GEOMETRY OF CERTAIN LAGRANGIAN SUBMANIFOLDS IN HERMITIAN SYMMETRIC SPACES

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ABSTRACT. In this paper we discuss a certain method to construct a family of Lagrangian submanifolds in a complex projective space $\mathbb{C}P^{n+1}$ from a Lagrangian submanifold in a complex hyperquadric $Q_n(\mathbb{C})$. We use a family of homogeneous isoparametric hypersurfaces in the unit standard sphere associated to the Riemannian symmetric pair $(SO(n+4), SO(2) \times SO(n+2))$ of rank 2.

INTRODUCTION

It is an interesting problem to study the geometric construction of Lagrangian submanifolds in Hermitian symmetric spaces from the viewpoint of Lie theory. We shall give attention to a relationship among Lagrangian submanifolds in different Hermitian symmetric spaces of compact type.

In this paper we discuss a certain method to construct a family of Lagrangian submanifolds in a complex projective space $\mathbb{C}P^{n+1}$ from a Lagrangian submanifold in a complex hyperquadric $Q_n(\mathbb{C})$. We use a family of homogeneous isoparametric hypersurfaces in the unit standard sphere $S^{2n+3}(1) \subset \mathbb{C}^{n+2}$ associated to the Riemannian symmetric pair $(SO(n+4), SO(2) \times SO(n+2))$ of rank 2. It induces a family of homogeneous isoparametric real hypersurfaces in a complex projective space $\mathbb{C}P^{n+1}$ whose focal manifolds are $Q_n(\mathbb{C})$ and $\mathbb{R}P^{n+1}$. Such a homogeneous real hypsersurface of a complex projective space is a *tube* over $Q_n(\mathbb{C})$ and $\mathbb{R}P^{n+1}$ and is called *of type B* by R. Takagi ([20, 21]). For each $t \in (-1, 1)$, we have the positive focal map $\nu_+ : M_t \to Q_n(\mathbb{C})$ and the negative focal map $\nu_- : M_t \to \mathbb{R}P^{n+1}$. By using the positive focal map $\nu_+ : M_t \to Q_n(\mathbb{C})$, we shall show that *for any Lagrangian submanifold L in $Q_n(\mathbb{C})$ we have a family of Lagrangian submanifolds $\hat{L}^{n+1} = \nu_+^{-1}(L) \, (\subset M_t)$ in $\mathbb{C}P^{n+1}$ for $t \in (-1, 1)$*. We shall discuss their structure and properties in detail.

We shall also mention about known results on compact homogeneous Lagrangian submanifolds in complex projective spaces and the Gauss map construction of Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbb{C})$ from

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isoparametric hypersurfaces in $S^{n+1}(1)$. By applying our result to nice Lagrangian submanifolds in complex hyperquadrics, we can obtain many interesting examples of Lagrangian submanifolds in complex projective space. We expect that our result provides a link from Lagrangian submanifold theory in complex hyperquadrics to Lagrangian submanifold thery in complex projective spaces.

1. Lagrangian submanifolds in Kähler manifolds

A smooth immersion $\varphi : L \to P$ of a smooth manifold L into a $2n$-dimensional symplectic manifold (P, ω) is called a *Lagrangian immersion* if $\varphi^*\omega = 0$ and $\dim L = n$. For each $v \in (\varphi^{-1}TP)_x = T_{\varphi(x)}P$, we define $\alpha_v \in T_x^*L$ by $\alpha_v(X) := \omega_{\varphi(x)}(v, (d\varphi)_x(X))$ for each $X \in T_xL$. If φ is a Lagrangian immersion, then we have the canonical isomorphisms $\varphi^{-1}TP/\varphi_*TL \cong T^*L$ and $C^\infty(\varphi^{-1}TP/\varphi_*TL) \cong \Omega^1(L)$. A smooth family of smooth immersions $\varphi_t : L \to P$ with $\varphi_0 = \varphi$ is called a *Lagrangian deformation* if $\varphi_t : L \to P$ is a Lagrangian immersion for each t. A Lagrangian deformation $\varphi_t : L \to P$ is characterized by the condition that α_{V_t} is closed for each t, where $V_t = \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1}TP)$ is the variational vector field of $\{\varphi_t\}$. If α_{V_t} is exact for each t, then we call $\{\varphi_t\}$ a *Hamiltonian deformation* of φ.

Suppose that (P, ω, g, J) is a Kähler manifold with a Kähler metric g and the complex structure J. Let $\varphi : L \to P$ be a Lagrangian immersion. Denote by B the second fundamental form of φ. We define a symmetric tensor field S on L by

$$S(X, Y, Z) := \omega(B(X, Y), Z) = g(JB(X, Y), Z)$$

for each $X, Y, Z \in TL$. The mean curvature vector field H of φ is defined by

$$H := \sum_{i=1}^{n} B(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis of T_xL. The 1-form α_H on L corresponding to the mean curvature vector field H is called the *mean curvature form* of φ, which satisfies the identity ([5])

(1.1) $$d\alpha_H = \varphi^*\rho_P,$$

where ρ_P denotes the Ricci form of a Käher manifold P.

The notion of Hamiltonian minimality for Lagrangian submanifolds in a Käher manifold P was introduced and investigated first by Y. G. Oh ([12]). A Lagrangian immersion $\varphi : L \to P$ into a Käher manifold P is called *Hamiltonian minimal* if for each compactly supported Hamiltonian deformation $\{\varphi_t\}$ of $\varphi_0 = \varphi$ the first variation of the volume vanishes:

$$\frac{d}{dt}\mathrm{Vol}(L, \varphi^*g)|_{t=0} = 0\,.$$

The Hamiltonian minimal Lagrangian submanifold equation is

(1.2) $$\delta\alpha_H = 0.$$

The usual minimal submanifold equation $H = 0$ is equivalent to $\delta\alpha_H = 0$ and in this case we call φ a *minimal Lagrangian immersion*. A Hamiltonian Lagrangian immersion $\varphi : L \to P$ into a Käher manifold P is called *Hamiltonian stable* if for each compactly supported Hamiltonian deformation $\{\varphi_t\}$ of $\varphi_0 = \varphi$ the second variation of the volume is nonnegative:

$$\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi^*g)|_{t=0} \geq 0\,.$$

The Lagrangian version of the second variational formula was given in [12].

By considering the second variation of the volume for Hamiltonian minimal Lagrangian submanifolds under Hamiltonian deformations, the notions of Hamiltonian stability and strict Hamiltonian stability are defined (cf. [12], [8]).

Let G be a connected Lie group acting on P preserving symplectic form ω. A Lagrangian submanifold L embedded in a symplectic manifold (P, ω) obtained as an orbit of G is called a *Lagrangian orbit* of G.

Assume that (P, ω, J, g) is a Kähler manifold and the action of G on P preserves the Kähler structure (ω, g, J), or G is a connected Lie subgroup of the automorphism group $\mathrm{Aut}(P, \omega, J, g)$ of the Kähler structure (ω, g, J). If a Lagrangian submanifold L embedded in P is obtained as an orbit of G, then L is called a *homogeneous Lagrangian submanifold* in a Kähler manifold P. Moreover if L is compact without boundary, then L is said to be a *compact homogeneous Lagrangian submanifold* in a Kähler manifold P. In this case, as $\delta\alpha_H$ is a constant function on L, by the divergence theorem we have $\delta\alpha_H = 0$. Hence we know

Proposition 1.1 ([8]). *Any compact homogeneous Lagrangian submanifold in a Käher manifold is always Hamiltonian minimal.*

Suppose that (P, ω, J, g) is a compact Kähler manifold with $\dim_{\mathbb{C}} H^{1,1}(P, \mathbb{C}) = 1$ and G is a compact connected Lie subgroup of the automorphism group $\mathrm{Aut}(P, \omega, J, g)$. Let $G^{\mathbb{C}} \subset \mathrm{Aut}(P, J)$ be the complexified Lie group of G. Then the following is a fundamental fact on the existence of compact homogeneous Lagrangian submanifolds in Kähler manifolds.

Theorem ([4]). *If $G \cdot x$ is a Lagrangian orbit of G, then the complexified orbit $G^{\mathbb{C}} \cdot x$ is Stein and Zariski open in P. Conversely if the complexified orbit $G^{\mathbb{C}} \cdot x$ is Stein and Zariski open in P, then $G^{\mathbb{C}} \cdot x$ contains a Lagrangian orbit of G.*

2. Lagrangian submanifolds in complex projective spaces

Let $\mathbb{C}^{n+1}$ be the complex Euclidean space and $S^{2n+1}(1)$ be the unit standard hypersphere of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Let $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ be the standard Hopf fibration over the n-dimensional complex projective space $\mathbb{C}P^n$ equipped with the standard Fubini-Study metric of constant holomorphic sectional curvature 4. Note that the standard S^1-action on $S^{2n+1}(1)$ preserves the induced metrics.

and $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ is a Riemannian submersion. The energy function $f : \mathbb{C}^{n+1} \to \mathbb{R}$ defined by $f(\mathbf{x}) := \|\mathbf{x}\|^2$ ($\mathbf{x} \in \mathbb{C}^{n+1}$) can be considered as a Hamiltonian function or a moment map with respect to the standard S^1-action on $\mathbb{C}^{n+1}$. Since $f^1 = S^{2n+1}(1)$ and $f^1/S^1 = \mathbb{C}P^n$, the complex projective space $\mathbb{C}P^n$ is a symplectic Kähler quotient of $\mathbb{C}^{n+1}$ by the moment map f. Then we know the following properties (cf. [6]):

Proposition 2.1. *Let L be a submanifold (immersed) in $\mathbb{C}P^n$. Then*

(1) *L is a Lagrangian submanifold (immersed) in $\mathbb{C}P^n$ if and only if the inverse image $\tilde{L} = \pi^{-1}(L)(\subset S^{2n+1}(1))$ by the Hopf fibration π is a Lagrangian submanifold (immersed) in $\mathbb{C}^{n+1}$.*

(2) *L is a Hamiltonian minimal in $\mathbb{C}P^n$ if and only if $\tilde{L} = \pi^{-1}(L)(\subset S^{2n+1}(1))$ is Hamiltonian minimal in $\mathbb{C}^{n+1}$.*

$$
\begin{array}{ccc}
\tilde{L}^{n+1} = \pi^{-1}(L) & \subset & S^{2n+1}(1) = f^{-1}(1) \subset \mathbb{C}^{n+1} \\
\pi \downarrow S^1 & & \pi \downarrow S^1 \\
L^n & \subset & \mathbb{C}P^n = f^{-1}(1)/S^1
\end{array}
$$

We mention about known results on the construction and classification of compact homogeneous Lagrangian submanifolds in complex projective spaces. In the case when L has parallel second fundamental form, that is, $\nabla S = 0$, $\pi^{-1}(L)$ also has parallel second fundamental form in both $S^{2n+1}(1)$ and $\mathbb{C}^{n+1}$. Such Lagrangian submanifolds with $\nabla S = 0$ in complex space forms were completely classified by Professors Hiroo Naitoh and Mararu Takeuchi in 1980's by theory of symmetric R-spaces (of type $U(r)$) and in particular they showed that any Lagrangian submanifolds with $\nabla S = 0$ in a complex space form is homogeneous. Compact Lagrangian submanifolds with $\nabla S = 0$ embedded in complex projective spaces and complex Euclidean spaces are strictly Hamiltonian stable (Amarzaya-Ohnita [1], [2]).

There are many examples of compact homogeneous Lagrangian submanifolds with $\nabla S \neq 0$ embedded in complex projective spaces and complex Euclidean spaces. The simplest non-trivial example can be given by a 3-dimensional minimal Lagrangian orbit in $\mathbb{C}P^2$ under an irreducible unitary representation of $SU(2)$ of degree 3 (cf. [3], [13]).

Bedulli-Gori ([4]) classified compact homogeneous Lagrangian submanifolds in complex projective spaces obtained as orbits of simple compact Lie groups of $PU(n+1)$. It is a crucial to use Theorem and theory of prehomogeneous vector spaces due to Professors Mikio Sato and Tatsuo Kimura. It is still an open problem to classify compact homogeneous Lagrangian submanifolds in $\mathbb{C}P^n$ obtained as orbits of *non-simple* compact Lie groups of $PU(n+1)$ (See also [19]).

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3. Lagrangian submanifolds in complex hyperquadrics and hypersurface geometry in spheres

Let N^n be an oriented hypersurface immersed in the unit standard sphere $S^{n+1}(1)$. Let $\mathbf{x}(p)$ denote the position vector of a point $p \in N^n$ and $\mathbf{n}(p)$ denote the unit normal vector at $p \in N^n$ in $S^{n+1}(1)$. The *Gauss map* $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ of the oriented hypersurface $N^n \subset S^{n+1}(1)$ is defined by

$$(3.1) \quad \mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}).$$

It is well-known that the Gauss map $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ is always a Lagrangian immersion.

We shall assume that N^n is an oriented *isoparametric* hypersurface immersed in the unit standard sphere $S^{n+1}(1)$, that is, a hypersurface with g distinct constant principal curvatures.

In general, an *isoparametric hypersurface* on a Riemannian manifold M is defined as a regular level hypersurface of an *isoparametric function*. A smooth function f is called an *isoparametric function* on a Riemannian manifold M if f satisfies the system of partial differential equations:

$$(3.2) \quad \Delta^M f = a(f), \quad \|\mathrm{grad} f\|^2 = b(f),$$

where a is a continuous function and b is a function of class C^2. It is well-known that each regular level hypersurface of an isoparametric function has constant mean curvature. Elie Cartan showed that a hypersurface immersed in real space forms is an isoparametric hypersurface if and only if it has constant principal curvatures. Let N^n be an isoparametric hypersurface in the standard hypersphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$ with g distinct principal curvatures $k_1 > k_2 > \cdots > k_g$ and the corresponding multiplicities m_α ($\alpha = 1, \cdots, g$). We express the principal curvatures $k_1 > k_2 > \cdots > k_g$ as $k_\alpha = \cot \theta_\alpha$ ($\alpha = 1, \cdots, g$) with $\theta_1 < \cdots < \theta_g$. Münzner ([10]) showed that $\theta_\alpha = \theta_1 + (\alpha - 1)\frac{\pi}{g}$ and $m_\alpha = m_{\alpha+2}$ indexed modulo g. Moreover he proved that N^n extends to a regular level hypersurface $f^{-1}(t)$ for some $t \in (-1, 1)$ defined by an isoparametric function $f : S^{n+1}(1) \to [-1, 1]$ which is a restriction of a homogeneous polynomial F of degree g satisfying

$$(3.3) \quad \Delta^{\mathbb{R}^{n+2}} F = c\,r^{g-2}, \quad \|\mathrm{grad}^{\mathbb{R}^{n+2}} F\|^2 = g^2\,r^{2g-2},$$

where $r = \sum_{i=1}^{n+2}(x_i)^2$ and $c = g^2(m_2 - m_1)/2$. Here the isoparametric function f is defined in a neighborhood of N^n as

$$(3.4) \quad t = f(q) = \cos(g\theta(q)) = \cos(g\theta_1),$$

where $\theta(q) := \theta_1$ for the maximal principal curvature $k_1 = \cot \theta_1$ of $f^{-1}(q)$. Such a homogeneous polynomial is called the *Cartan-Münzner polynomial*. His famous result is that g must be $1, 2, 3, 4$ or 6 ([11]) .

Let $k_1 > \cdots > k_g$ denote its distinct constant principal curvatures and $m_1, \cdots, m_g$ the corresponding multiplicities. Münzner ([10]) showed that $\theta_\alpha =$

$\theta_1 + (\alpha - 1)\frac{\pi}{g}$ and $m_\alpha = m_{\alpha+2}$ indexed modulo g. Hence we have

$$(3.5) \qquad \frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \geq 2, \\ 2m_1 & \text{if } g = 1. \end{cases}$$

The famous result of Münzner is that g must be $1, 2, 3, 4$ or 6 ([10], [11]).

The Lagrangian immersions $\mathcal{G}$ and the Gauss image $\mathcal{G}(N^n)$ have the following properties. It follows from the mean curvature form formula of Palmer ([18]) that

Proposition 3.1 ([18]). *The Gauss map* $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ *is a minimal Lagrangian immersion.*

Proposition 3.2 ([7], [14]). *The Gauss image* $\mathcal{G}(N^n)$ *is a compact smooth minimal Lagrangian submanifold embedded in* $Q_n(\mathbb{C})$ *and the Gauss map* $\mathcal{G} : N^n \to \mathcal{G}(N^n)$ *is a covering map with the deck transformation group* $\mathbb{Z}_g$. *Moreover,* $\mathcal{G}(N^n)$ *is a monotone and cyclic Lagrangian submanifold in* $Q_n(\mathbb{C})$ *with minimal Maslov number* $\frac{2n}{g}$. *Here* $\frac{2n}{g}$ *is even if and only if* $\mathcal{G}(N^n)$ *is orientable.*

In [7] we classified explicitly all compact homogeneous Lagrangian submanifolds in $Q_n(\mathbb{C})$. In particular we showed that all compact homogeneous *minimal* Lagrangian submanifolds in $Q_n(\mathbb{C})$ are obtained as the Guass images of homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$.

Palmer ([18]) showed that $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ is Hamiltonian stable if and only if $g = 1$, that is, N^n is a great or small sphere of $S^{n+1}(1)$. By my joint works [7], [9] with Hui Ma, we completely determined the strict Hamiltonian stability of the Gauss images $\mathcal{G}(N^n)$ of all homogeneous isoparametric hypersurfaces N^n in $S^{n+1}(1)$. Our main result was that the Gauss image $\mathcal{G}(N^n)$ of homogeneous isoparametric hypersurfaces N^n is Hamiltonian stable if and only if $|m_1 - m_2| \leq 2$ or N^n is a principal orbit of the isotropy representation of the Riemannian symmetric pair of type EIII (in this case $(m_1, m_2) = (6, 9)$).

4. Construction of a family of Lagrangian submanifolds in $\mathbb{C}P^{n+1}$ from an arbitrary Lagrangian submanifold in $Q_n(\mathbb{C})$

First we recall the structure of *tubes* M_t ($t \in (-1, 1)$) over a complex hyperquadric $Q_n(\mathbb{C})$ $(= M_1 = M_+)$ in $\mathbb{C}P^{n+1}$. They are also *tubes* M_t ($t \in (-1, 1)$) over a real projective subspace $\mathbb{R}P^n$ $(= M_{-1} = M_-)$ in $\mathbb{C}P^{n+1}$ at the same time. The tube M_t is a real hypersurface embedded in $\mathbb{C}P^{n+1}$, which is a homogeneous isoparametric real hypersurface of $\mathbb{C}P^{n+1}$ and its focal manifolds are $Q_n(\mathbb{C})$ and $\mathbb{R}P^n$. Let ν denote the unit normal vector field to M_t in $\mathbb{C}P^{n+1}$. The focal maps from M_t to focal manifolds are defined by using geodesics of $\mathbb{C}P^{n+1}$ normal to M_t. Let $\nu_+ : M_t \to Q_n(\mathbb{C})$ denote the positive focal map from M_t onto $Q_n(\mathbb{C})$ and $\nu_- : M_t \to \mathbb{R}P^n$ the negative focal map from M_t onto $\mathbb{R}P^n$:

$$
\begin{array}{ccccc}
\mathbb{C}P^{n+1} & = & \mathbb{C}P^{n+1} & = & \mathbb{C}P^{n+1} \\
\cup & & \cup & & \cup \\
M_{-1} = \mathbb{R}P^{n+1} & \xleftarrow{\ \nu_- \ } & M_t & \xrightarrow{\ \nu_+ \ } & Q_n(\mathbb{C}) = M_1
\end{array}
$$

The homogeneous isoparametric real hypersurfaces M_t in $\mathbb{C}P^{n+1}$ are obtained as the projection of homogeneous isoparametric real hypersurfaces $\tilde{M}_t$ in $S^{2n+3}(1)$ by the Hopf fibration $\pi : S^{2n+3}(1) \to \mathbb{C}P^{n+1}$. Let $(\tilde{G}, \tilde{U}) = (SO(n+4), SO(2) \times SO(n+2))$ be a compact rank 2 Riemannian symmetric pair of type BDII_2. Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{u}} + \tilde{\mathfrak{p}}$ be the corresponding canonical decomposition of its symmetric Lie algebra, where $\tilde{\mathfrak{g}} = \mathfrak{o}(n+4)$, $\tilde{\mathfrak{u}} = \mathfrak{o}(2) \oplus \mathfrak{o}(n+2)$ and the vector subspace $\tilde{\mathfrak{p}}$ is

$$
\tilde{\mathfrak{p}} = \left\{ \begin{bmatrix} 0 & \xi \\ -\xi^t & 0 \end{bmatrix} \ \Big| \ \xi \in M(2, n+2; \mathbb{R}) \right\}.
$$

Here $M(2, n+2; \mathbb{R})$ denotes the vector space of all real $2 \times (n+2)$-matrices. The standard complex structure $\tilde{J}$ of $\tilde{\mathfrak{p}}$ invariant under the isotropy representation of $\tilde{U}$ is given by

$$
J\left(\begin{bmatrix} 0 & \xi \\ -\xi^t & 0 \end{bmatrix} \right) = \mathrm{Ad}_{\tilde{\mathfrak{p}}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & \xi \\ -\xi^t & 0 \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} 0 & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi \\ -\xi^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^t & 0 \end{bmatrix}.
$$

Relative to the standard complex structure, the vector subspace $\tilde{\mathfrak{p}}$ is identified with

$$
M(2, n+2; \mathbb{R}) \cong \mathbb{R}^{2n+4} \cong \mathbb{C}^{n+2}.
$$

In particular, the standard S^1-action on $\mathbb{C}^{n+2}$ coincides with the isotropy action of $SO(2) \times \{\mathrm{I}_{n+2}\} \subset SO(2) \times SO(n+2)$ on $\tilde{\mathfrak{p}}$.

Let $S^{2n+3}(1) \subset \tilde{\mathfrak{p}}$ denote the unit standard hypersphere of $\tilde{\mathfrak{p}}$. Each principal orbit of $SO(2) \times SO(n+2)$ on $S^{2n+3}(1)$ is of codimension 1. Thus we have a family $\{\tilde{M}_t\}$ of homogeneous hypersurfaces of $S^{2n+3}(1)$ with constant principal curvatures, which is a family of so-called *isoparametric hypersurfaces*. Each M_t has $g = 4$ distinct principal curvatures and multiplicities $(m_1, m_2) = (1, n)$.

Let $\tilde{f} : S^{2n+3}(1) \to [-1, 1] \subset \mathbb{R}$ be such an isoparametric function defining $\{\tilde{M}_t\}$ and $F : \mathbf{C}^{n+2} \to \mathbb{R}$ denote the corresponding Cartan-Münzner polynomial of degree 4 so that $F|_{S^{2n+3}(1)} = \tilde{f}$. Explicitly the Cartan-Münzner polynomial F is given as follows ([17]):

$$
F(Z) = \frac{3}{4} \left(\mathrm{Tr}(Z^2) \right)^2 - 2 \, \mathrm{Tr}(Z^4)
$$

for each $Z \in \tilde{\mathfrak{p}}$, in other words

$$
F = r^4 - 2 \, F_0 \,,
$$

where the functions r and F are defined as

$$r(\xi) := \left(\sum_{i=1}^{n+2} |\xi_i|^2 \right)^{\frac{1}{2}}, \quad F_0(\xi) := \left| \sum_{i=1}^{n+2} \xi_i^2 \right|^2 = \left(\sum_{i=1}^{n+2} \xi_i^2 \right) \overline{\left(\sum_{i=1}^{n+2} \xi_i^2 \right)}$$

for each $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{n+2} \end{bmatrix} \in \mathbb{C}^{n+2}$.

Set $\tilde{M}_t := \tilde{f}^{-1}(t)$ for each $t \in [-1, 1]$ and $\tilde{M}_\pm := \tilde{M}_{\pm 1}$. Then for each $t \in (-1, 1)$ the level regular hypersurface $\tilde{M}_t$ is a compact homogeneous isoparametric hypersurface embedded in $S^{2n+3}(1)$ which is diffeomorphic to a compact homogeneous space $\dfrac{SO(2) \times SO(n+2)}{\mathbb{Z}_2 \times SO(n)}$. For $t = \pm 1$, $\tilde{M}_\pm$ are focal manifolds which are compact minimal submanifolds embedded in $S^{2n+3}(1)$. A focal manifold $\tilde{M}_+$ is diffeomorphic to a compact homogeneous space $\dfrac{SO(2) \times SO(n+2)}{SO(2) \times SO(n)}$ and another focal manifold $\tilde{M}_-$ is diffeomorphic to a compact homogeneous space $\dfrac{SO(2) \times SO(n+2)}{O(n+1)}$ ($\cong S^1 \cdot S^{n+1} \cong Q_{2,n+2}(\mathbb{R})$ a real hyperquadric).

Let $t \in (-1, 1)$ and set $\tilde{M} = \tilde{M}_t$. Let $k_1 > k_2 > k_3 > k_4$ be four distinct principal curvatures of M in $S^{2n+3}(1)$. Recall that $k_1 = \cot \theta_1$ and $t = \cos(4\theta_1)$. Denote by $\mathbf{x}$ the position vector of points on $\tilde{M}$ and by $\mathbf{n} := \dfrac{\operatorname{grad} \tilde{f}}{\|\operatorname{grad} \tilde{f}\|}$ the unit normal vector to $\tilde{M}$ in $S^{2n+3}(1)$. Let $A_{\mathbf{n}}$ denotes the shape operator of $\tilde{M}$ in $S^{2n+3}(1)$ in the direction of $\mathbf{n}$. The principal curvatures are nothing but eigenvalues of $A_{\mathbf{n}}$. For each $p \in \tilde{M}$, we express the eigenspace decomposition of the tangent vector space $T_p \tilde{M}$ corresponding to the four constant principal curvatures as follows:

$$\begin{aligned} T_p \tilde{M} &= \tilde{V}_1(p) \oplus \tilde{V}_2(p) \oplus \tilde{V}_3(p) \oplus \tilde{V}_4(p) \\ &= \mathbb{R} \, J\mathbf{x}(p) \oplus \mathbb{R} \, J\mathbf{n}(p) \oplus \tilde{V}_2(p) \oplus J(\tilde{V}_2(p)), \end{aligned}$$

where $\tilde{V}_1(p) = \mathbb{R}(-k_3 J\mathbf{x}(p) + J\mathbf{n}(p))$, $\tilde{V}_3 = \mathbb{R}(-k_1 J\mathbf{x}(p) + J\mathbf{n}(p))$, $\tilde{V}_4(p) = J(\tilde{V}_2(p))$.

The positive focal map $\tilde{\nu}_+ : \tilde{M}_t \to \tilde{M}_+$ is defined by

$$\tilde{\nu}_+(p) := \cos \theta_1 \, \mathbf{x}(p) + \sin \theta_1 \, \mathbf{n}(p) \in \tilde{M}_+$$

for each $p \in \tilde{M}_t$. The differential of the positive focal map $\tilde{\nu}_+$ is given as

$$\begin{aligned} (d\tilde{\nu}_+)_p(X) &= \cos \theta_1 \, X - \sin \theta_1 \, A_{\mathbf{n}}(X) \\ &= (\cos \theta_1 \, I - \sin \theta_1 \, A_{\mathbf{n}})(X) \end{aligned}$$

for each $X \in T_p\tilde{M}$. If $X \in \tilde{V}_i(p)$ $(i = 1, 2, 3, 4)$, then we have

$$
\begin{aligned}
(d\tilde{\nu}_+)_p(X) &= (\cos\theta_1 - \sin\theta_1\, k_i)(X) \\
&= (\cos\theta_1 - \sin\theta_1\, \cot\theta_i)(X) \\
&= \frac{\sin(\theta_i - \theta_1)}{\sin\theta_i}\, X.
\end{aligned}
$$

Note that

$$
\frac{\sin(\theta_i - \theta_1)}{\sin\theta_i} =
\begin{cases}
0 & \text{if } \theta_i = \theta_1. \\[2mm]
\dfrac{1}{\sqrt{2}\,\sin\theta_2} = \dfrac{1}{\cos\theta_1 + \sin\theta_1} & \text{if } \theta_i = \theta_2. \\[2mm]
\dfrac{1}{\sin\theta_3} = \dfrac{1}{\cos\theta_1} & \text{if } \theta_i = \theta_3. \\[2mm]
\dfrac{1}{\sqrt{2}\,\sin\theta_4} = \dfrac{1}{\sqrt{2}\,\cos\theta_2} = \dfrac{1}{\cos\theta_1 - \sin\theta_1} & \text{if } \theta_i = \theta_4.
\end{cases}
$$

The positive focal map $\tilde{\nu}_+ : \tilde{M}_t \to \tilde{M}_+$ is an $SO(2) \times SO(n+2)$-equivariant submersion

$$
\begin{aligned}
\nu_+ : \tilde{M}_t = \frac{SO(2) \times SO(n+2)}{\mathbb{Z}_2 \times SO(n)} &\longrightarrow \tilde{M}_+ = \frac{SO(2) \times SO(n+2)}{SO(2) \times SO(n)} \\
&\cong \frac{SO(n+2)}{SO(n)} \cong V_2(\mathbb{R}^{n+2})
\end{aligned}
$$

with fiber $\dfrac{SO(2) \times SO(n)}{\mathbb{Z}_2 \times SO(n)} \cong \dfrac{SO(2)}{\mathbb{Z}_2} \cong S^1$. Here $V_2(\mathbb{R}^{n+2})$ denotes the Stiefel manifold of all pairs of two orthonormal vectors in $\mathbb{R}^{n+2}$. Note that this map $\tilde{\nu}_+ : \tilde{M}_t \to \tilde{M}_+$ is *not* a Riemannian submersion.

Let $\pi : S^{2n+3}(1) \to \mathbb{C}P^{n+1}$ be the Hopf fibration with the $S^1 \cong SO(2)$-action. Since the isoparametric function $\tilde{f}$ on $S^{2n+3}(1)$ is invariant under the $S^1 \cong SO(2)$-action, a smooth function $f : \mathbb{C}P^{n+1} \to [-1, 1] \subset \mathbb{R}$ can be induced by $f \circ \pi = \tilde{f}$. Then f is an isoparametric function on $\mathbb{C}P^{n+1}$.

Set $M_t := f^{-1}(t)$ for each $t \in [-1, 1]$ and $M_\pm := M_{\pm 1}$. Then for each $t \in (-1, 1)$ the subset M_t is the level regular hypersurface, and is a compact homogeneous isoparametric hypersurface embedded in $\mathbb{C}P^{n+1}$ which is diffeomorphic to $\dfrac{SO(n+2)}{\mathbb{Z}_2 \times SO(n)}$. The subsets $M_\pm$ are focal manifolds which are compact minimal submanifolds embedded in $\mathbb{C}P^{n+1}$. The subset M_+ is a complex hyperquadric $Q_n(\mathbb{C})$ in $\mathbb{C}P^{n+1}$ which is diffeomorphic to $\dfrac{SO(n+2)}{SO(2) \times SO(n)}$ and the subset M_- is a real projective subspace $\mathbb{R}P^{n+1}$ in $\mathbb{C}P^{n+1}$ which is diffeomorphic to $\dfrac{SO(n+2)}{SO(1) \times SO(n+1)}$.

Each M_t $(t \in (-1, 1))$ is a homogeneous isoparametric real hypersurface in $\mathbb{C}P^{n+1}$ with *three* distinct constant principal curvatures

(4.1) $$\ell_1 = -2\tan 2\theta_1, \quad \ell_2 = \cot\theta_1, \quad \ell_3 = -\tan\theta_1,$$

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and the corresponding multiplicities

$$(4.2) \qquad m(\ell_1) = 1, \ m(\ell_2) = n, \ m(\ell_3) = n.$$

It is called a *homogeneous real hypersurface of type B* ([20, 21]). The vector field $\nu := \dfrac{\mathrm{grad} f}{\|\mathrm{grad} f\|}$ is a unit normal vector field on each M_t $(t \in (-1,1))$ and A_ν denotes the shape operator of M_t in the direction of ν. For each $x \in M_t$, we express the eigenspace decomposition of $T_x M_t$ corresponding to three principal curvatures ℓ_1, ℓ_2, ℓ_3 as follows:

$$(4.3) \qquad T_x M_t = V_1(x) \oplus V_2(x) \oplus V_3(x)$$

where $V_1(x) = \mathbb{R}\, J\nu(x)$, $V_2(x) = J(V_3(x))$.

Now we have a commutative diagram of positive focal maps and Hopf fibrations as follows:

$$
\begin{array}{ccc}
\mathbb{C}^{n+2} & = & \mathbb{C}^{n+2} \\
\cup & & \cup \\
S^{2n+3}(1) & = & S^{2n+3}(1) \\
\cup & & \cup \\
\tilde{M}_t & \xrightarrow{\ \tilde{\nu}_+\ } & \tilde{M}_+ \\
\pi \downarrow S^1 & & \pi \downarrow S^1 \\
M_t & \xrightarrow{\ \nu_+\ } & M_+ \\
\cap & & \cap \\
\mathbb{C}P^{n+1} & = & \mathbb{C}P^{n+1}
\end{array}
$$

Here

$$\tilde{M}_t = \pi^{-1}(M_t) \cong \frac{SO(2) \times SO(n+2)}{\mathbb{Z}_2 \times SO(n)} \quad (t \in (-1,1)),$$

$$\tilde{M}_+ = \pi^{-1}(M_+) \cong \frac{SO(2) \times SO(n+2)}{SO(2) \times SO(n)} \cong \frac{SO(n+2)}{SO(n)} \cong V_2(\mathbb{R}^{n+2}),$$

$$\tilde{M}_- = \pi^{-1}(M_-) \cong \frac{SO(2) \times SO(n+2)}{O(n+1)} \cong S^1 \cdot S^{n+1} \cong Q_{2,n+2}(\mathbb{R}),$$

and

$$M_t = \pi(\tilde{M}_t) \cong \frac{SO(n+2)}{\mathbb{Z}_2 \times SO(n)} \quad (t \in (-1,1)),$$

$$M_+ = \pi(\tilde{M}_+) \cong \frac{SO(n+2)}{SO(2) \times SO(n)} \cong Q_n(\mathbb{C}),$$

$$M_- = \pi(\tilde{M}_-) \cong \frac{SO(n+2)}{S(O(1) \times O(n+1))} \cong \mathbb{R}P^{n+1}.$$

Under a surjective linear map $(d\pi)_p : T_p\tilde{M}_t \to T_p M_t$, we have

$$\mathrm{Ker}((d\pi)_p) = \mathbb{R}\, J\mathbf{x}(p) \subset \tilde{V}_1(p) \oplus \tilde{V}_3(p) = \mathbb{R}\, J\mathbf{x}(p) \oplus \mathbb{R}\, J\mathbf{n}(p),$$

$$(d\pi)_p(\tilde{V}_1(p) \oplus \tilde{V}_3(p)) = V_1(p),$$

$$(d\pi)_p(\tilde{V}_2(p)) = V_2(p), \quad (d\pi)_p(\tilde{V}_4(p)) = V_3(p).$$

The positive focal map $\nu_+ : M_t \to M_+$ is an $SO(n+2)$-equivariant submersion

$$\nu_+ : M_t \cong \frac{SO(n+2)}{\mathbb{Z}_2 \times SO(n)} \longrightarrow M_+ = Q_n(\mathbb{C}) \cong \frac{SO(n+2)}{SO(2) \times SO(n)}.$$

Note that It is not a Riemannian submersion and the Reeb flow of a real hypersurface M_t as an almost contact metric manifold is not isometric.

We observe that the differential of the positive focal map ν_+ has kernel $\mathrm{Ker}(d\nu_+) = V_1$. Let $\omega_{Q_n(\mathbb{C})}$ denote the Kähler form of $Q_n(\mathbb{C}) \subset \mathbb{C}P^{n+1}$. Then a relation between the pull-back form of $\omega_{Q_n(\mathbb{C})}$ by ν_+ and the restriction of the Kähler form ω of $\mathbb{C}P^{n+1}$ to M_t is given as follows:

Lemma 4.1.

(4.4) $$\nu_+^*\omega_{Q_n(\mathbb{C})} = \frac{1}{\cos(2\theta_1)}\, \omega|_{M_t}.$$

Here note that $\cos(2\theta_1) \neq 0$.

The vector field $J(\mathrm{grad} f)$ on $\mathbb{C}P^{n+1}$ satisfies

$$\omega(J(\mathrm{grad} f), \cdot) = g(\mathrm{grad} f, \cdot) = -df,$$

and hence it is a Hamiltonian vector field X_{-f} on $\mathbb{C}P^{n+1}$ corresponding to the Hamiltonian $-f$. For each $t \in (-1, 1)$, the flow of a Hamiltonian vector field X_{-f} preserves the level set $f^{-1}(t) = M_t$ and $X_{-f}|_{M_t} = J(\mathrm{grad} f) = \|\mathrm{grad} f\|\, J\nu$ belongs to $V_1 \subset TM_t$. Note that $\|\mathrm{grad} f\|$ is constant on M_t. Hence we see that the vector field $X_{-f}|_{M_t}$ generates the S^1-action on M_t and the orbits of the S^1-action coincide with the fibers of the positive focal map $\nu_+ : M_t \to M_+$. Therefore $M_+ = Q_n(\mathbb{C})$ is regarded as a *symplectic quotient* $f^{-1}(t)/S^1$, but which is *not a Kähler quotient*, because the S^1-action on M_t is not isometric.

$$M_t \left(\cong \tfrac{SO(n+2)}{\mathbb{Z}_2 \times SO(n)}\right) = f^{-1}(t) \subset \mathbb{C}P^{n+1}$$

$$\nu_+ \Big\downarrow\, S^1 \left(\cong SO(2)/\mathbb{Z}_2\right)$$

$$M_+ = Q_n(\mathbb{C}) \left(\cong \tfrac{SO(n+2)}{SO(2) \times SO(n)}\right) = f^{-1}(t)/S^1$$

Lemma 4.1 gives the relation between the push-forward symplectic form $(\nu_+)_*\omega$ on the symplectic quotient $f^{-1}(t)/S^1$ and the original Kähler form $\omega_{Q_n(\mathbb{C})}$ of $Q_n(\mathbb{C})$.

The positive focal map $\nu_+ : M_t \to M_+ = Q_n(\mathbb{C})$ can be considered as a principle fiber bundle with structure group S^1 over $Q_n(\mathbb{C})$. We define a 1-form α_ν on M_t by

(4.5) $$\alpha_\nu(X) := \omega(\nu, X) = g(J\nu, X)$$

for each $X \in TM_t$. The 1-form α_ν satisfies

(4.6) $$(\nabla_X \alpha_\nu)(Y) = g(A_\nu(X), JY)$$

for each $X, Y \in TM_t$. Hence the exterior derivative of α_ν is

(4.7) $$\begin{aligned}(d\alpha_\nu)(X, Y) &= (\nabla_X \alpha_\nu)(Y) - (\nabla_Y \alpha_\nu)(X) \\ &= g(A_\nu(X), JY) - g(A_\nu(Y), JX) \\ &= -\omega(A_\nu(X), Y) + \omega(A_\nu(Y), X)\end{aligned}$$

for each $X, Y \in TM_t$. Hence by using (4.1), (4.3), we have a formula

Lemma 4.2.

(4.8) $$d\alpha_\nu = -2\tan(2\theta_1)\,\omega|_{M_t}.$$

The principal S^1-bundle $\nu_+ : M_t \to M_+ = Q_n(\mathbb{C})$ has a connection invariant under the left group action of $SO(n+2)$ whose horizontal subspaces are defined by $(J\nu)^\perp = V_2 \oplus V_4 = V_2 \oplus J(V_2)$. Then Lemma 4.2 means that its connection form is α_ν and its connection form is $-2\tan(2\theta_1)\,\omega|_{M_t}$. Moreover, by using Lemma 4.1 we have

(4.9) $$d\alpha_\nu = -2\tan(2\theta_1)\,\cos(2\theta_1)\,\nu_+^* \omega_{Q_n(\mathbb{C})} = -2\sin(2\theta_1)\,\nu_+^* \omega_{Q_n(\mathbb{C})}.$$

Note that $\sin(2\theta_1) \neq 0$.

Let $\varphi : L \to Q_n(\mathbb{C})$ be a Lagrangian immersion of an n-dimensional smooth manifold L^n into an n-dimensional complex hyperquadric $Q_n(\mathbb{C})$. Then we take a pull-back S^1-bundle $\hat{L}^{n+1} = \varphi^{-1} M_t$ by φ over L as

$$\begin{array}{ccccc} \hat{L}^{n+1} = \varphi^{-1} M_t & \xrightarrow{\hat{\varphi}_t} & M_t = f^{-1}(t) & \subset & \mathbb{C}P^{n+1} \\ \nu_+ \downarrow\, S^1 & & \nu_+ \downarrow\, S^1 & & \\ L^n & \xrightarrow{\varphi} & M_+ = Q_n(\mathbb{C}) & \subset & \mathbb{C}P^{n+1} \end{array}$$

Since φ is a Lagrangian immersion, i.e. $\varphi^* \omega_{Q_n(\mathbb{C})} = 0$, by Lemma 4.1 $\hat{\varphi}_t^* \omega = \cos(2\theta_1)\,\hat{\varphi}_t^* \nu_+^* \omega_{Q_n(\mathbb{C})} = \cos(2\theta_1)\,\nu_+^* \varphi^* \omega_{Q_n(\mathbb{C})} = 0$ and hence $\hat{\varphi}_t : L^n \to \mathbb{C}P^{n+1}$ is also a Lagrangian immersion. Moreover, $\nu_+ : \hat{L}^{n+1} = \varphi^{-1} M_t \to L^n$ is also a principal S^1-bundle over L^n and the pull-back connection is flat.

Theorem . *If $\varphi : L \to Q_n(\mathbb{C})$ is a Lagrangian immersion into an n-dimensional complex hyperquadric, then we have a family of Lagrangian immersions $\hat{\varphi}_t : \hat{L} = \varphi^{-1} M_t \to (M_t \subset) \mathbb{C}P^{n+1}$ $(t \in (-1, 1))$ into an $(n+1)$-dimensional complex projective space.*

Remark 1. The infinitesimal Lagrangian deformation given by the above family corresponds to the closed 1-form $\hat{\varphi}_t^* \alpha_\nu$.

Remark 2. If we take the Gauss images of compact isoparametric hypersurfaces in $S^{n+1}(1)$ as $L^n \subset Q_n(\mathbb{C})$, then by this theorem we obtain a class of compact Lagrangian submanifolds $\nu_+^*(L^n) (\subset M_t)$ in $\mathbb{C}P^{n+1}$ related to isoparametric hypersurfaces. If we take compact homogeneous Lagrangian submanifolds as

$L^n \subset Q_n(\mathbb{C})$, then by this theorem we obtain a class of compact Lagrangian orbits in $\mathbb{C}P^{n+1}$ which are not necessarily homogeneous in our sense.

We shall discuss the case when $L^n = S^n \subset Q_n(\mathbb{C})$ is a totally geodesic Lagrangian submanifold which is one of real forms in $Q_n(\mathbb{C})$. It also coincides with the Gauss image of a compact isoparametric hypersurface with $g = 1$ in $S^{n+1}(1)$ (i.e. a great or small hypersphere of $S^{n+1}(1)$) in Section 4.

$$\tilde{L}^{n+1} = (\nu_+)^{-1}(S^n) \quad \subset \quad M_t = f^{-1}(t) \quad \subset \quad \mathbb{C}P^{n+1}$$

$$\nu_+ \Big\downarrow S^1 \qquad \qquad \nu_+ \Big\downarrow S^1$$

$$L^n = S^n \quad \subset \quad M_+ = Q_n(\mathbb{C}) \quad \subset \quad \mathbb{C}P^{n+1}$$

In this case by direct computations we obtain

Theorem . *If $L^n = S^n \subset Q_n(\mathbb{C})$, then then we have a family of compact Lagrangian submanifolds $\hat{L}_t^{n+1} = (\nu_+)^{-1}(S^n)\,(\subset M_t)\,(t \in (-1,1))$ embedded in $\mathbb{C}P^{n+1}$ with the following properties:*

(1) *$\hat{L}_t^{n+1}$ is diffeomorphic but not isometric to the Riemannian product $S^n \times S^1$:*

$$\hat{L}_t^{n+1} = \nu_+^{-1}(S^n) \cong \frac{SO(n+1)}{SO(n)} \cdot (SO(2) \times \{\mathrm{I}_n\}) \cong S^n \times S^1 \,.$$

(2) *$\hat{L}_t^{n+1}$ is a compact Lagrangian orbit of $SO(n+1) \times R^1$ but not homogeneous in the sense of Section 2. Here R^1 denotes the flow on $\mathbb{C}P^{n+1}$ generated by the vector field $J(\mathrm{grad}f)$.*

(3) *$\hat{L}_t^{n+1}$ is not Hamiltonian minimal.*

Remark 3. $\hat{L}_t^{n+1}$ is isometric to a warped product $S^1 \times_\rho S^n$ for a smooth function $\rho(\gamma) = \cos^2\gamma \, \cos^2\theta_1 + \sin^2\gamma \, \sin^2\theta_1$ on S^1.

Finally we remark that the argument of this paper works also in the following other homogeneous isoparametric families:
(1) Construction from Lagrangian submanifold in $Q_n(\mathbb{C})$ to Lagrangian submanifolds in $Q_{n+1}(\mathbb{C})$:

$$
\begin{array}{ccccc}
Q_{n+1}(\mathbb{C}) & = & Q_{n+1}(\mathbb{C}) & = & Q_{n+1}(\mathbb{C}) \\
\cup & & \cup & & \cup \\
M_{-1} = S^{n+1} & \xleftarrow{\ \nu_-\ } & M_t & \xrightarrow{\ \nu_+\ } & Q_n(\mathbb{C}) = M_1
\end{array}
$$

(2) Construction from Lagrangian submanifold in $\mathbb{C}P^n$ to Lagrangian submanifolds in $\mathbb{C}P^{n+1}$:

$$
\begin{array}{ccccc}
\mathbb{C}P^{n+1} & = & \mathbb{C}P^{n+1} & = & \mathbb{C}P^{n+1} \\
\cup & & \cup & & \cup \\
M_{-1} = \{\text{a point}\} & \xleftarrow{\ \nu_-\ } & M_t & \xrightarrow{\ \nu_+\ } & \mathbb{C}P^n = M_1
\end{array}
$$

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