LOCAL ASYMPTOTIC NONDEGENERACY FOR MULTI-BUBBLE SOLUTIONS TO THE BIHARMONIC LIOUVILLE-GEL'FAND PROBLEM IN DIMENSION FOUR

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Dedicated to Professor Takashi Suzuki on the occasion of his sixties birthday

Abstract. We consider the biharmonic Liouville-Gel'fand problem under the Navier boundary condition in four space dimension. Under the nondegeneracy assumption of blow up points of multiple blowing-up solutions, we prove several estimates for the linearized equations and obtain some convergence result. The result can be seen as a weaker version of the asymptotic nondegeneracy of multi-bubble solutions, which was recently established by Grossi-Ohtsuka-Suzuki in two-dimensional Laplacian case.

1. Introduction.

In this paper, we consider the fourth order Liouville-Gel'fand problem with the Navier boundary conditions

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain, and $\lambda > 0$ is a parameter. Let $\{u_n\}$ be a solution sequence to (1.1) for $\lambda = \lambda_n \downarrow 0$ as $n \to \infty$. As for the asymptotic behavior of the solution sequence $\{u_n\}$, several studies have been done ([14], see [11] for the second order case and [9] for more general polyharmonic cases), and we have the following picture of the bubbling behavior of blowing-up solutions.

Proposition 1.1 ([14]). *Let λ_n be a sequence of positive numbers with $\lambda_n \downarrow 0$. Let $\Sigma_n = \lambda_n \int_\Omega e^{u_n} dx$ where u_n is a solution to (1.1) for $\lambda = \lambda_n$. Then as $n \to \infty$, there are three possibilities.*

Case(1): $\{\Sigma_n\}$ accumulates to 0. In this case, $\|u_n\|_{L^\infty(\Omega)} \to 0$.

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Case(2): $\{\Sigma_n\}$ accumulates to $64\pi^2 m$ for some $m \in \mathbb{N}$. In this case, u_n makes m-point blow ups, i.e., there exists a set $\mathcal{S} = \{p_1, \cdots, p_m\}$ which consists of m-interior points such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega \subset\subset \overline{\Omega} \setminus \mathcal{S}$, $\{u_n(x)\}$ has a limit for any $x \in \overline{\Omega} \setminus \mathcal{S}$ while $u_n|_{\mathcal{S}} \to +\infty$ as $n \to \infty$.

Case(3):$\{\Sigma_n\}$ accumulates to $+\infty$. In this case, $u_n(x) \to +\infty$ for all $x \in \Omega$.

Furthermore, in the Case(2), the limit function $u_\infty(x) = \lim_{n\to\infty} u_n(x)$ has the form

$$u_\infty(x) = 64\pi^2 \sum_{i=1}^m G(x, p_i) \tag{1.2}$$

where $G(x,z)$ denotes the Green function of Δ^2 under the Navier boundary condition

$$\begin{cases} \Delta^2 G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = \Delta G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Moreover, the blow up points $p_i \in \mathcal{S}$ must satisfy the following relation

$$\frac{1}{2}\nabla R(p_i) + \sum_{\substack{j=1 \\ j\neq i}}^m \nabla_x G(p_i, p_j) = 0, \quad (i = 1, \cdots, m) \tag{1.4}$$

where $H(x,z) = G(x,z) - \frac{1}{8\pi^2}\log|x-z|^{-1}$ denotes the regular part of G, and $R(x) = H(x,x)$ is the Robin function associated to G.

Note that, if we introduce the function

$$\mathcal{F} : \Omega^m = \overbrace{\Omega \times \cdots \times \Omega}^{m \text{ times}} \to \mathbb{R} \cup \{\pm\infty\},$$
$$\mathcal{F}(\xi_1, \cdots, \xi_m) = \frac{1}{2}\sum_{i=1}^m R(\xi_i) + \frac{1}{2}\sum_{\substack{1\le i,j\le m, \\ i\neq j}} G(\xi_i, \xi_j), \tag{1.5}$$

then the relation (1.4) is just saying that a point $(p_1, \cdots, p_m)$ with $p_i \in \mathcal{S}$ $(1 \le i \le m)$ is a critical point of $\mathcal{F}$ in Ω^m. The function $\mathcal{F}$ is called the *Hamiltonian* associated to the problem (1.1). Conversely, some existence results of the actual multiple-blowing up solutions to (1.1) are obtained by several authors [1] [3]. In particular, if $(p_1, \cdots, p_m)$ is a nondegenerate critical point of $\mathcal{F}$, then there exists a blowing-up solutions to (1.1) which blows up exactly at $\{p_1, \cdots, p_m\} \subset \Omega$, see [1]. We are interested in some qualitative properties of the multiple blowing-up solution u_n.

In order to state our result in this paper, we need some definitions. Let $\{u_n\}$ be a sequence of solutions to (1.1) for $\lambda = \lambda_n$ satisfying Case (2) of Proposition 1.1 as $n \to \infty$. Let $\mathcal{S} = \{p_1, \cdots, p_m\}$ be a blow up set of the sequence $\{u_n\}$. Then we have sufficiently small $\rho > 0$ and m sequences $\{x_n^i\}$ such that for each $p_i \in \mathcal{S}$,

$$u_n(x_n^i) = \max_{B_\rho(x_n^i)} u_n(x) \to \infty, \quad x_n^i \to p_i \quad (i = 1, \cdots, m)$$

as $n \to \infty$. Also let v_n be the solution to the linearized problem around u_n for $n \in \mathbb{N}$:

$$\begin{cases} \Delta^2 v_n = \lambda_n e^{u_n} v_n & \text{in } \Omega, \\ v_n = \Delta v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Recall that if (1.6) admits only the trivial solution $v_n \equiv 0$, the solution u_n of (1.1) is said *nondegenerate*. Let δ_n^i be the positive number so that

$$(\delta_n^i)^4 \lambda_n e^{u_n(x_n^i)} \equiv 1$$

and define the scaled function

$$\tilde{v}_n^i(y) = v_n(\delta_n^i y + x_n^i), \quad \text{for } y \in B_{\frac{\rho}{\delta_n^i}}(0)$$

for $i = 1, \cdots, m$ and $n \in \mathbb{N}$. Now, our result reads as follows:

Theorem 1.2. *Let $\{u_n\}$ be a multiple blowing-up solution sequence to (1.1) for $\lambda = \lambda_n$ whose set of blow up points is $\mathcal{S} = \{p_1, \cdots, p_m\}$. If $(p_1, \cdots, p_m) \in \Omega^m$ is a nondegenerate critical point of the Hamiltonian function $\mathcal{F}$, then by choosing a subsequence, we have,*

$$\tilde{v}_n^i \to 0 \quad \text{in } C^4_{loc}(\mathbb{R}^4) \quad \text{as } n \to \infty. \tag{1.7}$$

The result in Theorem 1.2 strongly suggests that the sequence $\{u_n\}$ is *asymptotically nondegenerate*, that is, $v_n \equiv 0$ for all large $n \in \mathbb{N}$. Unfortunately, we do not have the proof of this statement up to now, and we remain it as a future work. In the final section of this paper, we include some discussions and remarks on this issue. In this paper we call the property (1.7) as *local asymptotic nondegeneracy* of the multi-bubble solutions $\{u_n\}$ to (1.1).

Asymptotic nondegeneracy results of solutions to the second order Liouville-Gel'fand problem in two dimension can be found in the papers by Gladiali-Grossi [4] (one blow-up point case) and Grossi-Ohtsuka-Suzuki [6], Ohtsuka-Sato-Suzuki [12] (multiple blow-up points case). However, our fourth order Liouville-Gel'fand problem in four dimension is quite different from the second order case in several technical points. One of the main difficulties comes from the fact that the Kelvin transformation for the biharmonic operator Δ^2 does not preserve the

Navier boundary conditions, and this makes our result weaker than that for the second order case. In addition, the classification theorem by C.S. Lin [8] to the limit equation needs more restrictive growth assumption at infinity in our case. Such growth rate for a solution to the limit equation can be proved by following the argument by Ben Ayed-El Mehdi-Grossi [2], see Proposition 3.1. We also note that Theorem 1.2 applies to the multiple-blowing up solutions of (1.1) obtained in [1].

2. An integral identity for the Green function.

In this section, we prove an integral identity for the Green function with the Navier boundary condition, which is useful in the later.

First we recall an elementary integration by parts formula for Δ^2. For smooth functions g, h in a smooth subdomain $\omega \subset \Omega$, we have

$$\int_\omega \left((\Delta^2 g)h - g(\Delta^2 h)\right) dx = B_{\partial\omega}\left[g,h\right], \tag{2.1}$$

where

$$B_{\partial\omega}\left[g,h\right] = \int_{\partial\omega} \left(\left(\frac{\partial \Delta g}{\partial\nu}\right) h - \Delta g \left(\frac{\partial h}{\partial \nu}\right) + \left(\frac{\partial g}{\partial \nu}\right)\Delta h - g\left(\frac{\partial \Delta h}{\partial\nu}\right)\right) ds_x.$$

Here $\nu = \nu(x)$ denotes the unit outer normal at $x \in \partial\omega$. We note that $B_{\partial\omega}\left[g,h\right] = -B_{\partial\omega}\left[h,g\right]$ and $B_{\partial\omega}\left[g,h\right] = 0$ if $\Delta^2 g = \Delta^2 h = 0$ in ω.

Let $G = G(x,z)$ denote the Green function of Δ^2 under the Navier boundary condition in (1.3). We decompose G as

$$G(x,z) = N(|x-z|) + H(x,z),$$

where $N(r) = \frac{1}{8\pi^2}\log r^{-1}$ and $H(x,z)$ is the regular part of G.

Proposition 2.1. *For $a,b,c \in \Omega$ and $r > 0$ small such that $\overline{B_r(a)} \subset \Omega$, $b,c \notin \overline{B_r(a)}$ if $b,c \neq a$, put*

$$I_{ij}(a,b,c) = B_{\partial B_r(a)}\left[G_{x_i}(\cdot,b), G_{z_j}(\cdot,c)\right]$$

for $1 \le i,j \le 4$, where $G_{x_i}(x,z) = \frac{\partial G}{\partial x_i}(x,z)$ etc. Then I_{ij} does not depend on $r > 0$ small and we have

$$\begin{cases} I_{ij}(a,b,c) = 0, \quad \text{if } a \neq b \text{ and } a \neq c \\ I_{ij}(a,a,a) = -\frac{1}{2}R_{x_i x_j}(a), \\ I_{ij}(a,a,c) = -G_{x_i z_j}(a,c), \quad \text{if } a \neq c \\ I_{ij}(a,b,a) = -G_{x_i x_j}(a,b), \quad \text{if } a \neq b. \end{cases} \tag{2.2}$$

Proof. First, we see that $I_{ij}(a,b,c)$ does not depend on the choice of $r > 0$. This is because all functions in the integrand are smooth, biharmonic on an annular domain $B_{r_1}(a) \setminus B_{r_2}(a)$ for $r_1 > r_2$ and obviously

$$\begin{aligned} B_{\partial B_{r_1}(a)}[g,h] &= B_{\partial B_{r_2}(a)}[g,h] + B_{\partial\left(B_{r_1}(a)\setminus B_{r_2}(a)\right)}[g,h] \\ &= B_{\partial B_{r_2}(a)}[g,h] \end{aligned}$$

if $\Delta^2 g = \Delta^2 h = 0$ on $B_{r_1}(a) \setminus B_{r_2}(a)$. Similarly we obtain $I_{ij}(a,b,c) = 0$ when $a \neq b$ and $a \neq c$ because $G_{x_i}(\cdot,b)$ and $G_{z_j}(\cdot,c)$ are smooth biharmonic functions on $B_r(a)$ in this case.

We note that for a smooth function h on ω, $\Delta^2 G(\cdot,z) = \delta_z$ implies

$$h(z) = \int_\omega G(x,z)\Delta^2 h(x)dx + B_{\partial\omega}[G(\cdot,z),h(\cdot)] \tag{2.3}$$

for $z \in \omega$ (the Green's third identity for Δ^2, see [7] for example). Therefore

$$h_{x_j}(x)|_{x=z} = h_{z_j}(z) = \int_\omega G_{z_j}(x,z)\Delta^2 h(x)dx + B_{\partial\omega}\left[G_{z_j}(\cdot,z),h(\cdot)\right].$$

Oh the other hand, since $\Delta^2 G_{x_i}(x,z) = \partial_{x_i}\delta_z$, we have

$$-h_{x_i}(x)|_{x=z} = \int_\omega G_{x_i}(x,z)\Delta^2 h(x)dx + B_{\partial\omega}[G_{x_i}(\cdot,z),h(\cdot)].$$

Consequently we get

$$B_{\partial B_r(a)}\left[h(\cdot),G_{z_j}(\cdot,a)\right] = \int_{B_r(a)} G_{z_j}(x,a)\Delta^2 h(x)dx - h_{x_j}(a), \tag{2.4}$$

$$B_{\partial B_r(a)}[G_{x_i}(\cdot,a),h(\cdot)] = -\int_{B_r(a)} G_{x_i}(x,z)\Delta^2 h(x)dx - h_{x_i}(a) \tag{2.5}$$

for every smooth h. Therefore we get $I_{ij}(a,b,a) = -G_{x_i x_j}(a,b)$ when $a \neq b$ and $I_{ij}(a,a,c) = -G_{x_i z_j}(a,c)$ by inserting $h(x) = G_{x_i}(\cdot,b)$ to (2.4) and $h(x) = G_{z_j}(\cdot,c)$ to (2.5), respectively.

When $a = b = c$, we divide I_{ij} as follows:

$$\begin{aligned} I_{ij}(a,a,a) = {} & B_{\partial B_r(a)}\left[N(|x-a|)_{x_i}, N(|x-a|)_{z_j}\right] \\ & + B_{\partial B_r(a)}\left[N(|x-a|)_{x_i}, H_{z_j}(x,a)\right] + B_{\partial B_r(a)}\left[H_{x_i}(x,a), G_{z_j}(x,a)\right]. \end{aligned}$$

Here

$$B_{\partial B_r(a)}\left[H_{x_i}(x,a), G_{z_j}(x,a)\right] = -H_{x_i x_j}(a,a)$$

from (2.4). On the other hand, it is easy to see that (2.5) (and (2.4) as well) holds for $N(|x-z|)$ instead of $G(x,z)$. Therefore

$$B_{\partial B_r(a)}\left[N(|x-b|)_{x_i}, H_{z_j}(x,a)\right] = -H_{x_i z_j}(a,a).$$

Finally we get

$$B_{\partial B_r(a)}\left[N(|x-a|)_{x_i}, N(|x-a|)_{z_j}\right] = -B_{\partial B_r(a)}\left[N(|x-a|)_{x_i}, N(|x-a|)_{x_j}\right] = 0$$

from simple calculations

$$N(|x-a|)_{x_i} = -\frac{1}{8\pi^2}\frac{\nu_i(x)}{r}, \quad \frac{\partial N(|x-a|)_{x_i}}{\partial \nu} = \frac{1}{8\pi^2}\frac{\nu_i(x)}{r^2},$$

and

$$\Delta N(|x-a|) = -\frac{1}{4\pi^2}r^{-2}, \quad \Delta N(|x-a|)_{x_i} = -\frac{1}{2\pi^2}\frac{\nu_i(x)}{r^3},$$
$$\frac{\partial \Delta N(|x-a|)_{x_i}}{\partial \nu} = \frac{3}{2\pi^2}\frac{\nu_i(x)}{r^4},$$

where $\nu(x) = \frac{x-a}{|x-a|}$ for $x \in \partial B_r(a)$. □

3. Asymptotic behavior of solutions.

We will prove Theorem 1.2 by an argument as in [4] [6]. Since the result is trivial if $v_n \equiv 0$ for all n large, we assume that the existence of nontrivial solutions v_n to the linearized problem (1.6) for $n \in \mathbb{N}$. We may assume without loss of generality that

$$\|v_n\|_{L^\infty(\Omega)} \equiv 1.$$

Let δ_n^i be the positive number such that

$$(\delta_n^i)^4 \lambda_n e^{u_n(x_n^i)} \equiv 1 \tag{3.1}$$

for $i = 1, \cdots, m$ and $n \in \mathbb{N}$. By the fundamental pointwise estimate for blowing-up solutions to (1.1) due to [10], we have that $\delta_n^i = o(1)$ as $n \to \infty$; see Proposition 3.2 and Corollary 3.4 below. Using δ_n^i, define the rescaled functions

$$\tilde{u}_n^i(y) = u_n(\delta_n^i y + x_n^i) - u_n(x_n^i), \quad y \in B_{\frac{\rho}{\delta_n^i}}(0), \tag{3.2}$$

$$\tilde{v}_n^i(y) = v_n(\delta_n^i y + x_n^i), \quad y \in B_{\frac{\rho}{\delta_n^i}}(0) \tag{3.3}$$

around the local maximum x_n^i of u_n. Note that $\tilde{u}_n^i$ and $\tilde{v}_n^i$ satisfies

$$\begin{cases} \Delta^2 \tilde{u}_n^i(y) = e^{\tilde{u}_n^i(y)}, \quad y \in B_{\frac{\rho}{\delta_n^i}}(0), \\ \tilde{u}_n^i(y) \le \tilde{u}_n^i(0) = 0, \quad y \in B_{\frac{\rho}{\delta_n^i}}(0), \\ -\Delta \tilde{u}_n^i(y) > 0, \quad y \in B_{\frac{\rho}{\delta_n^i}}(0), \\ \int_{B_{\frac{\rho}{\delta_n^i}}(0)} e^{\tilde{u}_n^i(y)} dy = O(1), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \Delta^2 \tilde{v}_n^i(y) = e^{\tilde{u}_n^i(y)} \tilde{v}_n^i(y), \quad y \in B_{\frac{\rho}{\delta_n^i}}(0), \\ \|\tilde{v}_n^i\|_{L^\infty\left(B_{\frac{\rho}{\delta_n^i}}(0)\right)} \le 1, \end{cases}$$

respectively. By the standard elliptic estimates, we have the convergence (by choosing a subsequence if necessary)

$$\tilde{u}_n^i \to U, \quad \tilde{v}_n^i \to V^i \quad \text{as } n \to \infty \tag{3.5}$$

in $C^4_{loc}(\mathbb{R}^4)$ for some U and V^i. We see that U satisfies

$$\begin{cases} \Delta^2 U = e^U \quad \text{in } \mathbb{R}^4, \\ U(0) = \max_{\mathbb{R}^4} U(y) = 0, \\ \int_{\mathbb{R}^4} e^U dy < +\infty, \end{cases} \tag{3.6}$$

and also V^i satisfies

$$\Delta^2 V^i = e^U V^i \quad \text{in } \mathbb{R}^4.$$

Though the proof of this fact is now standard, we give a proof for the reader's convenience. Let $B_R(0)$ be the ball of radius R with center at the origin, and let ω_n be the unique solution of

$$\begin{cases} \Delta^2 \omega_n = e^{\tilde{u}_n^i} (\leq 1) \quad \text{in } B_R(0), \\ \omega_n = \Delta \omega_n = 0 \quad \text{on } \partial B_R(0). \end{cases}$$

By the maximum principle and the elliptic estimates, we have

$$0 < \omega_n \leq C, \quad 0 < -\Delta \omega_n \leq C \quad \text{in } B_R(0)$$

where $C > 0$ is a constant independent of n. Also we have $0 < -\Delta \tilde{u}_n^i \leq C$ in $B_{\frac{\rho}{\delta_n^i}}(0)$. Therefore we obtain

$$0 < -\Delta(\tilde{u}_n^i - \omega_n) \leq C \quad \text{in } B_R(0)$$

for n large such that $B_R(0) \subset B_{\frac{\rho}{\delta_n^i}}(0)$, since $-\Delta(\tilde{u}_n^i - \omega_n)$ is harmonic in $B_R(0)$ and $-\Delta(\tilde{u}_n^i - \omega_n) = -\Delta \tilde{u}_n^i$ on $\partial B_R(0)$. Now, let ϕ_n denote the unique solution of

$$-\Delta \phi_n = -\Delta(\tilde{u}_n^i - \omega_n) > 0 \text{ in } B_R(0), \quad \phi_n = 0 \text{ on } \partial B_R(0)$$

and set

$$\psi_n = \tilde{u}_n^i - \omega_n - \phi_n.$$

The maximum principle implies that $0 < \phi_n \leq C$ in $B_R(0)$ and ψ_n is a non-positive harmonic function on $B_R(0)$. Hence the Harnack alternative implies that

(i) $\psi_n \to -\infty$ uniformly on every compact sets on $B_R(0)$, or
(ii) ψ_n is bounded in $L^\infty_{\text{loc}}(B_R(0))$ if we choose a subsequence.

However, since $\psi_n(0) = -\omega_n(0) - \phi_n(0) \geq -2C$, the case (i) cannot happen. Thus a subsequence of ψ_n is bounded in $L^\infty_{\text{loc}}(B_R(0))$ and since ϕ_n, ω_n are uniformly bounded in $L^\infty(B_R(0))$ as noticed before, a subsequence of $\tilde{u}_n^i$ is also bounded in $L^\infty_{\text{loc}}(B_R(0))$ for any $R > 0$.

After this fact is established, the standard regularity theory assures the convergence of $\tilde{u}_n^i$ in $C^4_{\rm loc}(\mathbb{R}^4)$ to some U. Passing to the limit in (3.4) with the use of Fatou's lemma, we see that U satisfies (3.6). The proof for the convergence of $\tilde{v}_n^i$ is similar.

Here, we claim that

Proposition 3.1. *There holds*

$$|U(y)| = o(|y|^2) \quad as\ |y| \to \infty.$$

Proof. We argue as in the proof of Theorem 1.2 in [2]. See also Lemma 2.2 in [13]. Using a result of C-S. Lin ([8]:Theorem 1.2) applied to a solution of (3.6), we know that U can be represented by

$$U(y) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|z|}{|y-z|}\right) e^{U(z)}dz - 4\sum_{j=1}^4 a_j(y_j - y_j^0)^2 + c_0$$

$$= -4\sum_{j=1}^4 a_j(y_j - y_j^0)^2 - 4\alpha\log|y| + c_0 + O(|y|^{-\tau}) \tag{3.7}$$

for some $\tau > 0$ for $|y|$ large, and

$$\Delta U(y) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{e^{U(z)}}{|y-z|^2}\right) dz - 8\sum_{j=1}^4 a_j \tag{3.8}$$

where $a_j \ge 0$, c_0 are constants, $y^0 = (y_1^0, \cdots, y_4^0) \in \mathbb{R}^4$, and $\alpha = \frac{1}{32\pi^2}\int_{\mathbb{R}^4} e^{U(y)}dy$. Therefore, in order to obtain the desired estimate, it is enough to prove that $a_j = 0$ for all $j = 1, \cdots, 4$ in (3.7), (3.8).

Let

$$\overline{G_n^i}(y,z) = \frac{1}{4\pi^2}|y-z|^{-2} + \overline{H_n^i}(y,z)$$

denote the Green function of $-\Delta$ on the expanding domain $\Omega_n^i = \frac{\Omega - x_n^i}{\delta_n^i}$. Since $\tilde{u}_n^i$ satisfies

$$\begin{cases} \Delta^2 \tilde{u}_n^i = e^{\tilde{u}_n^i} & \text{in } \Omega_n^i, \\ \tilde{u}_n^i = \Delta\tilde{u}_n^i = 0 & \text{on } \partial\Omega_n^i, \end{cases}$$

we have

$$-\Delta\tilde{u}_n^i(0) = \int_{\Omega_n^i} \overline{G_n^i}(0,z) e^{\tilde{u}_n^i(z)}dz.$$

Fix any $R > 0$. For n large, we have

$$\begin{aligned}\int_{\Omega_n^i\setminus B_R(0)} \overline{G_n^i}(0,z)e^{\tilde{u}_n^i(z)}dz &\le \frac{1}{4\pi^2R^2}\int_{\Omega_n^i\setminus B_R(0)} e^{\tilde{u}_n^i(z)}dz \\ &\le \frac{1}{4\pi^2R^2}\int_{\Omega\setminus B_{\delta_n^i R}(x_n^i)} \lambda_n e^{u_n(x)}dx \le \frac{O(1)}{R^2}\end{aligned}$$

as $n \to \infty$. On the other hand, by the estimate

$$0 < -\overline{H_n^i}(y,z) \le -\frac{1}{4\pi^2}\cdot\frac{1}{d(y,\partial\Omega_n^i)^2}$$

for the regular part of $\overline{G_n^i}$ and $d(0,\partial\Omega_n^i) = O(\frac{1}{\delta_n^i})$, we have

$$\begin{aligned}&\int_{B_R(0)} \overline{G_n^i}(0,z)e^{\tilde{u}_n^i(z)}dz = \frac{1}{4\pi^2}\int_{B_R(0)} \frac{e^{\tilde{u}_n^i(z)}}{|z|^2}dz + \int_{B_R(0)} \overline{H_n^i}(0,z)e^{\tilde{u}_n^i(z)}dz \\ &= \frac{1}{4\pi^2}\int_{B_R(0)} \frac{e^{U(z)}}{|z|^2}dz + o(1) + O((\delta_n^i)^2)\end{aligned}$$

as $n \to \infty$. Thus, letting $n \to \infty$ first and then $R \to \infty$, we observe that

$$-\Delta\tilde{u}_n^i(0) \to \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{e^{U(z)}}{|z|^2}dz = -\Delta U(0) - 8\sum_{j=1}^4 a_j$$

where the last equality comes from (3.8). Now, since $-\Delta\tilde{u}_n^i(0) \to -\Delta U(0)$ holds by the $C^4_{loc}(\mathbb{R}^4)$ convergence of $\tilde{u}_n^i$, we obtain that $\sum_{j=1}^4 a_j = 0$, which leads to $a_j = 0$ for all $1 \le j \le 4$. □

By proposition 3.1, we can apply the uniqueness result of U ([8], Theorem 1.1) to (3.6), then we have

$$U(y) = -4\log\left(1 + \frac{|y|^2}{8\sqrt{6}}\right). \tag{3.9}$$

Also by the nondegeneracy result of U ([10], Lemma 2.6.), we obtain that V^i in (3.5) is written as

$$\begin{aligned}\lim_{n\to\infty}\tilde{v}_n^i = V^i &= \sum_{k=1}^4 a_k^i\frac{y_k}{(8\sqrt{6}+|y|^2)} + b^i\left(\frac{8\sqrt{6}-|y|^2}{8\sqrt{6}+|y|^2}\right) \\ &= \sum_{k=1}^4 a_k^i\left(-\frac{1}{8}\frac{\partial U}{\partial y_k}\right) + b^i\frac{1}{4}\left(y\cdot\nabla U + 4\right)\end{aligned} \tag{3.10}$$

for some $\boldsymbol{a}^i = (a_1^i, \cdots, a_4^i) \in \mathbb{R}^4$ and $b^i \in \mathbb{R}$.

Next is the strong pointwise estimate for the blowing-up solutions obtained by Lin-Wei [10].

Proposition 3.2. *([10], Theorem 3.1.) For a fixed $\rho \in (0,1)$, there exists a constant $C > 0$ independent of $i = 1, \cdots, m$ and $n \in \mathbb{N}$ such that*

(3.11)
$$\left| u_n(x) - \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n^{1/2}}{8\sqrt{6}} e^{\frac{1}{2} e^{u_n(x_n^i)}} |x - x_n^i|^2\right)^4} \right| \le C \quad \text{for } x \in B_\rho(x_n^i)$$

holds true.

In terms of $\tilde{u}_n^i$, the above estimate implies

Corollary 3.3. *For a fixed $\rho \in (0,1)$, there exists a constant $C > 0$ independent of i, n such that*

$$|\tilde{u}_n^i(y) - U(y)| \le C \quad \text{for } y \in B_{\frac{\rho}{\delta_n^i}}(0).$$

From this corollary, we have

Corollary 3.4. *For any $i = \{1, \cdots, m\}$, there exists a constant $C_i > 0$ such that, if we choose a subsequence of δ_n^i if necessary, we have*

$$\delta_n^i = C_i \lambda_n^{\frac{1}{4}} + o\left(\lambda_n^{\frac{1}{4}}\right), \tag{3.12}$$

$$u_n(x_n^i) = -2 \log \lambda_n + O(1) \tag{3.13}$$

as $n \to \infty$.

Proof. Since u_n is uniformly bounded for $x \in \partial B_R(x_n^i)$ for small $R > 0$, we have

$$\left| \log \frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n^{1/2}}{8\sqrt{6}} e^{\frac{1}{2} e^{u_n(x_n^i)}} R^2\right)^4} \right| = O(1)$$

by the sup + inf estimate (3.11). Since

$$\frac{e^{u_n(x_n^i)}}{\left(1 + \frac{\lambda_n^{1/2}}{8\sqrt{6}} e^{\frac{1}{2} e^{u_n(x_n^i)}} R^2\right)^4} = \frac{1}{\left(e^{-\frac{1}{4} u_n(x_n^i)} + \frac{\lambda_n^{1/4}}{8\sqrt{6}} (\delta_n^i)^{-1} R^2\right)^4}$$

by the relation (3.1), this implies there exist constants $c, C > 0$ such that $c \le \lambda_n^{\frac{1}{4}} (\delta_n^i)^{-1} \le C$. Thus we have (3.12) if we choose a subsequence. Also, by (3.1) and (3.12), it holds

$$\lambda_n^2 e^{u_n(x_n^i)} \left(C_i^4 + o(1)\right) = 1$$

which implies (3.13). □

Using above lemmas, we obtain the following key proposition of our argument. See [6] Lemma 2.1 for the second order Laplacian case.

Proposition 3.5. *There exists a subsequence of* $\{v_n\}$ *and* $C_i > 0$ $(i = 1, \cdots, m)$ *such that*

$$\frac{v_n}{\lambda_n^{1/4}} \to 8\pi^2 \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z G(x, p_i), \quad \frac{\Delta v_n}{\lambda_n^{1/4}} \to 8\pi^2 \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z \Delta_x G(x, p_i)$$

in $C^1_{\mathrm{loc}}\left(\overline{\Omega} \setminus \bigcup_{i=1}^m B_{2\rho}(p_i)\right)$ *holds true. Here* $\boldsymbol{a}^i = (a^i_1, \cdots, a^i_4) \in \mathbb{R}^4$ *is as in (3.10).*

To prove Proposition 3.5, we adopt the argument used in the proof of [5, Proposition 6.4]. First we decompose v_n as follows:

$$\begin{aligned} v_n(x) &= \int_\Omega G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz \\ &= \sum_{i=1}^m \int_{B_\rho(x_n^i)} G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz + \int_{\Omega \setminus \cup_{i=1}^m B_\rho(x_n^i)} G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz \\ &=: \sum_{i=1}^m \psi_n^i + \psi_n^0. \end{aligned}$$

Also we have $\Delta v_n(x) = \sum_{i=1}^m \Delta\psi_n^i + \Delta\psi_n^0$, where

$$\begin{aligned} \Delta\psi_n^i &= \int_{B_\rho(x_n^i)} \Delta_x G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz \quad (i = 1, \cdots, m), \\ \Delta\psi_n^0 &= \int_{\Omega \setminus \cup_{i=1}^m B_\rho(x_n^i)} \Delta_x G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz. \end{aligned}$$

Recall u_n is bounded outside of points $p_1, .., p_m$, then we derive $\|\psi_n^0\|_{L^\infty(\Omega)} = O(\lambda_n)$ and $\|\Delta\psi_n^0\|_{L^\infty(\Omega)} = O(\lambda_n)$. Therefore we get

$$\frac{\psi_n^0}{\lambda_n^{1/4}} = O\left(\lambda_n^{3/4}\right) = o(1) \quad \text{and} \quad \frac{\Delta\psi_n^0}{\lambda_n^{1/4}} = O\left(\lambda_n^{3/4}\right) = o(1) \tag{3.14}$$

uniformly in $\overline{\Omega}$. Here we prove

Proposition 3.6. *As* $n \to \infty$,

$$\begin{aligned} \psi_n^i(x) &= G(x, x_n^i)\gamma_n^i + 8\pi^2 \boldsymbol{a}^i \cdot \nabla_z G(x, x_n^i)\delta_n^i + o(\delta_n^i), \\ \Delta\psi_n^i(x) &= \Delta_x G(x, x_n^i)\gamma_n^i + 8\pi^2 \boldsymbol{a}^i \cdot \nabla_z \Delta_x G(x, x_n^i)\delta_n^i + o(\delta_n^i) \end{aligned}$$

as $n \to \infty$ *holds uniformly for all* $x \in \overline{\Omega} \setminus B_\rho(x_n^i)$ *for each* i, *where*

$$\gamma_n^i = \int_{B_\rho(x_n^i)} \lambda_n e^{u_n(z)} v_n(z)dz.$$

Proof. For simplicity, we shall omit i in several characters, e.g., x_n as x_n^i, ψ_n as $\psi_n^i, \cdots$. Without loss of generality, we may assume $p_i = 0$. We prove the formula for ψ_n only since the proof for $\Delta\psi_n$ is exactly the same. Taking sufficiently small $\varepsilon > 0$ determined later, we divide ψ_n into two parts:

$$\psi_n(x) = \int_{B_\rho(x_n)} G(x,z)\lambda_n e^{u_n(z)} v_n(z)dz = \int_{B_\rho(x_n)\setminus B_\varepsilon(x_n)} + \int_{B_\varepsilon(x_n)} =: I_1 + I_2$$

Proposition 3.2 implies

$$0 \le e^{u_n(z)} \le \frac{Ce^{u_n(x_n)}}{\left(1 + \frac{\lambda_n^{\frac{1}{2}}}{8\sqrt{6}} e^{\frac{1}{2}u_n(x_n)}|z-x_n|^2\right)^4} \quad \text{in } B_\rho(x_n). \tag{3.15}$$

Then it follows that

$$\begin{aligned}
|I_1| &\le \int_{B_\rho(x_n)\setminus B_\varepsilon(x_n)} |G(x,z)|\lambda_n \frac{Ce^{u_n(x_n)}}{\left(1 + \frac{\lambda_n^{\frac{1}{2}}}{8\sqrt{6}} e^{\frac{1}{2}u_n(x_n)}|z-x_n|^2\right)^4} dz \\
&\le \frac{C'\delta_n^{-4}}{\left(1 + \frac{1}{8\sqrt{6}}\frac{\varepsilon^2}{\delta_n^2}\right)^4}
\end{aligned}$$

for $C' = C\sup_{x\in\Omega}\int_\Omega |G(x,z)|dz$, and hence

$$|I_1| \le \frac{C'}{\left(\delta_n + \frac{1}{8\sqrt{6}}\frac{\varepsilon^2}{\delta_n}\right)^4} = \delta_n \frac{C'}{\left(\delta_n^{\frac{5}{4}} + \frac{1}{8\sqrt{6}}\varepsilon^2\delta_n^{-\frac{3}{4}}\right)^4} = o(\delta_n)$$

if we choose $\varepsilon_n = \delta_n^k$ for some k satisfying $2k - \frac{3}{4} < 0$, that is, $0 < k < \frac{3}{8}$. Henceforth such k is fixed.

For every $x \in \overline{\Omega}\setminus B_\rho(x_n)$ and $z \in B_\varepsilon(x_n)$, Taylor's theorem guarantees

$$G(x,z) = G(x,x_n) + \nabla_z G(x,x_n)\cdot(z-x_n) + s(x,\eta,z-x_n)$$

with

$$s(x,\eta,z-x_n) = \frac{1}{2}\sum_{1\le j,l\le 4} G_{z_j z_l}(x,\eta)(z-x_n)_j(z-x_n)_l$$

and $\eta = \eta(n,z) \in B_\varepsilon(x_n)$. Since $\varepsilon = \delta_n^k \to 0$ as $n \to \infty$, we have $B_\varepsilon(x_n) \Subset B_\rho(x_n)$ for $n \gg 1$, and hence we can apply this formula to

I_2. Thus

$$
\begin{aligned}
I_2 &= G(x,x_n)\int_{B_\varepsilon(x_n)} \lambda_n e^{u_n(z)} v_n(z)dz \\
&\quad + \nabla_z G(x,x_n)\cdot \int_{B_\varepsilon(x_n)} (z-x_n)\lambda_n e^{u_n(z)} v_n(z)dz \\
&\quad + \int_{B_\varepsilon(x_n)} s(x,\eta,z-x_n)\lambda_n e^{u_n(z)} v_n(z)dz \\
&=: I_{2,1}+I_{2,2}+I_{2,3}
\end{aligned}
$$

Using the estimate of I_1, we obtain

$$I_{2,1} = G(x,x_n)\left\{\gamma_n + o(\delta_n)\right\} = G(x,x_n)\gamma_n + o(\delta_n)$$

since $x \in \overline{\Omega}\backslash B_\rho(x_n)$. Similarly

$$I_{2,2} = \nabla_z G(x,x_n)\cdot \int_{B_{\frac{\varepsilon}{\delta_n}}(0)} \delta_n \widetilde{z} e^{\widetilde{u}_n(\widetilde{z})}\widetilde{v}_n(\widetilde{z})d\widetilde{z}$$

by (3.2), where $\widetilde{z} = \frac{z-x_n}{\delta_n}$. Here $\varepsilon/\delta_n = \delta_n^{k-1} \to \infty$ as $n\to\infty$ since $k < \frac{3}{8}$. Using Corollary 3.3 and (3.5), we can apply the dominated convergence theorem to obtain

$$
\begin{aligned}
&\int_{B_{\frac{\varepsilon}{\delta_n}}(0)} \widetilde{z}_j e^{\widetilde{u}_n(\widetilde{z})}\widetilde{v}_n(\widetilde{z})d\widetilde{z} \\
&\to \int_{\mathbb{R}^4} \widetilde{z}_j\left\{\boldsymbol{a}\cdot\nabla\left(-\frac{1}{8}e^U\right) + b\,\mathrm{div}\left(\frac{1}{4}\widetilde{z}e^U\right)\right\}d\widetilde{z} = 8\pi^2 a_j
\end{aligned}
$$

for $j\in\{1,\cdots,4\}$, which in turn implies

$$I_{2,2} = 8\pi^2\boldsymbol{a}\cdot\nabla_z G(x,x_n)\delta_n + o(\delta_n).$$

Finally we use

$$\sup_{\substack{x\notin B_\rho(0),\,\eta\in B_\varepsilon(0)\\ 1\le k,l\le 4}} |G_{z_k z_l}(x,\eta)| \le C < \infty$$

for some constant C independent of $\varepsilon \ll 1$ to estimate

$$|I_{2,3}| \leqq C\lambda_n \int_{B_\varepsilon(x_n)} |z-x_n|^2 e^{u_n(z)}dz \le C\varepsilon\delta_n \int_{B_{\frac{\varepsilon}{\delta_n}}(0)} |\widetilde{z}|e^{\widetilde{u}_n(\widetilde{z})}dz.$$

Using Corollary 3.3 and (3.5) again, we assure the following convergence

$$\int_{B_{\frac{\varepsilon}{\delta_n}}(0)} |\widetilde{z}|e^{\widetilde{u}_n(\widetilde{z})}d\widetilde{z} \to \int_{\mathbb{R}^4} |\widetilde{z}|e^{U(\widetilde{z})}d\widetilde{z} < \infty.$$

Consequently we get $I_{2,3} = o(\delta_n)$ and the conclusion. $\square$

Since $B_{2\rho}(p_i) \supset B_\rho(x_n^i)$ for every i and $n \gg 1$, Proposition 3.6 and Corollary 3.4 imply the following *pre*-asymptotic formula: As $n \to \infty$, it holds

$$v_n(x) = \sum_{i=1}^m \gamma_n^i G(x, x_n^i) + 8\pi^2 \lambda_n^{\frac{1}{4}} \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z G(x, x_n^i) + o\left(\lambda_n^{\frac{1}{4}}\right),$$

$$\Delta v_n(x) = \sum_{i=1}^m \gamma_n^i \Delta_x G(x, x_n^i) + 8\pi^2 \lambda_n^{\frac{1}{4}} \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z \Delta_x G(x, x_n^i) + o\left(\lambda_n^{\frac{1}{4}}\right)$$

uniformly in $x \in \overline{\Omega} \backslash \cup_{i=1}^m B_{2\rho}(p_i)$ and consequently in $C^1\left(\overline{\Omega} \backslash \cup_{i=1}^m B_{2\rho}(p_i)\right)$ from the elliptic regularity theory.

To get the finer asymptotic formula (Proposition 3.5), we need to show

$$\gamma_n^i = o\left(\lambda_n^{\frac{1}{4}}\right) \tag{3.16}$$

for some subsequence. Now we complete the proof of Proposition 3.5.

Proof of Proposition 3.5. We argue by contradiction. If (3.16) does not hold then there exists i satisfying

$$\limsup_{n\to\infty} \frac{\lambda_n^{\frac{1}{4}}}{|\gamma_n^i|} < \infty.$$

We may assume $i = 1$ and put

$$r_i = \lim_{n\to\infty} \frac{\gamma_n^i}{\gamma_n^1}, \ (i = 2, \cdots, m) \quad \text{and} \quad c = \lim_{n\to\infty} \frac{\lambda_n^{\frac{1}{4}}}{\gamma_n^1}.$$

Without loss of generality we may also assume

$$1 = r_1 \geq r_2 \geq \cdots \geq r_m \geq -1$$

for some subsequence. Then we get

$$\frac{v_n(x)}{\gamma_n^1} \to \sum_{i=1}^m r_i G(x, p_i) + 8\pi^2 c \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z G(x, p_i), \tag{3.17}$$

$$\frac{\Delta v_n(x)}{\gamma_n^1} \to \sum_{i=1}^m r_i \Delta_x G(x, p_i) + 8\pi^2 c \sum_{i=1}^m C_i \boldsymbol{a}^i \cdot \nabla_z \Delta_x G(x, p_i)$$

uniformly in $x \in \Omega \backslash \cup_{i=1}^m B_{2\rho}(p_i)$. We take $r > 2\rho$ satisfying

$$B_r(p_i) \subset\subset \Omega, \quad B_r(p_i) \cap B_r(p_j) = \emptyset \quad (i \neq j).$$

Since $\Delta(x \cdot \nabla) = (x \cdot \nabla + 2)\Delta$ and $\Delta^2(x \cdot \nabla) = (x \cdot \nabla + 4)\Delta^2$, we see

$$w_n = (x - p) \cdot \nabla u_n + 4$$

satisfies (1.6) except for the boundary condition where $p \in \mathbb{R}^4$ is arbitrary, and $\Delta w_n = (x-p)\cdot\nabla\Delta u_n + 2\Delta u_n$. Thus by Proposition 1.1, we have

(3.18)
$$\begin{aligned} w_n &\to (x-p)\cdot\nabla\left(64\pi^2\sum_{i=1}^m G(x,p_i)\right)+4,\\ \Delta w_n &\to (x-p)\cdot\nabla\left(64\pi^2\sum_{i=1}^m \Delta_x G(x,p_i)\right)+128\pi^2\sum_{i=1}^m \Delta_x G(x,p_i)\end{aligned}$$

uniformly in $x \in \Omega\setminus\cup_{i=1}^m B_{2\rho}(p_i)$. Now, taking $p = x_n^1$, using (3.17), (3.18) and Green's formula again, we get

$$\begin{aligned} 0 &= B_{\partial B_r(x_n^1)}\left[w_n, v_n/\gamma_n^1\right]\\ &\to 64\pi^2\sum_{k,l=1}^m r_l B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G(x,p_l)\right]\\ &\qquad + 512\pi^4 C\sum_{k,l=1}^m\sum_{i=1}^4 C_l a_i^l B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G_{z_i}(x,p_l)\right]\\ &\qquad + \sum_{l=1}^m r_l B_{\partial B_r(p_1)}\left[4, G(x,p_l)\right] + 8\pi^2\sum_{l=1}^m\sum_{i=1}^4 C_l a_i^l B_{\partial B_r(p_1)}\left[4, G_{z_i}(x,p_l)\right]\end{aligned} \tag{3.19}$$

as $n\to\infty$.

On the other hand, from the identities (2.3) and (2.4), we obtain

$$\begin{aligned} &\sum_{l=1}^m r_l B_{\partial B_r(p_1)}\left[4, G(x,p_l)\right] = r_1 B_{\partial B_r(p_1)}\left[4, G(x,p_1)\right] = -4r_1,\\ &\sum_{l=1}^m\sum_{i=1}^4 C_l a_i^l B_{\partial B_r(p_1)}\left[4, G_{z_i}(x,p_l)\right] = \sum_{i=1}^4 C_1 a_i^1 B_{\partial B_r(p_1)}\left[4, G_{z_i}(x,p_1)\right] = 0.\end{aligned}$$

Also, since

$$(x-p_1)\cdot\nabla G(x,p_1) = -\frac{1}{8\pi^2} + (x-p_1)\cdot\nabla H(x,p_1),$$

and $(x-p_1)\cdot\nabla G(x,p_k)$ for $k\neq 1$ are biharmonic in $B_r(p_1)$, we get

$$\begin{aligned}
&\sum_{k,l=1}^{m} r_l B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G(x,p_l)\right]\\
&\qquad= \sum_{k=1}^{m} r_1 B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G(x,p_1)\right]\\
&\qquad= r_1 B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_1), G(x,p_1)\right] = \frac{r_1}{8\pi^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k,l=1}^{m}\sum_{i=1}^{4} C_l a_i^l B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G_{z_i}(x,p_l)\right]\\
&\quad= \sum_{k=1}^{m}\sum_{i=1}^{4} C_1 a_i^1 B_{\partial B_r(p_1)}\left[(x-p_1)\cdot\nabla G(x,p_k), G_{z_i}(x,p_1)\right]\\
&\quad= -\sum_{k=1}^{m}\sum_{i=1}^{4} C_1 a_i^1 \frac{\partial}{\partial x_i}\left((x-p_1)\cdot\nabla G(x,p_k)\right)\Big|_{x=p_1}\\
&\quad= -C_1\sum_{i=1}^{4} a_i^1\left(H_{x_i}(p_1,p_1)+\sum_{k=2}^{m} G_{x_i}(p_1,p_k)\right)\\
&\quad= -C_1\boldsymbol{a}^1\cdot\nabla_{\xi_1}\mathcal{F}(p_1,\cdots,p_m)=0,
\end{aligned}$$

since $(p_1,\cdots,p_m)$ is a critical point of the Hamiltonian $\mathcal{F}$; see (1.4). Therefore, returning to (3.19), we obtain

$$0 = 8r_1 + 0 - 4r_1 + 0$$

which leads to a desired contradiction $1 = r_1 = 0$. Thus we have proved Proposition 3.5. $\square$

4. Proof of Theorem 1.2.

In this section, we prove Theorem 1.2. We just need to assure that V^i in (3.10) is identically zero. We divide the proof into two steps.

Step 1. First, we prove that all coefficients a_k^i $(i = 1,\cdots,m,\ k = 1,\cdots,4)$ in (3.10) must be zero. Here we will use the assumption that $(p_1,\cdots,p_m)\in\Omega^m$ is a nondegenerate critical point of $\mathcal{F}$.

Fix $p_j\in\mathcal{S}$ and take $r > 2\rho > 0$ small such that $\overline{B_r(p_j)}\subset\Omega$ and $B_r(p_j)\cap\mathcal{S} = p_j$. Differentiating the equation $\Delta^2 u_n = \lambda_n e^{u_n}$ with respect to x_i $(i = 1,\cdots,4)$, we have

$$\Delta^2 (u_n)_{x_i} = \lambda_n e^{u_n}(u_n)_{x_i}.$$

Since v_n is a solution of $\Delta^2 v_n = \lambda_n e^{u_n} v_n$, Green's formula for Δ^2 (2.1) implies that

$$0 = \int_{B_r(p_j)} \big((\Delta^2 (u_n)_{x_i}) v_n - (u_n)_{x_i} (\Delta^2 v_n) \big)\, dx = B_{\partial B_r(p_j)} \left[(u_n)_{x_i}, v_n \right]$$

From Proposition 1.1 (1.2) and elliptic estimates, we have

$$(u_n)_{x_i} \to 64\pi^2 \sum_{i=1}^{m} G_{x_i}(x, p_i), \tag{4.1}$$

$$\Delta (u_n)_{x_i} \to 64\pi^2 \sum_{i=1}^{m} \Delta_x G_{x_i}(x, p_i) \tag{4.2}$$

uniformly on every $\omega \subset\subset \overline{\Omega} \setminus \mathcal{S}$. By Proposition 3.5, (4.1) and (4.2), we have

(4.3)

$$0 = B_{\partial B_r(p_j)} \left[(u_n)_{x_i}, v_n / \lambda_n^{1/4} \right] \to 512\pi^4 \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{i'=1}^{4} C_l a^l_{i'} I_{ii'}(p_j, p_k, p_l)$$

where $I_{ii'}(a,b,c)$ is defined in Proposition 2.1. By Proposition 2.1, we obtain

$$\sum_{k=1}^{m} I_{ii'}(p_j, p_k, p_l) = \begin{cases} -\frac{1}{2} R_{x_i x_{i'}}(p_j) - \displaystyle\sum_{\substack{1 \le k \le m \\ k \ne j}} G_{x_i x_{i'}}(p_j, p_k), & (j = l), \\ -G_{x_i z_{i'}}(p_j, p_l), \quad (j \ne l). \end{cases}$$

In other words,

$$\sum_{k=1}^{m} I_{ii'}(p_j, p_k, p_l) = -\frac{\partial^2}{\partial (\xi_j)_i \partial (\xi_l)_{i'}} \mathcal{F}(\xi_1, \cdots, \xi_m) \Big|_{(\xi_1, \cdots, \xi_m) = (p_1, \cdots, p_m)}$$

for any $i, i' \in \{1, 2, 3, 4\}$, where $\mathcal{F}$ is the Hamiltonian function in (1.5). Note that $C_l > 0$ in (4.3). Also by our assumption, $(\mathrm{Hess}\mathcal{F})(p_1, \cdots, p_m)$ is invertible. Thus we obtain all $a^l_{i'} = 0$ for any $l = 1, \cdots, m$ and $i' = 1, \cdots, 4$ from (4.3).

Step 2. Next, we prove $b^i = 0$ for all $i = 1, \cdots, m$. Fix $i \in \{1, \cdots, m\}$ and choose $r > 0$ small such that $B_r(x_n^i) \subset\subset \Omega$. By Green's formula (2.1) on $B_r(x_n^i)$, we have

$$\int_{B_r(x_n^i)} \big((\Delta^2 u_n) v_n - u_n (\Delta^2 v_n) \big)\, dx = B_{\partial B_r(x_n^i)} \left[u_n, v_n \right]$$

The RHS is $O(\lambda_n^{1/4}) = o(1)$ by Proposition 1.1 and Proposition 3.5. The LHS is

$$\int_{B_r(x_n^i)} \left(\lambda_n e^{u_n} v_n - \lambda_n e^{u_n} v_n (u_n - u_n(x_n^i)) - \lambda_n e^{u_n} v_n u_n(x_n^i)\right) dx$$
$$= (1 - u_n(x_n^i)) \int_{B_r(x_n^i)} \lambda_n e^{u_n} v_n dx - \int_{B_r(x_n^i)} \lambda_n e^{u_n} v_n (u_n - u_n(x_n^i)) dx$$
$$= (1 - u_n(x_n^i)) \int_{B_r(x_n^i)} \lambda_n e^{u_n} v_n dx - \int_{B_{\frac{r}{\delta_n^i}}(0)} e^{\tilde{u}_n^i} \tilde{v}_n^i \tilde{u}_n^i dy.$$

Note that

$$\int_{B_r(x_n^i)} \lambda_n e^{u_n} v_n dx = \int_{B_r(x_n^i)} \Delta^2 v_n dx = \int_{\partial B_r(x_n^i)} \frac{\partial \Delta v_n}{\partial \nu} = O(\lambda_n^{1/4})$$

by Proposition 3.5. On the other hand, by Corollary 3.4 (3.13), we have

$$(1 - u_n(x_n^i)) \int_{B_r(x_n^i)} \lambda_n e^{u_n} v_n dx = (-2 \log \lambda_n + O(1)) o(\lambda_n^{1/4}) = o(1).$$

Finally,

$$\int_{B_{\frac{r}{\delta_n^i}}(0)} e^{\tilde{u}_n^i} \tilde{v}_n^i \tilde{u}_n^i dy \to \int_{\mathbb{R}^4} e^U U V^i dy = 64\pi^2 b^i$$

by (3.9) and (3.10). All together, we conclude $b^i = 0$ for all $i = 1, \cdots, m$. □

5. Some discussions about the asymptotic nondegeneracy.

In order to prove the asymptotic nondegeneracy of the multi-bubble solutions u_n, that is, the solution v_n to (1.6) satisfies $v_n \equiv 0$ for all $n \in \mathbb{N}$ large, one of the possible way of arguments is as follows.

Assume that there would exist non trivial solutions v_n of (1.6) for large $n \in \mathbb{N}$. Since we have assured Theorem 1.2, it suffices to derive some contradiction from the fact $V^i \equiv 0$. The next lemma is obtained easily.

Lemma 5.1. *We have*

$$v_n \to 0, \quad \Delta v_n \to 0$$

locally uniformly on $\overline{\Omega} \setminus \mathcal{S}$.

Proof. Since $\lambda_n e^{u_n} \to 0$ locally uniformly on $\overline{\Omega} \setminus \mathcal{S}$ and $\|v_n\|_{L^\infty(\Omega)} = 1$, the elliptic regularity implies that there exists $\overline{v}_\infty$ such that, if we choose a subsequence, $\Delta v_n \to \overline{v}_\infty$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$ and $\overline{v}_\infty$ is a solution of

$$\Delta \overline{v}_\infty = 0 \text{ in } \overline{\Omega} \setminus \mathcal{S}, \quad \overline{v}_\infty = 0 \text{ on } \partial\Omega.$$

Thus $\overline{v}_\infty \equiv 0$ on $\overline{\Omega}$, because $\mathcal{S}$ is a set of finite points which are negligible. Again elliptic regularity implies $v_n \to 0$. Since the limits are unique, above convergences hold for the full sequence. □

Since Lemma 5.1 holds, it is sufficient to show

$$v_n \to 0 \quad \text{uniformly in } B_r(p_i) \tag{5.1}$$

for each i to obtain the desired result, where $0 < r \ll 1$. In the sequel, we abbreviate i in several characters and assume $p_i = 0$ as before.

By Theorem 1.2, we have already obtained

$$\tilde{v}_n \to 0 \quad \text{local uniformly in } \mathbb{R}^4. \tag{5.2}$$

Therefore suppose (5.1) does not hold. Then we get

$$\limsup_{n\to\infty} \max_{x \in \overline{B_r(0)}} |v_n(x)| = M > 0,$$

and therefore, up to subsequences (denoted by the same symbol), we have $\widetilde{x}_n \in B_{\frac{r}{\delta_n}}(0)$ such that

$$|\widetilde{v}_n(\widetilde{x}_n)| = \max_{x \in \overline{B_r(0)}} |v_n| \to M, \quad |\widetilde{x}_n| \to \infty.$$

Now, we take the Kelvin transformed functions

$$\hat{u}_n(z) = \widetilde{u}_n\left(\frac{z}{|z|^2}\right), \quad \hat{v}_n(z) = \widetilde{v}_n\left(\frac{z}{|z|^2}\right),$$

which satisfy the equation

$$\Delta^2 \hat{v}_n(z) = |z|^{-8} e^{\hat{u}_n} \hat{v}_n \quad \text{for } z \in \left(B_{\frac{\delta_n}{r}}(0)\right)^c.$$

Note that

$$\hat{x}_n := \frac{\widetilde{x}_n}{|\widetilde{x}_n|^2} \to 0, \quad \hat{v}_n(\hat{x}_n) = \widetilde{v}_n(\widetilde{x}_n) \to M. \tag{5.3}$$

Next we take the unique solution w_n of the following problem:

$$\Delta^2 w_n = f_n := \begin{cases} |x|^{-8} e^{\hat{u}_n} \hat{v}_n, & \text{in } B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}, \\ 0, & \text{in } B_{\frac{\delta_n}{r}}(0), \end{cases}$$
$$w_n = \Delta w_n = 0 \quad \text{on } \partial B_1(0).$$

Using Proposition 3.11, we get

$$0 \le |x|^{-8} e^{\hat{u}_n} \le C < \infty$$

in $B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}$, where C is a constant independent of n. On the other hand, $\hat{v}_n(x) \to 0$ for every $x \in B_r(0) \setminus \{0\}$ by (5.2), therefore,

$$\|f_n\|_{L^p(B_1(0))} \to 0 \quad \text{for each } p \in [1, \infty)$$

by the dominated convergence theorem. Consequently,

$$w_n \quad \text{and} \quad \Delta w_n \to 0 \quad \text{uniformly in } B_1(0) \tag{5.4}$$

follows from the elliptic regularity. We also take the unique solution ϕ_n of the problem

$$\Delta \phi_n = \begin{cases} \Delta(\hat{v}_n - w_n), & \text{in } B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}, \\ 0, & \text{in } B_{\frac{\delta_n}{r}}(0), \end{cases}$$
$$\phi_n = 0 \quad \text{on } \partial B_1(0).$$

The elliptic regularity theory guarantees

$$\|\phi_n\|_{L^\infty(B_1(0))} \le C \|\Delta(\hat{v}_n - w_n)\|_{L^\infty\left(B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}\right)}. \tag{5.5}$$

Note that $\Delta(\hat{v}_n - w_n)$ is harmonic in $B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}$. Therefore the maximum principle gives

$$\begin{aligned} &\|\Delta(\hat{v}_n - w_n)\|_{L^\infty\left(B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}\right)} \\ &\quad \le \|\Delta(\hat{v}_n - w_n)\|_{L^\infty(\partial B_1(0))} + \|\Delta(\hat{v}_n - w_n)\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} \\ &\quad \le \|\Delta \hat{v}_n\|_{L^\infty(\partial B_1(0))} + \|\Delta \hat{v}_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} + \|\Delta w_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)}, \end{aligned}$$

where $\|\Delta w_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} = o(1)$. Since

$$\Delta_z \hat{v}_n(z) = |y|^4 \Delta_y \widetilde{v}_n(y) - 4|y|^2 (y \cdot \nabla_y) \widetilde{v}_n(y),$$

for $y = z/|z|^2$, we get

$$\|\Delta \hat{v}_n\|_{L^\infty(\partial B_1(0))} = \|\Delta_y \widetilde{v}_n(y) - 4(y \cdot \nabla_y) \widetilde{v}_n(y)\|_{L^\infty(\partial B_1(0))} = o(1)$$

from (5.2).

Assume for the moment that

$$\|\Delta \hat{v}_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} = o(1) \tag{5.6}$$

as $n \to \infty$. Then we can get the desired contradiction as follows: The estimates (5.5) and (5.6) imply

$$\|\phi_n\|_{L^\infty(B_1(0))} = o(1). \tag{5.7}$$

Now we consider the difference $\hat{v}_n - w_n - \phi_n$. This function is harmonic on $B_1(0) \setminus \overline{B_{\frac{\delta_n}{r}}(0)}$, so the maximum principle guarantees

$$\begin{aligned}
&\|\hat{v}_n - w_n - \phi_n\|_{L^\infty\left(B_1(0)\setminus\overline{B_{\frac{\delta_n}{r}}(0)}\right)} \\
&\quad\le \|\hat{v}_n - w_n - \phi_n\|_{L^\infty(\partial B_1(0))} + \|\hat{v}_n - w_n - \phi_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} \\
&\quad\le \|\hat{v}_n\|_{L^\infty(\partial B_1(0))} \\
&\quad+ \|\hat{v}_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} + \|w_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} + \|\phi_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)}
\end{aligned}$$

where the estimates

$$\begin{aligned}
&\|\hat{v}_n\|_{L^\infty(\partial B_1(0))} = \|\widetilde{v}_n\|_{L^\infty(\partial B_1(0))} = o(1), \\
&\|\hat{v}_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} = \|\widetilde{v}_n\|_{L^\infty\left(\partial B_{\frac{r}{\delta_n}}(0)\right)} = \|v_n\|_{L^\infty(\partial B_r(x_n))} = o(1)
\end{aligned}$$

follow by (5.2) and Lemma 5.1. Hence it follows that

$$\begin{aligned}
&\|\hat{v}_n\|_{L^\infty\left(B_1(0)\setminus\overline{B_{\frac{\delta_n}{r}}(0)}\right)} \\
&\le \|w_n\|_{L^\infty(B_1(0))} + \|\phi_n\|_{L^\infty(B_1(0))} + \|\hat{v}_n - w_n - \phi_n\|_{L^\infty\left(B_1(0)\setminus\overline{B_{\frac{\delta_n}{r}}(0)}\right)} \\
&= 2\|\phi_n\|_{L^\infty(B_1(0))} + o(1) = o(1)
\end{aligned}$$

by (5.4) and (5.7), which contradicts to (5.3).

However, an easy way of estimation using Proposition 3.5 can only provide

$$\begin{aligned}
\|\Delta\hat{v}_n\|_{L^\infty\left(\partial B_{\frac{\delta_n}{r}}(0)\right)} &= \|(r/\delta_n)^4\Delta_y\widetilde{v}_n(y) - 4(r/\delta_n)^2(y\cdot\nabla_y)\widetilde{v}_n(y)\|_{L^\infty\left(\partial B_{\frac{r}{\delta_n}}(0)\right)} \\
&\le \left(r^4/\delta_n^2\right)\|\Delta v_n\|_{L^\infty(\partial B_r(x_n))} + \left(4r^2/\delta_n^2\right)\|(x\cdot\nabla)v_n\|_{L^\infty(\partial B_r(x_n))} \\
&= O(1/\delta_n),
\end{aligned}$$

which is far from the needed decay (5.6).

Another possible way to obtain the asymptotic nondegeneracy is to refine Proposition 3.5. This will be a future subject.

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