chi_{B_R(x_0)}) f_n$ where $\chi_{B_R(x_0)}$ denotes the characteristic function of $B_R(x_0)$. Split $v_n = v_{1n} + v_{2n}$, where v_{1n}, v_{2n} is a solution to

$$\begin{cases} -\Delta v_{1n} + v_{1n} = 0 & \text{in } \Omega, \\ \frac{\partial v_{1n}}{\partial \nu} = a_n & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_{2n} + v_{2n} = 0 & \text{in } \Omega, \\ \frac{\partial v_{2n}}{\partial \nu} = b_n & \text{on } \partial\Omega \end{cases}$$

respectively. By the maximum principle, we have $v_{1n}, v_{2n} > 0$. Since $b_n = 0$ on $B_R(x_0)$, elliptic estimates imply that

$$\|v_{2n}\|_{L^\infty(B_{R/2}(x_0)\cap\Omega)} \le C\|v_{2n}\|_{L^1(B_R(x_0)\cap\Omega)} \le C,$$

where we used the fact $\|v_{2n}\|_{L^1(\Omega)} = \|\Delta v_{2n}\|_{L^1(\Omega)} = \|b_n\|_{L^1(\partial\Omega)} \le C$ for the last inequality. Thus we have to consider v_{1n} only.

Claim: For any $x \in \partial\Omega$, we have

$$f_n(x) \le \exp\left((L_0 + \delta/2) v_n(x)\right) \tag{4.4}$$

for n large.

Indeed, put

$$\alpha_n = \frac{\|u_n\|_{L^\infty(\partial\Omega)}}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}}.$$

Then by Lemma 7 and Corollary 8, we have

$$\limsup_{n\to\infty} \alpha_n \le \sqrt{e}.$$

Since the function $s \mapsto \frac{\log s}{s}$ is monotone increasing if $0 < s < e$, and $\frac{u_n(x)}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}} \le \alpha_n$ for any $x \in \partial\Omega$, we observe that for fixed $\varepsilon > 0$,

$$\frac{\log \frac{u_n(x)}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}}}{\frac{u_n(x)}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}}} \le \frac{\log \alpha_n}{\alpha_n} \le \frac{1}{2\sqrt{e}} + \varepsilon$$

holds for large n. Thus

$$
\begin{aligned}
f_n(x) &= \exp\left(p_n \log \frac{u_n(x)}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}}\right) \le \exp\left(\frac{p_n u_n(x)}{\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1/p_n}}\left(\frac{1}{2\sqrt{e}}+\varepsilon\right)\right) \\
&= \exp\left(p_n v_n(x)\left(\int_{\partial\Omega} u_n^{p_n} ds_x\right)^{1-1/p_n}\left(\frac{1}{2\sqrt{e}}+\varepsilon\right)\right) \\
&\le \exp\left(\left(\limsup_{n\to\infty} p_n \int_{\partial\Omega} u_n^{p_n} ds_x\right) v_n(x)\left(\frac{1}{2\sqrt{e}}+2\varepsilon\right)\right) \\
&= \exp\left(\left(\frac{1}{2\sqrt{e}}+2\varepsilon\right) 2\sqrt{e} L_0 v_n(x)\right) = \exp\left(\left(L_0 + 4\varepsilon\sqrt{e} L_0\right) v_n(x)\right).
\end{aligned}
$$

Thus if we choose $\varepsilon > 0$ so small, we have the claim (4.4).

By this claim and the fact that v_{2n} is uniformly bounded in $B_{R/2}(x_0)$, for sufficiently small $\delta_0 > 0$ so that $(1+\delta_0)\frac{L_0+\delta/2}{L_0+\delta} < 1$, we have

$$
\begin{aligned}
\int_{B_{R/2}(x_0)\cap\partial\Omega} f_n^{1+\delta_0} ds_x &\le \int_{B_{R/2}(x_0)\cap\partial\Omega} \exp\left((1+\delta_0)(L_0+\delta/2)v_n(x)\right) ds_x \\
&\le C \int_{B_{R/2}(x_0)\cap\partial\Omega} \exp\left((1+\delta_0)(L_0+\delta/2)v_{1n}(x)\right) ds_x \\
&\le C \int_{B_{R/2}(x_0)\cap\partial\Omega} \exp\left(\pi(1+\delta_0)\frac{L_0+\delta/2}{L_0+\delta} v_{1n}(x)\right) ds_x \\
&= C \int_{B_{R/2}(x_0)\cap\partial\Omega} \exp\left(\pi(1-\varepsilon_0)v_{1n}(x)\right) ds_x,
\end{aligned}
$$

where $1-\varepsilon_0 = (1+\delta_0)\frac{L_0+\delta/2}{L_0+\delta}$. Thus by Lemma 9, we have

$$
\int_{B_{R/2}(x_0)\cap\partial\Omega} f_n^{1+\delta_0} ds_x \le C
$$

for some $C > 0$ independent of n. This fact and elliptic estimates imply that

$$
\limsup_{n\to\infty} \|v_n\|_{L^\infty(\Omega\cap B_{R/4}(x_0))} \le C,
$$

which proves Lemma. □

Now, we estimate the cardinality of the set S. By Theorem 1, we have

$$
v_n(x_n) = \frac{\|u_n\|_{L^\infty(\partial\Omega)}}{\int_{\partial\Omega} u_n^{p_n} ds_x} \ge \frac{C_1}{\int_{\partial\Omega} u_n^{p_n} ds_x} \to \infty
$$

for a sequence $x_n \in \partial\Omega$ such that $u_n(x_n) = \|u_n\|_{L^\infty(\partial\Omega)}$. Thus by Lemma 11, we see $x_0 = \lim_{n\to\infty} x_n \in S$ and $\sharp S \geq 1$. On the other hand, by (4.3) we have

$$1 = \lim_{n\to\infty} \|f_n\|_{L^1(\partial\Omega)} \geq \mu(\partial\Omega) \geq \frac{\pi}{L_0 + 2\delta} \sharp S,$$

which leads to

$$1 \leq \sharp S \leq \frac{L_0 + 2\delta}{\pi} \leq \sqrt{e} + \frac{2\delta}{\pi} \simeq 1.64 \cdots + \frac{2\delta}{\pi}.$$

Thus we have $\sharp S = 1$ if $\delta > 0$ is chosen small.

Let $S = \{x_0\}$ for some point $x_0 \in \partial\Omega$. By Lemma 11, we can conclude easily that $f_n \overset{*}{\rightharpoonup} \delta_{x_0}$ in the sense of Radon measures on $\partial\Omega$:

$$\int_{\partial\Omega} f_n \varphi ds_x \to \varphi(x_0), \quad \text{as } n \to \infty$$

for any $\varphi \in C(\partial\Omega)$, since v_n is locally uniformly bounded on $\partial\Omega \setminus \{x_0\}$ and $f_n \to 0$ uniformly on any compact sets of $\partial\Omega \setminus \{x_0\}$.

Now, by the L^1 estimate in Lemma 10, we have v_n is uniformly bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. Thus, by choosing a subsequence, we have a function $\bar{G}$ such that $v_n \rightharpoonup \bar{G}$ weakly in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$, $v_n \to \bar{G}$ strongly in $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leq t < \infty$. The last convergence follows by the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^t(\Omega)$ for any $1 \leq t < \frac{q}{2-q}$. Thus by taking the limit in the equation

$$\int_\Omega (-\Delta\varphi + \varphi) v_n dx = \int_{\partial\Omega} f_n \varphi ds_x - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} v_n ds_x$$

for any $\varphi \in C^1(\overline{\Omega})$, we obtain

$$\int_\Omega (-\Delta\varphi + \varphi) \bar{G} dx + \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \bar{G} ds_x = \varphi(x_0),$$

which implies $\bar{G}$ is the solution of (1.4) with $y = x_0$.

Finally, we prove the statement (3) of Theorem 2. We borrow the idea of [6] and derive Pohozaev-type identities in balls around the peak point. We may assume $x_0 = 0$ without loss of generality. As in [6], we use a conformal diffeomorphism $\Psi : H \cap B_{R_0} \to \Omega \cap B_r$ which flattens the boundary $\partial\Omega$, where $H = \{(y_1, y_2) \mid y_2 > 0\}$ denotes the upper half space and $R_0 > 0$ is a radius sufficiently small. We may choose Ψ is at least C^3, up to $\partial H \cap B_{R_0}$, $\Psi(0) = 0$ and $D\Psi(0) = Id$. Set $\tilde{u}_n(y) = u_n(\Psi(y))$ for $y = (y_1, y_2) \in H \cap B_{R_0}$. Then by the conformality of Ψ, $\tilde{u}_n$ satisfies

$$\begin{cases} -\Delta \tilde{u}_n + b(y)\tilde{u}_n = 0 & \text{in } H \cap B_{R_0}, \\ \frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} = h(y)\tilde{u}_n^{p_n} & \text{on } \partial H \cap B_{R_0}, \end{cases} \tag{4.5}$$

where $\tilde{\nu}$ is the unit outer normal vector to $\partial(H \cap B_{R_0})$, b and h are defined

$$b(y) = |\det D\Psi(y)|, \quad h(y) = |D\Psi(y)e|$$

with $e = (0, -1)$. Note that $\tilde{\nu}(y) = \nu(\Psi(y))$ for $y \in \partial H \cap B_{R_0}$. Note also that, by using a clever idea of [6], we can modify Ψ to prescribe the number

$$\alpha = \frac{\left(\frac{\partial h}{\partial y_1}\right)}{h(y)^2}\Big|_{y=0} = \left(\frac{\partial h}{\partial y_1}\right)(0).$$

Let $D \subset \mathbb{R}^N$ be a bounded domain and recall the Pohozaev identity for the equation $-\Delta u = f(y, u), y \in D$:

$$\begin{aligned} &N \int_D F(y,u)dy - \left(\frac{N-2}{2}\right) \int_D |\nabla u|^2 dy + \int_D (y - y_0, \nabla_y F(y,u))\, dy \\ &= \int_{\partial D} (y - y_0, \nu) F(y,u) ds_y + \int_{\partial D} (y - y_0, \nabla u) \left(\frac{\partial u}{\partial \nu}\right) ds_y \\ &- \frac{1}{2} \int_{\partial D} (y - y_0, \nu) |\nabla u|^2 ds_y \end{aligned}$$

for any $y_0 \in \mathbb{R}^N$, where u is a smooth solution. Applying this to (4.5) for $N = 2$, $D = H \cap B_R$ for $0 < R < R_0$, $f(y, \tilde{u}_n) = -b(y)\tilde{u}_n$ and $F(y, \tilde{u}_n) = -\frac{b(y)}{2}\tilde{u}_n^2$, we obtain

$$\begin{aligned} &\int_{H\cap B_R} b(y)\tilde{u}_n^2(y)dy + \int_{H\cap B_R} (y - y_0, \nabla b(y))\frac{1}{2}\tilde{u}_n^2(y)dy \\ &= \int_{\partial(H\cap B_R)} (y - y_0, \tilde{\nu})\frac{1}{2}b(y)\tilde{u}_n^2(y)ds_y - \int_{\partial(H\cap B_R)} (y - y_0, \nabla \tilde{u}_n(y)) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}}\right) ds_y \\ &+ \frac{1}{2} \int_{\partial(H\cap B_R)} (y - y_0, \tilde{\nu}) |\nabla \tilde{u}_n|^2 ds_y, \end{aligned}$$

where and from now on, $\tilde{\nu}$ will be used again to denote the unit normal to $\partial(H \cap B_R)$. Differentiating with respect to y_0, we have, in turn,

$$\begin{aligned}&\int_{\partial(H\cap B_R)} \nabla \tilde{u}_n(y) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y \\ &= \frac{1}{2} \int_{\partial(H\cap B_R)} \left(|\nabla \tilde{u}_n|^2 + b(y)\tilde{u}_n^2 \right) \tilde{\nu} ds_y - \frac{1}{2} \int_{H\cap B_R} \nabla b(y) \tilde{u}_n^2(y) dy.\end{aligned}$$

Since $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) = (0, -1)$ on $\partial H \cap B_R$, the first component of the above vector equation reads

$$\begin{aligned}&\int_{\partial H\cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y + \int_{H\cap \partial B_R} (\tilde{u}_n)_{y_1}(y) \left(\frac{\partial \tilde{u}_n}{\partial \tilde{\nu}} \right) ds_y \\ &= \frac{1}{2} \int_{H\cap \partial B_R} \left(|\nabla \tilde{u}_n|^2 + b(y)\tilde{u}_n^2 \right) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H\cap B_R} b_{y_1}(y) \tilde{u}_n^2(y) dy,\end{aligned} \tag{4.6}$$

where $(\quad)_{y_1}$ denotes the derivative with respect to y_1. Let $\gamma_n = \int_{\partial\Omega} u_n^{p_n} ds_x$. From the fact that $\tilde{f}_n(y) = \frac{\tilde{u}_n^{p_n}}{\gamma_n} \overset{*}{\rightharpoonup} \delta_0$ in the sense of Radon measures on $\partial H \cap B_R$, Corollary 8 and $\|\tilde{u}_n\|_{L^\infty(\partial H \cap B_R)} = O(1)$ uniformly in n, we see

$$\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n + 1} = \frac{1}{(p_n+1)\gamma_n} \tilde{f}_n(y) \tilde{u}_n(y)$$

satisfies that $supp(\tilde{g}_n) \to \{0\}$ and $\int_{\partial H\cap B_R} \tilde{g}_n ds_y = O(1)$ as $n \to \infty$. Thus, by choosing a subsequence, we have the convergence

$$\tilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}(y)}{p_n + 1} \overset{*}{\rightharpoonup} C_0 \delta_0$$

in the sense of Radon measures on $\partial H \cap B_R$, where $C_0 = \lim_{n\to\infty} \int_{\partial H\cap B_R} \tilde{g}_n ds_y$ (up to a subsequence). By using this fact, we have

$$\begin{aligned}&\frac{1}{\gamma_n^2} \int_{\partial H\cap B_R} (\tilde{u}_n)_{y_1} h(y) \tilde{u}_n^{p_n}(y) ds_y \\ &= \left[\frac{h(y)}{\gamma_n^2} \frac{\tilde{u}_n^{p_n+1}}{p_n+1} \right]_{y_1=-R}^{y_1=R} - \int_{\partial H\cap B_R} h_{y_1}(y) \frac{\tilde{u}_n^{p_n+1}(y)}{(p_n+1)\gamma_n^2} ds_y \\ &\to 0 - C_0 h_{y_1}(0) = -C_0 \alpha\end{aligned}$$

as $n \to \infty$. Thus after dividing (4.6) by γ_n^2 and then letting $n \to \infty$, we obtain

$$
\begin{aligned}
&- C_0\alpha + \int_{H\cap\partial B_R} \tilde{G}_{y_1}(y) \left(\frac{\partial \tilde{G}}{\partial \tilde{\nu}} \right) ds_y \qquad (4.7) \\
&= \frac{1}{2} \int_{H\cap\partial B_R} \left(|\nabla \tilde{G}|^2 + b(y)\tilde{G}^2 \right) \tilde{\nu}_1 ds_y - \frac{1}{2} \int_{H\cap B_R} b_{y_1}(y)\tilde{G}^2(y)dy,
\end{aligned}
$$

where $\tilde{G}(y) = G(\Psi(y), 0)$ is a limit function of $\tilde{v}_n(y) = v_n(\Psi(y)) = \frac{\tilde{u}_n(y)}{\gamma_n}$. At this point, we have the same formula as the equation (117) in [6], thus we obtain the result. Indeed, decompose $G(x, 0) = s(x) + w(x)$ where

$$
s(x) = \frac{1}{\pi} \log |x|^{-1}, \quad w(x) = H(x, 0),
$$

and put $\tilde{s}(y) = s(\Psi(y))$, $\tilde{w}(y) = H(\Psi(y), 0)$ so that $\tilde{G} = \tilde{s} + \tilde{w}$. Then after some computation using the fact that $\tilde{w}$ satisfies

$$
-\Delta \tilde{w} + b(y)\tilde{w} = -b(y)\tilde{s}(y) \quad \text{in } H \cap B_R,
$$

we have from (4.7) that

$$
\begin{aligned}
&- C_0\alpha + \int_{H\cap\partial B_R} (\tilde{s}_{\tilde{\nu}}\tilde{s}_{y_1} + \tilde{s}_{\tilde{\nu}}\tilde{w}_{y_1} + \tilde{s}_{y_1}\tilde{w}_{\tilde{\nu}})\, ds_y \\
&= \int_{H\cap\partial B_R} \left(\frac{1}{2}|\nabla \tilde{s}|^2 + \nabla\tilde{s}\cdot\nabla\tilde{w} \right) \tilde{\nu}_1 ds_y + \int_{H\cap\partial B_R} \left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w} \right) b(y)\tilde{\nu}_1 ds_y \\
&- \int_{\partial H\cap B_R} b_{y_1}(y) \left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w} \right) ds_y + \int_{\partial H\cap B_R} \tilde{w}_{\tilde{\nu}}\tilde{w}_{y_1} ds_y \\
&- \int_{H\cap B_R} b(y)\tilde{s}(y)\tilde{w}_{y_1} dy. \qquad (4.8)
\end{aligned}
$$

By Lemma 9.3 in [6], we know estimates

$$
\begin{aligned}
&\lim_{R\to 0} \int_{H\cap\partial B_R} \tilde{s}_{\tilde{\nu}}\tilde{s}_{y_1} ds_x = \frac{3\alpha}{4\pi}, \quad \lim_{R\to 0} \int_{H\cap\partial B_R} \tilde{s}_{\tilde{\nu}}\tilde{w}_{y_1} ds_x = -\tilde{w}_{y_1}(0), \\
&\lim_{R\to 0} \frac{1}{2} \int_{H\cap\partial B_R} |\nabla \tilde{s}|^2 \tilde{\nu}_1 ds_x = \frac{\alpha}{4\pi}, \quad \lim_{R\to 0} \int_{H\cap\partial B_R} \nabla\tilde{s}\cdot\nabla\tilde{w}\tilde{\nu}_1 ds_x = -\frac{1}{2}\tilde{w}_{y_1}(0)
\end{aligned}
$$

and other terms in (4.8) go to 0 as $R \to 0$. Thus we take the limit in (4.8) as $R \to 0$ to obtain the relation

$$-C_0\alpha + \frac{3\alpha}{4\pi} - \tilde{w}_{y_1}(0) = \frac{\alpha}{4\pi} - \frac{1}{2}\tilde{w}_{y_1}(0),$$

which leads to

$$\alpha\left(\frac{1}{2\pi} - C_0\right) = \frac{1}{2}\tilde{w}_{y_1}(0).$$

Since $\alpha \in \mathbb{R}$ can be chosen arbitrary, we conclude that $C_0 = \frac{1}{2\pi}$ and $\tilde{w}_{y_1}(0) = 0$. This last equation means the desired conclusion of Theorem 2 (3). □

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