ROTATION NUMBER OF PRIMITIVE VECTOR SEQUENCES

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Abstract. We give a formula on the rotation number of a sequence of primitive vectors, which is a generalization of the formula on the rotation number of a unimodular sequence in [2].

1. Introduction

Let $v_1, \dots, v_d \in \mathbb{Z}^2$ be a sequence of primitive vectors such that $\varepsilon_i = \det(v_i, v_{i+1}) \neq 0$ for all $i = 1, \dots, d$, and let $a_i = \varepsilon_{i-1}^{-1}\varepsilon_i^{-1}\det(v_{i+1}, v_{i-1})$, where $v_0 = v_d$ and $v_{d+1} = v_1$. The *rotation number* of the sequence $v_1, \dots, v_d$ around the origin is defined by

$$\frac{1}{2\pi}\sum_{i=1}^{d}\int_{L_i}\frac{-ydx + xdy}{x^2 + y^2},$$

where L_i is the line segment from v_i to v_{i+1}. The sequence is called *unimodular* if $|\varepsilon_i| = 1$ for all $i = 1, \dots, d$. Recently A. Higashitani and M. Masuda [2] proved the following:

Theorem 1 ([2]). *The rotation number of a unimodular sequence $v_1, \dots, v_d$ around the origin is given by*

$$\frac{1}{12}\sum_{i=1}^{d}(3\varepsilon_i + a_i).$$

When $\varepsilon_i = 1$ for all i and the rotation number is one, Theorem 1 is well known and formulated as $3d + \sum_{i=1}^{d} a_i = 12$. It can be proved in an elementary way, but interestingly it can also be proved using toric geometry, to be more precise, by applying Nöther's formula to complete non-singular toric varieties of complex dimension two, see [1]. When $\varepsilon_i = 1$ for all i but the rotation number is not necessarily one, Theorem 1 was proved in [4] using toric topology. The proof is a generalization of the proof above using toric geometry. The original proof of Theorem 1 by Higashitani and Masuda was a slight modification of the proof in [4] but then they found an elementary proof. Another elementary proof of Theorem 1 is given by R. T. Zivaljevic [3].

Theorem 1 does not hold when the unimodularity condition is dropped. In this paper, we give a formula on the rotation number of a (not necessarily unimodular) sequence of primitive vectors $v_1, \dots, v_d$ with $\varepsilon_i \neq 0$ for all i, see Theorem 4. The proof is done by adding primitive vectors in an appropriate way to the given sequence so that the enlarged sequence is unimodular and then by applying Theorem 1 to the enlarged unimodular sequence. This combinatorial process, that is,

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making the given sequence to a unimodular sequence by adding primitive vectors corresponds to resolution of singularity by blow-up in geometry, see [1].

The structure of the paper is as follows: In Section 2, we state the main theorem and give an example. In Section 3, we discuss Hirzebruch-Jung continued fractions used in our proof of the main theorem. In Section 4, we give a proof of the main theorem.

2. The main theorem

Let $v_1, \dots, v_d \in \mathbb{Z}^2$ be a sequence of vectors such that $\varepsilon_i = \det(v_i, v_{i+1}) \neq 0$ for all $i = 1, \dots, d$. We define $v_0 = v_d$ and $v_{d+1} = v_1$. We assume that each vector is *primitive*, i.e. its components are relatively prime.

Lemma 2. *For each $i = 1, \dots, d$, there exists a unique non-negative integer $x_i < |\varepsilon_i|$ such that x_i and $|\varepsilon_i|$ are relatively prime and*

$$P_i = (v_i, v_{i+1}) \begin{pmatrix} 1 & -x_i \\ 0 & |\varepsilon_i| \end{pmatrix}^{-1}$$

is a unimodular matrix.

Proof. Let $v_i = \begin{pmatrix} a \\ b \end{pmatrix}$ and $v_{i+1} = \begin{pmatrix} c \\ d \end{pmatrix}$. We assume that $\varepsilon_i > 0$. Since v_i is primitive, there exist $p, q \in \mathbb{Z}$ such that $ap + bq = 1$. Then we have

$$\begin{pmatrix} p & q \\ -b & a \end{pmatrix} (v_i, v_{i+1}) = \begin{pmatrix} 1 & cp + dq \\ 0 & |\varepsilon_i| \end{pmatrix}.$$

There exists a unique $n \in \mathbb{Z}$ satisfying $-|\varepsilon_i| < cp + dq + n|\varepsilon_i| \leq 0$. So we put $x_i = -(cp + dq + n|\varepsilon_i|)$. Then we have

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ -b & a \end{pmatrix} (v_i, v_{i+1}) = \begin{pmatrix} 1 & -x_i \\ 0 & |\varepsilon_i| \end{pmatrix}.$$

Hence

$$P_i = \begin{pmatrix} a & -q \\ b & p \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

is a unimodular matrix. When $\varepsilon_i < 0$, we can show that the assertion holds by a similar argument. Since $\begin{pmatrix} -x_i \\ |\varepsilon_i| \end{pmatrix} = P_i^{-1} v_{i+1}$ is primitive, x_i and $|\varepsilon_i|$ are relatively prime. □

Note that $\det(P_i) = \dfrac{\varepsilon_i}{|\varepsilon_i|}$. Similarly, there exists a unique non-negative integer $y_i < |\varepsilon_i|$ such that

$$Q_i = (v_{i+1}, v_i) \begin{pmatrix} 1 & -y_i \\ 0 & |\varepsilon_i| \end{pmatrix}^{-1} \tag{2.1}$$

is a unimodular matrix.

Since x_i and $|\varepsilon_i|$ are relatively prime, $x_i > 0$ when $|\varepsilon_i| \geq 2$ and $x_i = 0$ when $|\varepsilon_i| = 1$. For i such that $|\varepsilon_i| \geq 2$, let

$$\frac{|\varepsilon_i|}{x_i} = n_1^{(i)} - \cfrac{1}{n_2^{(i)} - \cfrac{1}{\ddots - \cfrac{1}{n_{l_i}^{(i)}}}}, \quad n_j^{(i)} \geq 2. \tag{2.2}$$

be the Hirzebruch-Jung continued fraction expansion. This continued fraction expansion is unique. We define $l_i = 0$ when $|\varepsilon_i| = 1$.

Lemma 3. *Let $a_i = \varepsilon_{i-1}^{-1}\varepsilon_i^{-1}\det(v_{i+1}, v_{i-1})$. Then a_i satisfies*

$$\varepsilon_{i-1}^{-1}v_{i-1} + \varepsilon_i^{-1}v_{i+1} + a_i v_i = 0. \tag{2.3}$$

Proof. It is easy to check that

$$\det(v_i, v_{i+1})v_{i-1} + \det(v_{i-1}, v_i)v_{i+1} + \det(v_{i+1}, v_{i-1})v_i = 0.$$

Dividing both sides by $\varepsilon_{i-1}\varepsilon_i = \det(v_{i-1}, v_i)\det(v_i, v_{i+1})$, we obtain (2.3). □

The following is our main theorem:

Theorem 4. *Let $v_1, \ldots, v_d$ be a sequence of primitive vectors and $\varepsilon_i = \det(v_i, v_{i+1})$, $a_i = \varepsilon_{i-1}^{-1}\varepsilon_i^{-1}\det(v_{i+1}, v_{i-1})$. Let x_i, y_i, l_i, and $n_j^{(i)}$ be the integers defined in Lemma 2, (2.1), and (2.2). Then the rotation number of the sequence $v_1, \ldots, v_d$ around the origin is given by*

$$\frac{1}{12}\sum_{i=1}^{d}\left(\left(3(l_i+1) - \sum_{j=1}^{l_i} n_j^{(i)}\right)\frac{\varepsilon_i}{|\varepsilon_i|} + a_i - \frac{x_i + y_i}{\varepsilon_i}\right). \tag{2.4}$$

Example 5. Let $d = 5$ and

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Then we have the following:

i	ε_i	a_i	x_i	y_i	l_i	$n_1^{(i)}$	$n_2^{(i)}$
1	3	-2	2	2	2	2	2
2	5	$\frac{1}{15}$	2	3	2	2	3
3	-4	$\frac{7}{20}$	1	1	1	4	
4	1	$\frac{11}{4}$	0	0	0		
5	3	$\frac{1}{3}$	1	1	1	3	

So we have

$$\begin{aligned} & \sum_{i=1}^{d}\left(3(l_i+1) - \sum_{j=1}^{l_i} n_j^{(i)}\right)\frac{\varepsilon_i}{|\varepsilon_i|} \\ = \; & (3(2+1)-4) + (3(2+1)-5) - (3(1+1)-4) + 3 + (3(1+1)-3) = 13, \\ & \sum_{i=1}^{d} a_i = -2 + \frac{1}{15} + \frac{7}{20} + \frac{11}{4} + \frac{1}{3} = \frac{3}{2}, \\ & \sum_{i=1}^{d}\frac{x_i+y_i}{\varepsilon_i} = \frac{2+2}{3} + \frac{2+3}{5} + \frac{1+1}{-4} + \frac{0+0}{1} + \frac{1+1}{3} = \frac{5}{2}. \end{aligned}$$

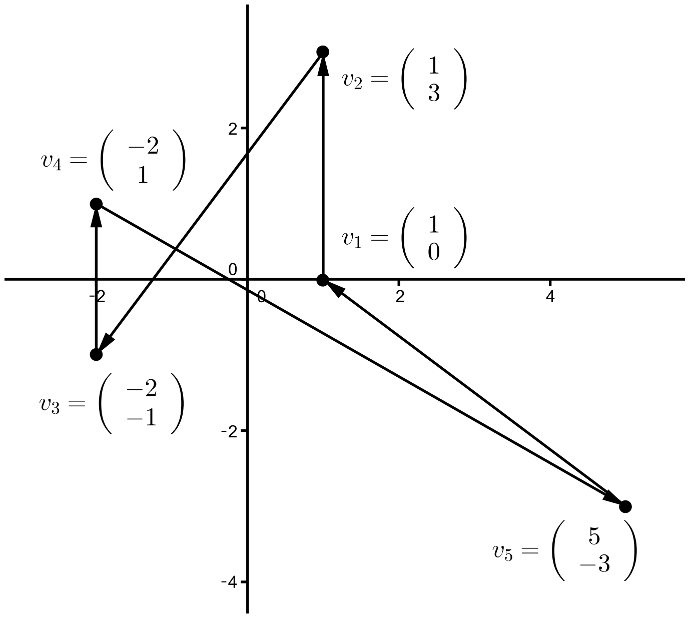

FIGURE 1. A sequence of primitive vectors

Therefore the value (2.4) is $\frac{1}{12}\left(13+\frac{3}{2}-\frac{5}{2}\right)=1$, while the rotation number of the sequence $v_1, \ldots, v_5$ in Figure 1 is clearly one.

3. CONTINUED FRACTIONS

Let $m \geq 2$ and $x(< m)$ be a positive integer prime to m, and let

$$\frac{m}{x} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}, \ n_j \geq 2$$

be the continued fraction expansion. This continued fraction expansion is unique, and is called a *Hirzebruch-Jung continued fraction.*

Lemma 6. *Let $m \geq 2$ and $x(< m)$ be a positive integer prime to m, and let*

$$\frac{m}{x} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}, \ n_j \geq 2$$

be the continued fraction expansion. Let $y(< m)$ be a unique positive integer such that $xy \equiv 1 \pmod{m}$. Then the following identity holds:

$$\begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & n_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & -1 \\ 1 & n_l \end{pmatrix} = \begin{pmatrix} \frac{1-xy}{m} & -x \\ y & m \end{pmatrix}.$$

Proof. We prove this by induction on l.

If $l = 1$, then we must have $x = 1, n_1 = m$ and $y = 1$. So the lemma holds when $l = 1$.

Suppose that $l \geq 2$ and the lemma holds for $l - 1$. We have

$$\frac{x}{n_1 x - m} = n_2 - \cfrac{1}{n_3 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}.$$

Since $l \geq 2$, we have $x \geq 2$. Since the right hand side in the identity above is greater than 1, we have $0 < n_1 x - m < x$. Since m and x are relatively prime, x and $n_1 x - m$ are relatively prime. Moreover $\frac{xy-1}{m}$ is a positive integer less than x and $(n_1 x - m)\frac{xy-1}{m} \equiv 1 \pmod{x}$. Hence by the hypothesis of induction, we obtain

$$\begin{aligned}
\begin{pmatrix} 0 & -1 \\ 1 & n_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_l \end{pmatrix} &= \begin{pmatrix} \dfrac{1 - (n_1 x - m)\frac{xy-1}{m}}{x} & -(n_1 x - m) \\ \dfrac{xy-1}{m} & x \end{pmatrix} \\
&= \begin{pmatrix} y - n_1 \dfrac{xy-1}{m} & -(n_1 x - m) \\ \dfrac{xy-1}{m} & x \end{pmatrix}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_l \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} \begin{pmatrix} y - n_1 \dfrac{xy-1}{m} & -(n_1 x - m) \\ \dfrac{xy-1}{m} & x \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1 - xy}{m} & -x \\ y & m \end{pmatrix},
\end{aligned}$$

proving the lemma for l. □

Proposition 7. *The following identity holds:*

$$\frac{m}{y} = n_l - \cfrac{1}{n_{l-1} - \cfrac{1}{\ddots - \cfrac{1}{n_1}}}.$$

Proof. Let $f : M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$ be the antihomomorphism defined by

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}.$$

By Lemma 6, we have

$$\begin{aligned}\begin{pmatrix} 0 & -1 \\ 1 & n_l \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} &= f\left(\begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_l \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} \frac{1-xy}{m} & -x \\ y & m \end{pmatrix}\right) = \begin{pmatrix} \frac{1-xy}{m} & -y \\ x & m \end{pmatrix},\end{aligned}$$

proving the proposition. □

Remark 8. A similar assertion holds for regular continued fractions. Let

$$\frac{m}{x} = n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_l}}},\ n_j \geq 1$$

be a continued fraction expansion, and let $y(< m)$ be a unique positive integer such that $xy \equiv (-1)^{l+1} \pmod{m}$. Then the following identity holds:

$$\begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & n_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & n_l \end{pmatrix} = \begin{pmatrix} \frac{xy+(-1)^l}{m} & x \\ y & m \end{pmatrix}.$$

The proof is similar to Lemma 6. The following identity can be deduced by taking transpose at the identity above:

$$\frac{m}{y} = n_l + \cfrac{1}{n_{l-1} + \cfrac{1}{\ddots + \cfrac{1}{n_1}}}.$$

4. Proof of Theorem 4

In this section, we give a proof of Theorem 4. We will use the notation in Section 2 freely. We need the following lemma.

Lemma 9. *For each $i = 1, \dots, d$, the following identity holds:*

$$\begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i)} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & n_2^{(i)} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_{l_i}^{(i)} \end{pmatrix} = \begin{pmatrix} \frac{1-x_iy_i}{|\varepsilon_i|} & -x_i \\ y_i & |\varepsilon_i| \end{pmatrix}.$$

Proof. If $|\varepsilon_i| = 1$, then $x_i = y_i = l_i = 0$ and the left hand side above is understood to be the identity matrix. Assume $|\varepsilon_i| \geq 2$. By Lemma 2, x_i and $|\varepsilon_i|$ are relatively prime. Since

$$\begin{aligned}Q_i^{-1}P_i &= \begin{pmatrix} 1 & -y_i \\ 0 & |\varepsilon_i| \end{pmatrix}(v_{i+1}, v_i)^{-1}(v_i, v_{i+1})\begin{pmatrix} 1 & -x_i \\ 0 & |\varepsilon_i| \end{pmatrix}^{-1} \\ &= \frac{1}{|\varepsilon_i|}\begin{pmatrix} 1 & -y_i \\ 0 & |\varepsilon_i| \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |\varepsilon_i| & x_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -y_i & \frac{1-x_iy_i}{|\varepsilon_i|} \\ |\varepsilon_i| & x_i \end{pmatrix}\end{aligned}$$

is a unimodular matrix, x_iy_i is congruent to 1 modulo $|\varepsilon_i|$. Therefore the lemma follows from Lemma 6. □

Proof of Theorem 4. For $j = 0, \dots, l_i + 1$, we define

$$w_j^{(i)} = \begin{cases} P_i \begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i)} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_j^{(i)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & (0 \le j \le l_i), \\ P_i \begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i)} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_{j-1}^{(i)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & (1 \le j \le l_i + 1). \end{cases} \tag{4.1}$$

Note that both expressions at the right hand side of (4.1) are equal if $1 \le j \le l_i$. By the definition of $w_j^{(i)}$, it follows that

$$\det(w_j^{(i)}, w_{j+1}^{(i)}) = \det(P_i)\det\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \frac{\varepsilon_i}{|\varepsilon_i|} \in \{\pm 1\} \tag{4.2}$$

for any $j = 0, \dots, l_i$. So the sequence

$$\dots, v_i = w_0^{(i)}, w_1^{(i)}, \dots, w_{l_i+1}^{(i)} = v_{i+1}, \dots \tag{4.3}$$

is unimodular.

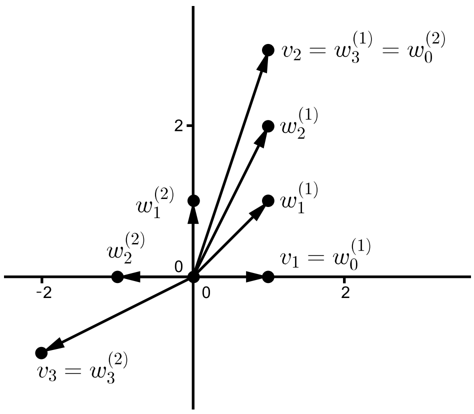

FIGURE 2. Adding $w_j^{(i)}$ to the given vector sequence

Hence by Theorem 1, the rotation number of $v_1, \dots, v_d$ is given by

$$\begin{aligned} &\frac{1}{4}\sum_{i=1}^{d}\sum_{j=0}^{l_i} \det(w_j^{(i)}, w_{j+1}^{(i)}) \\ &+ \frac{1}{12}\sum_{i=1}^{d} \frac{\det(w_1^{(i)}, w_{l_{i-1}}^{(i-1)})}{\det(w_{l_{i-1}}^{(i-1)}, v_i)\det(v_i, w_1^{(i)})} \\ &+ \frac{1}{12}\sum_{i=1}^{d}\sum_{j=1}^{l_i} \frac{\det(w_{j+1}^{(i)}, w_{j-1}^{(i)})}{\det(w_{j-1}^{(i)}, w_j^{(i)})\det(w_j^{(i)}, w_{j+1}^{(i)})}. \end{aligned} \tag{4.4}$$

As for the first summand in (4.4), it follows from (4.2) that

$$\sum_{j=0}^{l_i} \det(w_j^{(i)}, w_{j+1}^{(i)}) = (l_i + 1)\frac{\varepsilon_i}{|\varepsilon_i|}.$$

As for the second summand in (4.4), we first observe that it follows from Lemma 3 that

$$\begin{aligned}
P_{i-1}^{-1}P_i &= \begin{pmatrix} 1 & -x_{i-1} \\ 0 & |\varepsilon_{i-1}| \end{pmatrix} (v_{i-1}, v_i)^{-1}(v_i, v_{i+1}) \begin{pmatrix} 1 & -x_i \\ 0 & |\varepsilon_i| \end{pmatrix}^{-1} \\
&= \frac{1}{|\varepsilon_i|} \begin{pmatrix} 1 & -x_{i-1} \\ 0 & |\varepsilon_{i-1}| \end{pmatrix} (v_{i-1}, v_i)^{-1}(v_i, -\varepsilon_i(\varepsilon_{i-1}^{-1}v_{i-1} + a_i v_i)) \begin{pmatrix} |\varepsilon_i| & x_i \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{|\varepsilon_i|} \begin{pmatrix} 1 & -x_{i-1} \\ 0 & |\varepsilon_{i-1}| \end{pmatrix} \begin{pmatrix} 0 & -\varepsilon_i\varepsilon_{i-1}^{-1} \\ 1 & -a_i\varepsilon_i \end{pmatrix} \begin{pmatrix} |\varepsilon_i| & x_i \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{|\varepsilon_i|} \begin{pmatrix} -|\varepsilon_i|x_{i-1} & -\varepsilon_i\varepsilon_{i-1}^{-1} - x_{i-1}x_i + a_i\varepsilon_i x_{i-1} \\ |\varepsilon_{i-1}||\varepsilon_i| & |\varepsilon_{i-1}|(x_i - a_i\varepsilon_i) \end{pmatrix}.
\end{aligned}$$

So it follows from (4.3), (4.1), (4.2), and Lemma 9 that

$$\begin{aligned}
&\frac{\det(w_1^{(i)}, w_{l_{i-1}}^{(i-1)})}{\det(w_{l_{i-1}}^{(i-1)}, v_i)\det(v_i, w_1^{(i)})} = \frac{\det(w_1^{(i)}, w_{l_{i-1}}^{(i-1)})}{\det(w_{l_{i-1}}^{(i-1)}, w_{l_{i-1}+1}^{(i-1)})\det(w_0^{(i)}, w_1^{(i)})} \\
&= \frac{|\varepsilon_{i-1}||\varepsilon_i|}{\varepsilon_{i-1}\varepsilon_i}\det\left(P_i\begin{pmatrix} 0 \\ 1 \end{pmatrix}, P_{i-1}\begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i-1)} \end{pmatrix}\cdots\begin{pmatrix} 0 & -1 \\ 1 & n_{l_{i-1}}^{(i-1)} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= \frac{|\varepsilon_{i-1}||\varepsilon_i|}{\varepsilon_{i-1}\varepsilon_i}\det(P_{i-1})\det\left(P_{i-1}^{-1}P_i\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \dfrac{1 - x_{i-1}y_{i-1}}{|\varepsilon_{i-1}|} \\ y_{i-1} \end{pmatrix}\right) \\
&= \frac{1}{\varepsilon_i}\det\begin{pmatrix} -\dfrac{\varepsilon_i}{\varepsilon_{i-1}} - x_{i-1}x_i + a_i\varepsilon_i x_{i-1} & \dfrac{1 - x_{i-1}y_{i-1}}{|\varepsilon_{i-1}|} \\ |\varepsilon_{i-1}|(x_i - a_i\varepsilon_i) & y_{i-1} \end{pmatrix} \\
&= a_i - \frac{x_i}{\varepsilon_i} - \frac{y_{i-1}}{\varepsilon_{i-1}}.
\end{aligned}$$

As for the last summand in (4.4), it follows from (4.1) and (4.2) that

$$\begin{aligned}
&\frac{\det(w_{j+1}^{(i)}, w_{j-1}^{(i)})}{\det(w_{j-1}^{(i)}, w_j^{(i)})\det(w_j^{(i)}, w_{j+1}^{(i)})} \\
&= \det(P_i)\det\left(\begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i)} \end{pmatrix}\cdots\begin{pmatrix} 0 & -1 \\ 1 & n_j^{(i)} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & n_1^{(i)} \end{pmatrix}\cdots\begin{pmatrix} 0 & -1 \\ 1 & n_{j-1}^{(i)} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= \det(P_i)\det\left(\begin{pmatrix} 0 & -1 \\ 1 & n_j^{(i)} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= \frac{\varepsilon_i}{|\varepsilon_i|}\det\begin{pmatrix} -1 & 1 \\ n_j^{(i)} & 0 \end{pmatrix} = -n_j^{(i)}\frac{\varepsilon_i}{|\varepsilon_i|}.
\end{aligned}$$

Therefore (4.4) reduces to

$$\begin{aligned}
&\frac{1}{4}\sum_{i=1}^{d}(l_i + 1)\frac{\varepsilon_i}{|\varepsilon_i|} + \frac{1}{12}\sum_{i=1}^{d}\left(a_i - \frac{x_i}{\varepsilon_i} - \frac{y_{i-1}}{\varepsilon_{i-1}}\right) + \frac{1}{12}\sum_{i=1}^{d}\sum_{j=1}^{l_i}\left(-n_j^{(i)}\frac{\varepsilon_i}{|\varepsilon_i|}\right) \\
= \ &\frac{1}{12}\sum_{i=1}^{d}\left(\left(3(l_i + 1) - \sum_{j=1}^{l_i} n_j^{(i)}\right)\frac{\varepsilon_i}{|\varepsilon_i|} + a_i - \frac{x_i + y_i}{\varepsilon_i}\right),
\end{aligned}$$

proving the theorem. □

Remark 10. It sometimes happens that a sequence of primitive vectors $v_1, \dots, v_d$ is not unimodular but is unimodular with respect to the sublattice of $\mathbb{Z}^2$ generated by vectors $v_1, \dots, v_d$. Such a sequence is called an l-reflexive loop and studied in [5]. Theorem 1 can be applied to an l-reflexive loop with respect to the sublattice generated by the vectors in the l-reflexive loop, but it is unclear whether the resulting formula can be obtained from Theorem 4.

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