On the Brezis-Nirenberg problem with a Kirchhoff type perturbation

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Abstract

In this paper, we consider a nonlinear elliptic problem,

$$\begin{cases} -(1+b\|u\|^2)\Delta u = \lambda u + u^5, \ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda \in \mathbb{R}$ and $b \geq 0$. We give an extension of the result by Brezis-Nirenberg in 1983 to the case $b > 0$.

Keywords: Kirchhoff, nonlocal, elliptic, critical, variational method

1. Introduction

In this paper, we investigate the existence of solutions of the problem,

$$\begin{cases} -(1+b\|u\|^2)\Delta u = \lambda u + u^5 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

here Ω is a 3 dimensional open ball and $\|\cdot\|$ denotes the usual $H_0^1(\Omega)$ norm. We regard $\lambda \in \mathbb{R}$ as a given constant and $b \geq 0$ as a parameter. (P) has the following two features.

The characteristic principal term of the equation of (P) has its origin in the theory of nonlinear vibrations. See the book by Kirchhoff [14] and the survey [6]. Recently this Kirchhoff type elliptic problem is investigated extensively. We refer to [2]-[4][9][10][15]-[17][19]-[21][24][26] and so on. Among their works, the effects of the nonlocal coefficient, on the existence of solutions are investigated.

On the other hand, the right hand side of the equation of (P) has the critical term u^5. Thus (P) has the typical difficulty in proving the existence of solutions.

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This difficulty is caused by the lack of compactness of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$. See the results for several critical problems, the pioneering work [7] and [1][5][8][11][12][25], etc.

1.1. Main result

Here we note a result by Brezis-Nirenberg [7]. In [7], (P) with $b = 0$ is treated. One of the results can be read, "If and only if $\lambda_1/4 < \lambda < \lambda_1$ there exists a solution of (P) if $b = 0$", here $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ on the open ball. In this paper, we extend the result above to the case $b > 0$. Our result is the following.

Theorem 1.1. *Let $\lambda \in \mathbb{R}$ be a given constant. Then the following assertions hold.*

(i) If $\lambda \leq \lambda_1/4$, (P) has no solution for all $b \geq 0$.
(ii) If $\lambda_1/4 < \lambda < \lambda_1$, there exists a constant $B_2 = B_2(\lambda) > 0$ such that (P) has a solution for all $0 \leq b < B_2$.
(iii) If $\lambda = \lambda_1$, there exists a constant $B_3 = B_3(\lambda_1) > 0$ such that (P) has a solution for all $0 < b < B_3$ and (P) has no solution for $b = 0$.
(iv) If $\lambda_1 < \lambda$, there exists a constant $B_4 = B_4(\lambda) > 0$ such that (P) has a solution for all $b \geq B_4$.

We focus on Theorem 1.1 (iii) and (iv). As we state before, in the case $\lambda \geq \lambda_1$, (P) has no solution if $b = 0$. But our theorem says that even if $\lambda \geq \lambda_1$, (P) do have a solution thanks to $b > 0$ in the appropriate region. Such existence phenomena induced by the nonlocal perturbation is a new knowledge among the other recent researches in Kirchhoff type elliptic problems.

On the other hand, we still have some questions, (i) the existence or nonexistence for the cases $\lambda_1/4 < \lambda < \lambda_1$ with large $b > 0$, (ii) similarly, for the case $\lambda > \lambda_1$ with small $b > 0$, and (iii) the existence of the second solution for the case $\lambda > \lambda_1$ with large $b > 0$. These are left for our future works.

Several results associated to the Kirchhoff type elliptic problems with critical nonlinearity are obtained in [2][4][9][10][17][20][24][26]. In particular, [9], [20] and [24] are closely related to (P). They consider (P) with nonlinear term $\lambda u^q + u^5$ (but note that they consider $b \geq 0$ as a given constant and $\lambda > 0$ as a parameter in contrast to us). In [9], Figueiredo considers the case $1 < q < 5$. He gets the existence if $\lambda > 0$ is sufficiently large. In a part of [20], Naimen treats the case $3 < q < 5$. He concludes the existence for all $\lambda > 0$. In [24], Sun and Liu consider the case $0 < q < 1$. They ensure the solvability for sufficiently small $\lambda > 0$. But to our best knowledge, there is no previous work for the case $q = 1$. In this paper, we consider such a case and conclude Theorem 1.1.

1.2. Organization of this paper

This paper is organized as follows. In Section 2, we prove Theorem 1.1 (i). In Section 3, we demonstrate Theorem 1.1 (ii) and (iii). Finally in Section 4, we conclude Theorem 1.1 (iv). In the proof, we use a same character C to denote several positive constants. Note also that we denote $B(x,r)$ as an 3 dimensional open ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$ or an open ball in $H_0^1(\Omega)$ topology centered at $x \in H_0^1(\Omega)$ with radius r.

1.3. The weak solutions of (P)

Here we give the definition of the weak solutions of (P). We say $u \in H_0^1(\Omega)$ is a weak solution of (P), if and only if u satisfies

$$(1 + b\|u\|^2) \int_\Omega \nabla u \cdot \nabla h dx - \lambda \int_\Omega u_+ h dx - \int_\Omega u_+^5 h dx = 0, \tag{1}$$

for all $h \in H_0^1(\Omega)$, where $u_+ := \max\{0, u\}$. By the analogue with the usual elliptic regularity argument, we can conclude that every weak solution of (P) is a classical solution of (P) even if $b > 0$.

1.4. A priori estimate

We can get a priori estimate for the solutions u of (P) if $b > 0$ as follows. Let $\lambda_1/4 < \lambda$ and $b > 0$ be in the appropriate region. We have if $\lambda < \lambda_1$,

$$0 < \|u\| < \left(\frac{4\lambda - \lambda_1}{b\lambda_1}\right)^{\frac{1}{2}}, \tag{2}$$

and if $\lambda \geq \lambda_1$,

$$\left(\frac{\lambda - \lambda_1}{b\lambda_1}\right)^{\frac{1}{2}} < \|u\| < \left(\frac{4\lambda - \lambda_1}{b\lambda_1}\right)^{\frac{1}{2}}. \tag{3}$$

In fact, put $C := (1 + b\|u\|^2)^{-1/4}$. Then $v := Cu$ satisfies

$$\begin{cases} -\Delta v = \frac{\lambda}{1+b\|u\|^2} v + v^5, \ v > 0 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases}$$

From [7], it follows that

$$\frac{\lambda_1}{4} < \frac{\lambda}{1 + b\|u\|^2} < \lambda.$$

This proves (2) and (3).

2. The case $\lambda \leq \lambda_1/4$

In this section, we prove Theorem 1.1 (i). The argument is strictly based on that in [7]. Firstly we consider the case $\lambda \leq 0$. In this case, we use the following Pohozaev type identity [22] (see also [23]) for the solutions of (P).

Lemma 2.1. *Let u be a solution of (P) and put $g(t) := \lambda t + t^5$. Then the following identity holds.*

$$\frac{(1+b\|u\|^2)}{2}\int_\Omega |\nabla u|^2 dx - 3\int_\Omega G(u)dx + \frac{(1+b\|u\|^2)}{2}\int_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma = 0 \tag{4}$$

where $G(t) := \int_0^u g(s)ds$, ν and $\partial/\partial\nu$ denote the outer normal vector and the outer normal derivative on $\partial\Omega$ respectively, and further $d\sigma$ is the 2-dimensional surface measure on $\partial\Omega$.

Proof. Let u be a solution of (P). Then applying the well known procedure, the proof is straightforward. □

We give the following theorem.

Theorem 2.2. *Let Ω be strictly star-shaped and $\lambda \leq 0$. Then (P) has no solution for all $b \geq 0$.*

Proof. Utilizing (4), we can clearly conclude the theorem by the usual argument. □

Next we consider the case $0 < \lambda \leq \lambda_1/4$.

Theorem 2.3. *If Ω is an open ball and $0 < \lambda \leq \lambda_1/4$, there exists no solution of (P) for all $b \geq 0$.*

Proof. For simplicity we assume $\Omega = B(0,1)$. Then by careful reading of the argument in [13], we can confirm that every solution of (P) is radially symmetric even if $b > 0$. Consequently every solution of (P) satisfies

$$\begin{cases} -A\left(u'' + \frac{2}{r}u'\right) = \lambda u + u^5 \text{ in } [0,1), \\ u > 0 \text{ in } [0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{K_r}$$

where we put $A = A(\|u\|^2) = 1 + b\|u\|^2$ for simplicity. Now let $0 < \lambda \leq \lambda_1/4$ and u be a solution of (K_r). We take a smooth function ψ in $[0,1]$ such that $\psi(0) = 0$. By a similar procedure to that in [7], we have a variant of Pohozaev type identity,

$$\int_0^1 u^2\left(\frac{A}{4}\psi''' + \lambda\psi'\right) r^2 dr = \frac{2}{3}\int_0^1 u^6\left(r\psi - r^2\psi'\right) dr + \frac{A}{2}|u'(1)|^2\psi(1). \tag{5}$$

We take

$$\psi(r) = \sin\left((4\lambda/A)^{1/2} r\right),$$

so that $\psi(0) = 0$ and $\psi(1) \geq 0$. Then noting $A \geq 1$ we get a contradiction by (5). □

The proof of Theorem 1.1 (i). Assume Ω is an open ball. Then the proof is obvious by Theorem 2.2 and Theorem 2.3. □

3. The case $\lambda_1/4 < \lambda \le \lambda_1$

In this section, we prove Theorem 1.1 (ii) and (iii). We suppose $\Omega = B(0,1)$ for simplicity and fix $\lambda_1/4 < \lambda \le \lambda_1$. If $b = 0$, the conclusion is in [7]. Hence here, we give the proof for the case $b > 0$. But if $b > 0$, the minimizing argument in [7] does not seem to work because of the existence of the nonlocal coefficient. Thus we apply the mountain pass theorem here. We define the energy functional associated to (P) so that

$$I_b(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega u_+^2\,dx - \frac{1}{6}\int_\Omega u_+^6\,dx.$$

Clearly I_b is well-defined and continuously Fréchet differentiable on $H_0^1(\Omega)$. We shall ensure the existence of a nontrivial critical point of I_b. The main argument lies in ensuring the local PS condition for I_b. To this aim, we give the following lemma.

Lemma 3.1. *Let $\{u_j\} \subset H_0^1(\Omega)$ be a (PS)$_c$ sequence with*

$$c < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

where $C_K := \left(bS^3 + \sqrt{(bS^3)^2 + 4S^3}\right)/2$. Then there exists a function $u \in H_0^1(\Omega)$ such that $(u_j)_+ \to u_+$ in $L^6(\Omega)$ up to subsequences.

Proof. We first claim that $\{u_j\}$ is bounded in $H_0^1(\Omega)$. In fact, since $I_b(u_j) \to c$ and $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$, the Poincare inequality and our assumption $\lambda \le \lambda_1$ confirm

$$\begin{aligned} c + 1 &\ge I_b(u_j) - \frac{1}{6}\langle I_b'(u_j), u_j\rangle + \frac{1}{6}\langle I_b'(u_j), u_j\rangle \\ &\ge \frac{1}{3}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_j\|^2 + \frac{b}{12}\|u_j\|^4 - \|u_j\| \\ &\ge \frac{b}{12}\|u_j\|^4 - \|u_j\|, \end{aligned}$$

for large $j \in \mathbb{N}$. As $b > 0$, this proves the claim. Hence by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embedding, we have

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_j \to u \text{ in } L^2(\Omega), \\ &u_j \to u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequence but still denoted $\{u_j\}$. Furthermore by the second concentration compactness lemma [18], we can assume that there exist an at most countable set J, points $\{x_k\}_{k\in J} \subset \overline{\Omega}$, and values $\{\mu_k\}_{k\in J}, \{\nu_k\}_{k\in J} \subset \mathbb{R}^+$ with

$$S\nu_k^{\frac{1}{3}} \le \mu_k \ (k \in J) \tag{6}$$

such that,

$$|\nabla u_j|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \sum_{k\in J} \mu_k \delta_{x_k},$$
$$(u_j)_+^6 \rightharpoonup d\nu = u_+^6 + \sum_{k\in J} \nu_k \delta_{x_k},$$

in the measure sense, here δ_x denotes the Dirac delta measure concentrated at $x \in \mathbb{R}^3$ with mass 1. Now we claim $J = \emptyset$. To show this, we assume $J \neq \emptyset$ to the contrary. Then fix $k \in J$. Define a smooth test function ϕ in $\mathbb{R}^3$ such that $\phi = 1$ on $B(x_k, \varepsilon)$, $\phi = 0$ on $B(x_k, 2\varepsilon)^c$ and $0 \leq \phi \leq 1$ otherwise. We also assume $|\nabla \phi| \leq 2/\varepsilon$. Again since $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$ and $\{u_j\}$ is bounded, we have after some calculations,

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle I_b'(u_j), u_j\phi\rangle \\ &= \left(1 + b\int_\Omega d\mu\right)\int_\Omega \phi d\mu - \int_\Omega \phi d\nu + o(1) \end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Taking $\varepsilon \to 0$, we conclude

$$0 \geq (1 + b\mu_k)\mu_k - \nu_k.$$

Finally using (6) we reach to an estimate,

$$\mu_k \geq \frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 + 4S^3}\right) = C_K. \tag{7}$$

Then since $I_b(u_j) \to c$ and $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$, we have

$$\begin{aligned} c &= \lim_{j\to\infty}\left\{I_b(u_j) - \frac{1}{4}\langle I_b'(u_j), u_j\rangle\right\} \\ &\geq \lim_{j\to\infty}\left\{\left(\frac{1}{2} - \frac{1}{4}\right)\|u_j\|^2 - \lambda\left(\frac{1}{2} - \frac{1}{4}\right)\int_\Omega (u_j)_+^2 dx + \left(\frac{1}{4} - \frac{1}{6}\right)\int_\Omega (u_j)_+^6 dx\right\} \\ &\geq \left(\frac{1}{2} - \frac{1}{4}\right)\mu_k + b\left(\frac{1}{4} - \frac{1}{4}\right)\mu_k^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\nu_k \\ &\geq \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3 \end{aligned}$$

where for the last inequality we use (6), (7), and the fact that $C_K + bC_K^2 - C_K^3/S^3 = 0$. This contradicts our hypothesis on c. It follows that

$$\int_\Omega (u_j)_+^6 dx \to \int_\Omega u_+^6 dx \text{ as } j \to \infty.$$

This leads us to the proof. □

Remark 3.2. We can easily confirm that u is nonnegative. In fact, since $\{u_j\}$ is bounded, we can assume that $\|u_j\|^2 \to B$ for some value $B \geq 0$. If $B = 0$, we finish the proof. If not, noting that we can further suppose $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $(u_j)_+ \to u_+$ in $L^p(\Omega)$ for all $1 \leq p < 6$, we have

$$(1 + bB) \int_\Omega \nabla u \cdot \nabla h dx = \lambda \int_\Omega u_+ h dx + \int_\Omega u_+^5 h dx,$$

for all $h \in H_0^1(\Omega)$. Taking $h = -\min\{u, 0\} =: u_-$, we ensure the claim.

Here as in [7], for all $\varepsilon > 0$ we introduce the cut off Tarenti function function,

$$u_\varepsilon(x) := \frac{\varepsilon^2 \tau(|x|)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}$$

where $\tau \in C_0^\infty([0,1])$ is a cut off function with $\tau(0) = 1$ and $\tau'(0) = \tau(1) = 0$. Then we estimate

$$\begin{cases} \int_\Omega |\nabla u_\varepsilon|^2 dx = K_1 + \omega\varepsilon \int_0^1 |\tau'(r)|^2 dr + O(\varepsilon^2), \\ \int_\Omega u_\varepsilon^6 dx = K_2^3 + O(\varepsilon^2), \\ \int_\Omega u_\varepsilon^2 dx = \omega\varepsilon \int_0^1 |\tau(r)|^2 dr + O(\varepsilon^2), \end{cases} \tag{8}$$

where ω is the area of a 2 dimensional unit disc and $K_1, K_2 > 0$ are some constants with $K_1/K_2 = S$. Here $S > 0$ is the usual Sobolev constant defined by

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega u^6 dx)^{\frac{1}{3}}}.$$

The next lemma shows that a mountain pass level of I_b is below the desired value.

Lemma 3.3. *For every $\lambda_1/4 < \lambda \leq \lambda_1$, there exists a constant $B = B(\lambda) > 0$ such that for all $0 < b < B$, we can choose a constant $\varepsilon_1 = \varepsilon_1(\lambda, b) > 0$ so that*

$$\sup_{t \geq 0} I_b(tu_\varepsilon) < \frac{1}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3$$

for all $0 < \varepsilon < \varepsilon_1$.

Proof. Let u_ε be as defined before with $\tau(r) := \cos(\pi r/2)$. Take $t > 0$. Then we have after some calculation that

$$\begin{aligned} I_b(tu_\varepsilon) &= \frac{t^2}{2} \left(\|u_\varepsilon\|^2 - \lambda \int_\Omega u_\varepsilon^2 dx \right) + \frac{bt^4}{4} \|u_\varepsilon\|^4 - \frac{t^6}{6} \int_\Omega u_\varepsilon^6 dx \\ &\leq \frac{1}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3 - C \left\{ \left(\lambda - \frac{\lambda_1}{4} \right) S^{\frac{1}{2}} + O(b) \right\} \varepsilon + O(\varepsilon^2), \end{aligned}$$

for some constant $C > 0$. Notice that since $\lambda > \lambda_1/4$, there exists a constant $B = B(\lambda) > 0$ such that

$$\left(\lambda - \frac{\lambda_1}{4} \right) S^{\frac{1}{2}} + O(b) > 0$$

for all $0 < b < B$. Fix such a b. Then we can take a constant $\varepsilon_1 = \varepsilon_1(\lambda, b) > 0$ so that

$$I_b(tu_\varepsilon) < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3$$

for all $t > 0$ and all $\varepsilon \in (0, \varepsilon_1)$. This completes the proof. □

Proof of Theorem 1.1 (ii) and (iii). Fix $\lambda_1/4 < \lambda \leq \lambda_1$. The conclusion for the case $b = 0$ is in [7]. Let us consider the case $b > 0$. Take $B = B(\lambda) > 0$ from lemma 3.3. For all $0 < b < B$, we shall ensure the followings.

(1) There exist constants $\alpha, \rho > 0$ such that $I_b(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$.
(2) There exists a function $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| > \rho$ and $I_b(v_0) \leq 0$.

We confirm (1). Let $\rho > 0$ and $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. Then the Poincare inequality and the Sobolev embedding imply

$$\begin{aligned} I_b(u) &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^6 \\ &\geq \frac{b}{4}\rho^4 - C\rho^6 \end{aligned}$$

for some constant $C > 0$, where for the last inequality we use the assumption $\lambda \leq \lambda_1$. Noting $b > 0$, we obtain the conclusion. We can also prove (2) as usual. Now we define

$$\Gamma_b := \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0\ ,\ \gamma(1) = v_0\}$$

and

$$c_b := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_b(\gamma(t)).$$

Noting Lemma 3.3 and choosing v_0 appropriately, we get

$$c_b < \frac{1}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3.$$

Finally let us conclude the $(\mathrm{PS})_{c_b}$ condition for I_b. Let $\{u_j\} \subset H_0^1(\Omega)$ be a $(\mathrm{PS})_{c_b}$ sequence. Then by Lemma 3.1 and Remark 3.2, we can assume that there exists a nonnegative function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_j \to u \text{ in } L^2(\Omega), \\ &(u_j)_+ \to u \text{ in } L^6(\Omega). \end{aligned}$$

Furthermore since $I_b'(u_j) \to 0$ in $H^{-1}(\Omega)$, we have

$$\langle I_b'(u_j), u_j - u \rangle = o(1)$$

where $o(1) \to 0$ as $j \to \infty$. Then the $L^2(\Omega)$ convergence of $\{u_j\}$ and the $L^6(\Omega)$ convergence of $\{(u_j)_+\}$ show

$$(1 + b\|u_j\|^2) \int_\Omega \nabla u_j \cdot \nabla(u_j - u)dx = o(1) \text{ as } j \to \infty.$$

Finally by the weak convergence, we obtain $u_j \to u$ in $H_0^1(\Omega)$. The mountain pass theorem concludes the proof. □

4. The case $\lambda > \lambda_1$

In this section we prove Theorem 1.1 (iv). We remark that for the conclusion, it is enough if we assume Ω is a bounded domain with sufficiently smooth boundary. Fix $\lambda > \lambda_1$. As in the previous section, we define the energy functional,

$$I_b(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega u_+^2 dx - \frac{1}{6}\int_\Omega u_+^6 dx.$$

We shall prove the existence of a nontrivial critical point, in particular a minimal point, of I_b. To this aim, inspired by [24], we apply the method of the Nehari manifold. We also refer to the original work [25]. To the beginning, we define

$$\Lambda := \{u \in H_0^1(\Omega) \mid \langle I_b'(u), u\rangle = 0\},$$

and split Λ into 3 parts,

$$\Lambda^+ := \{u \in \Lambda \mid b\|u\|^4 > 2\int_\Omega u_+^6 dx\}, \tag{9}$$

$$\Lambda^0 := \{u \in \Lambda \mid b\|u\|^4 = 2\int_\Omega u_+^6 dx\}, \tag{10}$$

$$\Lambda^- := \{u \in \Lambda \mid b\|u\|^4 < 2\int_\Omega u_+^6 dx\}. \tag{11}$$

Now we choose

$$b > \max\left\{4\left(\frac{\lambda}{\lambda_1} - 1\right)^{\frac{1}{2}} S^{-\frac{3}{2}}, 4\left(\frac{\lambda}{\lambda_1} - 1\right) S^{-3}\right\}, \tag{12}$$

where S is the Sobolev constant as defined in previous section. Since we fix $b > 0$ as above, we denote I_b as I for simplicity. Let us begin with the following lemmas.

Lemma 4.1. $\Lambda^+, \Lambda^- \neq \emptyset$ *and* $\Lambda^0 = \{0\}$.

Proof. We consider the principal eigenfunction $\phi_1 > 0$ of $-\Delta$ on Ω with $\|\phi_1\| = 1$. Then for $t > 0$, we put a function

$$f(t) := -\left(\frac{\lambda}{\lambda_1} - 1\right)t^2 + bt^4 - \left(\int_\Omega \phi_1^6 dx\right)t^6$$

Noting the fact $\int_\Omega \phi_1^6 dx \le S^{-3}$ and (12), there exist constants $0 < t^- < t^+$ such that $f(t^\pm) = 0$ and $f'(t^\pm) \lessgtr 0$. This implies $t^\pm\phi_1 \in \Lambda^\mp$. In addition, we assume that there exists a nontrivial function $u \in \Lambda^0$ to the contrary. Firstly suppose $\|u\|^2 \ge \lambda \int_\Omega u_+^2 dx$. Then we obtain by (10),

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda \int_\Omega u_+^2 dx - \int_\Omega u_+^6 dx \\ &\ge \frac{b}{2}\|u\|^4. \end{aligned}$$

This is impossible. Next we assume $\|u\|^2 < \lambda \int_\Omega u_+^2 dx$. Then (12) and the Sobolev and the Poincare inequalities implies

$$\begin{aligned} 0 &< \left(\frac{b^2}{4\left(\frac{\lambda}{\lambda_1} - 1\right) S^{-3}} - 1 \right) \int_\Omega u_+^6 dx \\ &\le \frac{b^2\|u\|^8}{4\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2} - \int_\Omega u_+^6 dx \\ &\le \frac{b^2\|u\|^8}{4\left(\lambda \int_\Omega u_+^2 dx - \|u\|^2\right)} - \frac{b}{2}\|u\|^4 \\ &= 0, \end{aligned}$$

where for the last equality, we use (10). This is impossible. This concludes the proof. □

Lemma 4.2. *For any $u \in \Lambda^+$, there holds*

$$\|u\| < \left(\frac{S^{\frac{3}{2}}}{2} \right)^{\frac{1}{2}}.$$

Proof. For all $u \in \Lambda^+$, we have by the Poincare inequality and (9),

$$\begin{aligned} 0 &= \|u\|^2 + b\|u\|^4 - \lambda \int_\Omega u_+^2 dx - \int_\Omega u_+^6 dx \\ &> \frac{b}{2}\|u\|^2 \left\{ \|u\|^2 - \frac{2}{b}\left(\frac{\lambda}{\lambda_1} - 1 \right) \right\}. \end{aligned}$$

Noting our assumption $\lambda > \lambda_1$ and (12), we get the conclusion. □

Lemma 4.3. *For all $u \in \Lambda^+$, there exists a constant $\varepsilon > 0$ and a C^1 functional t on $B(0,\varepsilon) \subset H_0^1(\Omega)$ such that $t(0) = 1$, $t(w) > 0$, $t(w)(u - w) \in \Lambda$ for all $w \in B(0,\varepsilon)$, and further,*

$$\langle t'(0), h \rangle = \frac{(1 + 2b\|u\|^2)\int_\Omega \nabla u \cdot \nabla h dx - \lambda \int_\Omega u_+ h dx - 3\int_\Omega u_+^5 h dx}{b\|u\|^4 - 2\int_\Omega u_+^6 dx}, \tag{13}$$

for all $h \in H_0^1(\Omega)$.

Proof. For all $u \in \Lambda^+$, we define a C^1 map,

$$g(w,t) := \|u-w\|^2 + bt^2\|u-w\|^4 - \lambda \int_\Omega (u-w)_+^2 dx - t^4 \int_\Omega (u-w)_+^6 dx.$$

Then by (9), we can easily verify that

$$g(0,1) = 0 \text{ and } g_t(0,1) \neq 0.$$

The implicit function theorem concludes the proof. □

Lemma 4.4. $-\infty < \inf_{u\in\Lambda^+\cup\Lambda^0} < 0$

Proof. For all $u \in \Lambda^+ \cup \Lambda^0$, using the Poincare inequality we get

$$\begin{aligned} I(u) &= I(u) - \frac{1}{6}\langle I'(u), u\rangle \\ &\geq -\frac{1}{3}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u\|^2 + \frac{b}{12}\|u\|^4. \end{aligned} \tag{14}$$

On the other hand, we fix a function $u \in \Lambda^+$. Then we have from (9),

$$I(u) < -\frac{b}{12}\|u\|^4. \tag{15}$$

(14) and (15) ensure the proof. □

Now we put $c_0 := \inf_{u\in\Lambda^+\cup\Lambda^0} I(u)$. From the Ekeland variational principle, there exists a sequence $\{u_n\} \subset \Lambda^+ \cup \Lambda^0$ such that

$$I(u_n) \leq \frac{1}{n} + c_0 \text{ and } I(w) > I(u_n) - \frac{1}{n}\|w-u_n\| \ (w \in \Lambda^+ \cup \Lambda^0). \tag{16}$$

Lemma 4.5. *Let $\{u_n\}$ be given as above. Then*

$$I'(u_n) \to 0 \text{ in } H^{-1}(\Omega), \tag{17}$$

up to subsequences.

Proof. We first claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Actually noting (16), we have similarly to (14),

$$\begin{aligned} c_0 + 1 &\geq I(u_n) - \frac{1}{6}\langle I'(u_n), u_n\rangle \\ &\geq -\frac{1}{3}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u_n\|^2 + \frac{b}{12}\|u_n\|^4. \end{aligned}$$

for large $n \in \mathbb{N}$. Since $b > 0$, this inequality implies the claim. Then by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embeddings, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ &u_n \to u \text{ in } L^2(\Omega), \\ &u_n \to u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences but still denoted $\{u_n\}$. Next we propose that $u \neq 0$. If not, since $\{u_n\} \subset \Lambda^+ \cup \Lambda^0$, by the $L^2(\Omega)$ convergence, we have

$$o(1) = \|u_n\|^2 + b\|u_n\|^4 - \int_\Omega (u_n)_+^6 dx$$
$$\geq \|u_n\|^2 + \frac{b}{2}\|u_n\|^4.$$

It follows that $u_n \to 0$ in $H_0^1(\Omega)$. This is a contradiction. Consequently we can assume $\{u_n\} \subset \Lambda^+$. Now let us conclude (17). To this end, we follow the argument in the proof of Theorem 1 in [25]. We assume $\|I'(u_n)\| > 0$ for large $n \in \mathbb{N}$. For such a $n \in \mathbb{N}$ and $u_n \in \Lambda^+$, we take a constant $\varepsilon > 0$ and a C^1 functional t on $B(0, \varepsilon) \subset H_0^1(\Omega)$ from Lemma 4.3. For all $0 < \delta < \varepsilon$, we define $t_n(\delta) := t(\delta I'(u_n)/\|I'(u_n)\|)$ and

$$w_\delta := t_n(\delta)\left(u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right).$$

Note that $w_\delta \in \Lambda^+$ for sufficiently small $\delta > 0$. Recalling (16), we have

$$\frac{1}{n}\|w_\delta - I(u_n)\| > I(w_\delta) - I(u_n)$$
$$= (1 - t_n(\delta))\langle I'(w_\delta), u_n\rangle + \delta t_n(\delta)\langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle + o(\delta).$$

Dividing by $\delta > 0$ and taking $\delta \to 0$, we get

$$\frac{C(|t_n'(0)| + 1)}{n} \geq -\langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle\langle I'(u_n), u_n\rangle + \langle I'(u_n), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle$$
$$= \|I'(u_n)\|,$$

for some constant $C > 0$, where $|t_n'(0)| := |\langle t_n'(0), I'(u_n)/\|I'(u_n)\|\rangle|$. Thus the proof is done once we confirm that $|t_n'(0)|$ is bounded. Let us show this. From (13), there exists a constant $C > 0$ which is independent of $n \in \mathbb{N}$ such that

$$|t_n'(0)| \leq \frac{C}{b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx}.$$

We claim that we can extract a subsequence so that

$$b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx > C$$

for some constant $C > 0$. To confirm this, it is enough to show that

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) > C \tag{18}$$

for a value $C > 0$. Since $\{u_n\} \subset \Lambda^+$, obviously

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) \geq 0.$$

Now we suppose

$$\lim_{n\to\infty}\left(b\|u_n\|^4 - 2\int_\Omega (u_n)_+^6 dx\right) = 0 \tag{19}$$

to the contrary. Then since $\{u_n\}$ is bounded, we can assume that there exists a constant $B > 0$ such that

$$\|u_n\|^2 \to B \tag{20}$$

and thus by (19),

$$\int_\Omega (u_n)_+^6 dx \to \frac{b}{2}B^2. \tag{21}$$

In addition as $\{u_n\} \subset \Lambda^+$, (20) and (21) shows

$$\lambda \int_\Omega u_+^2 dx = B + \frac{b}{2}B^2. \tag{22}$$

Again using (12) and applying the Sobolev and the Poincare inequalities, we have

$$\begin{aligned} 0 &< \left(\frac{b^2}{16\left(\frac{\lambda}{\lambda_1} - 1\right)S^{-3}} - 1\right)\int_\Omega (u_n)_+^6 dx \\ &= \frac{b^2\|u_n\|^8}{16\left(\frac{\lambda}{\lambda_1} - 1\right)\|u_n\|^2} - \int_\Omega (u_n)_+^6 dx \\ &\leq \frac{b^2\|u_n\|^8}{16\left(\lambda\int_\Omega u_+^2 dx - \|u_n\|^2\right)} - \int_\Omega (u_n)_+^6 dx. \end{aligned}$$

Taking $n \to \infty$ and noting (20), (21) and (22), we conclude

$$0 \leq -\frac{3bB^2}{8}.$$

This is impossible. Thus (18) holds. This completes the proof. □

Finally we ensure the existence of a nontrivial critical point of I.

Lemma 4.6. *Let $\{u_n\}$ be a minimizing sequence of c_0 as in the paragraph above Lemma 4.5. Then there exists a nontrivial critical point $u \in \Lambda^+$ of I.*

Proof. Our argument is based on that in [24]. As in the proof of Lemma 4.5, we have that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then by the second concentration compactness lemma, there exist a nonnegative function $u \in H_0^1(\Omega)$, an at most countable set J, points $\{x_k\}_{k\in J} \subset \overline{\Omega}$, and values $\{\mu_k\}_{k\in J}, \{\nu_k\}_{k\in J} \subset \mathbb{R}^+$ with $S\nu_k^{1/3} \leq \mu_k$ for all $k \in J$ such that up to subsequences,

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u| + \sum_{k\in J}\mu_k\delta_{x_k}$$

$$(u_n)_+^6 \rightharpoonup d\nu = u^6 + \sum_{k\in J}\nu_k\delta_{x_k}$$

in the measure sense. Similarly to the proof of Lemma 3.1, we shall show $J = \emptyset$. To do this, we assume $J \neq \emptyset$. Choose $k \in J$. Define a smooth cut off function ϕ such that $\phi = 1$ on $B(x_k, \varepsilon)$, $\phi = 0$ on $B(x_k, 2\varepsilon)^c$ and $0 \leq \phi \leq 1$ otherwise. Moreover we suppose $|\nabla \phi| \leq 2/\varepsilon$. Since $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ by Lemma 4.5, we estimate similarly to (7),

$$\mu_k \geq S^{\frac{3}{2}}.$$

This inequality implies

$$\|u_n\|^2 \geq S^{\frac{3}{2}} + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. But since $\{u_n\} \subset \Lambda^+$, there holds

$$\|u_n\|^2 \leq \frac{S^{\frac{3}{2}}}{2},$$

by Lemma 4.2. This is a contradiction. It follows that

$$(u_n)_+ \to u \text{ in } L^6(\Omega).$$

Then analogously with the proof of Theorem 1.1, we have

$$u_n \to u \text{ in } H_0^1(\Omega).$$

As a consequence, u is a critical point of I and clearly $u \in \Lambda^+ \cup \Lambda^0$. Furthermore since $u \neq 0$, Lemma 4.1 concludes that $u \in \Lambda^+$. This finishes the proof. □

The proof of Theorem 1.1 (iv). Let $\lambda > \lambda_1$. We put

$$B_4 := \max \left\{ 4 \left(\frac{\lambda}{\lambda_1} - 1 \right)^{\frac{1}{2}} S^{-\frac{3}{2}}, 4 \left(\frac{\lambda}{\lambda_1} - 1 \right) S^{-3} \right\}.$$

Then from Lemma 4.6, (P) has a nontrivial weak solution for all $b > B_4$. This completes the proof. □

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