SINGULAR EXTREMAL SOLUTIONS TO A LIOUVILLE-GELFAND TYPE PROBLEM WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. We consider a Liouville-Gelfand type problem

$$-\Delta u = e^u + \lambda f(x) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth bounded domain, $f \geq 0$, $f \not\equiv 0$ is a given smooth function, and $\lambda \geq 0$ is a parameter. We are concerned with the regularity property of extremal solutions to the problem, and prove that there exists a domain Ω and a smooth nonnegative function f such that the extremal solution of the problem is singular when the dimension $N \geq 10$. This result is sharp in the sense that the extremal solution is always regular (bounded) for any f and Ω when $1 \leq N \leq 9$.

1. Introduction.

In this paper, we consider a Liouville-Gelfand type problem with the exponential nonlinearity:

$$\begin{cases} -\Delta u = e^u + \lambda f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a smooth bounded domain, $f \in C^\infty(\Omega)$ is a nonnegative function, not identically equal to zero, and $\lambda \geq 0$ is a parameter.

First, we recall the notion of a *weak solution* to (1.1); see Brezis et al. [2].

Definition 1.1. *A function $u \in L^1(\Omega)$ is called a weak solution to (1.1) if $u > 0$ in Ω, $e^u \delta \in L^1(\Omega)$, and*

$$-\int_\Omega u \Delta\zeta dx = \int_\Omega \left(e^u + \lambda f\right) \zeta dx \tag{1.2}$$

Date: June 19, 2014.
2010 Mathematics Subject Classification: 35J25. 35J60.
Key words: weak solution, extremal solution, exponential nonlinearity.

holds for any $\zeta \in C^2(\overline{\Omega})$ *such that* $\zeta = 0$ *on* $\partial\Omega$*, where* $\delta(x) = dist(x, \partial\Omega)$.

Note that since $|\zeta| \le C\delta$ for any $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$, the integral of the right hand side of (1.2) is well-defined.

By the methods in [2], [3] and [8], we can prove the following basic facts concerning the problem $(1.1)_\lambda$.

Proposition 1.2. *Let* $f \in C^\infty(\Omega)$, $f \ge 0, f \not\equiv 0$ *be a given function. Then there exists* $\lambda^* \in (0, +\infty)$*, called an extremal parameter, such that the followings hold true.*

(i) For $\lambda \in (0, \lambda^*)$*, there exists a minimal solution* u_λ *to* $(1.1)_\lambda$. u_λ *is smooth, stable in the sense that*

$$\int_\Omega |\nabla\phi|^2 dx \ge \int_\Omega e^{u_\lambda}\phi^2 dx \tag{1.3}$$

holds for any $\phi \in C_0^1(\Omega)$*. Furthermore,* u_λ *depends continuously and monotone increasingly on* $\lambda \in (0, \lambda^*)$*.*

(ii) For $\lambda = \lambda^*$*, there exists a unique weak solution* u^* *to* $(1.1)_\lambda$. u^* *is called the extremal solution and is obtained as an increasing limit of the minimal solutions* u_λ*:*

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x) \quad (x \in \Omega).$$

(iii) For $\lambda > \lambda^*$*, there is no solution to* $(1.1)_\lambda$*, even in the weak sense.*

In this paper, we concern the regularity issue of the extremal solution u^* in Proposition 1.2 (ii). In some cases, u^* may be singular (i.e., $u^* \notin L^\infty(\Omega)$), but little is known about the singular extremal solutions.

For the well-studied problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

we have also the extremal parameter $\lambda^* \in (0, +\infty)$ for which there is a minimal, strict stable solution for $0 < \lambda < \lambda^*$, the unique extremal solution (may be singular) for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$ even in the weak sense [2] [8]. If $\Omega = B$, the unit ball in $\mathbb{R}^N$, and $N \ge 10$, then the explicit radial function $v(x) = -2\log|x|$ becomes the singular extremal solution of (1.4) for $\lambda = 2(N-2)$ [3]. Note that $v \in H_0^1(B)$ if $N \ge 3$. On the other hand, the extremal solution of (1.4) is bounded on any bounded smooth domain Ω when $1 \le N \le 9$ [4], [9]. The readers are recommended to refer to the recent book by Dupaigne [7] and its references for these results. Concerning the existence of singular solutions, Dávila and Dupaigne [6] prove that there exists an 1-parameter family of singular solutions $(u(t), \lambda(t))_{t>0}$ to (1.4) for $\lambda = \lambda(t)$ with the property

$$\|u(t) - \log \frac{1}{|\cdot - \xi(t)|^2}\|_{L^\infty(\Omega)} + |\lambda(t) - 2(N-2)| \to 0 \quad (t \to 0)$$

for some $\xi(t) \in \Omega$, where the domain Ω is a small perturbation of a ball in an appropriate sense in $\mathbb{R}^N, N \geq 4$. The authors also prove that these singular solutions correspond to the extremal solutions when $N \geq 11$. Recently, Miyamoto [10] studies the perturbed Liouville-Gelfand problem on the unit ball B in $\mathbb{R}^N$, $N \geq 3$:

$$\begin{cases} -\Delta u = \lambda(e^u + g(u)) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $g \in C^1$ is an appropriate nonlinearity which is "small" compared to e^u. The author proves the existence of radial singular solution (u^*, λ^*) with the property

$$u^*(|x|) \sim -2\log|x| - \log\lambda^* + \log 2(N-2) \quad (|x| \to 0),$$

and if $N \geq 10$, this singular radial solution corresponds to the extremal solution.

For other nonlinearities, Dávila [5] studies the regularity and singularity issue of extremal solutions to the problem

$$\begin{cases} -\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a smooth bounded domain, $f \in C^\infty(\Omega)$ is a nonnegative function, not identically equal to zero, and $\lambda > 0$. The results in this paper correspond to the ones in [5] for the exponential nonlinear case.

This paper is organized as follows: In §2, we prove that the extremal solutions are regular for any f and Ω when $1 \leq N \leq 9$. In §3, we examine the sharpness of this regularity theorem in terms of the dimension of the domain, and prove that there exists a bounded domain Ω and a smooth $f \geq 0, f \not\equiv 0$ such that the extremal solution u^* is not bounded when $N \geq 10$. This means that the assumption $1 \leq N \leq 9$ in the regularity theorem in §2 is sharp and cannot be relaxed in general. Finally in §4, we treat the case when the domain is a ball.

2. Extremal solutions are regular for $1 \le N \le 9$.

First, we prove the boundedness of the extremal solution to (1.1) in lower dimensions.

Theorem 2.1. *Let Ω be any smooth bounded domain in $\mathbb{R}^N$ and let $f \in C^\infty(\Omega)$, $f \ge 0, f \not\equiv 0$ be any given function. If $1 \le N \le 9$, then there exists a constant $C > 0$ such that for any $0 < \lambda < \lambda^*$, it holds*

$$\|u_\lambda\|_{L^\infty(\Omega)} \le C$$

for the minimal solution u_λ to $(1.1)_\lambda$. Consequently, the extremal solution u^* is bounded, hence smooth.

Proof. We follow the arguments in [4], [9] with some modifications for our context. Recall the minimal solution $u = u_\lambda$ satisfies the stability inequality

$$\int_\Omega |\nabla \phi|^2 dx \ge \int_\Omega e^u \phi^2 dx, \quad \forall \phi \in C_0^1(\Omega)$$

and the weak form of the equation

$$\int_\Omega \nabla \psi \cdot \nabla u dx = \int_\Omega (e^u + \lambda f)\, \psi dx, \quad \forall \psi \in C_0^1(\Omega).$$

We put $\phi = e^{tu} - 1$ and $\psi = \frac{t}{2}(e^{2tu} - 1)$, where $t > 0$. Testing with them, we have

$$\int_\Omega t^2 e^{2tu} |\nabla u|^2 dx \ge \int_\Omega e^u (e^{tu} - 1)^2 dx$$

and

$$\int_\Omega t^2 e^{2tu} |\nabla u|^2 dx = \frac{t}{2} \int_\Omega (e^u + \lambda f)\left(e^{2tu} - 1\right) dx.$$

Combining these, we obtain

$$\int_\Omega e^u (e^{tu} - 1)^2 dx \le \frac{t}{2} \int_\Omega (e^u + \lambda f)\left(e^{2tu} - 1\right) dx,$$

which in turn implies

$$\begin{aligned}
\left(1 - \frac{t}{2}\right) \int_\Omega e^{(2t+1)u} dx &\le \int_\Omega \left(2e^{(t+1)u} - \left(\frac{t}{2} + 1\right) e^u + \frac{\lambda t}{2}\left(e^{2tu} - 1\right) f \right) dx \\
&\le 2 \int_\Omega e^{(t+1)u} dx + \frac{\lambda t}{2} \int_\Omega e^{2tu} f dx \\
&\le 2 \left(\int_\Omega e^{(2t+1)u} dx \right)^{\frac{t+1}{2t+1}} |\Omega|^{\frac{t}{2t+1}} \\
&+ \frac{t\lambda^*}{2} \left(\int_\Omega e^{(2t+1)u} dx \right)^{\frac{2t}{2t+1}} \left(\int_\Omega f^{2t+1} dx \right)^{\frac{1}{2t+1}}.
\end{aligned}$$

We may assume that

$$\int_\Omega e^{(2t+1)u}dx > 1,$$

because on the contrary, we have $\|e^u\|_{L^{2t+1}(\Omega)} \le 1$, and the estimate is independent of $\lambda \in (0,\lambda^*)$. In this case, if $1-\frac{t}{2} > 0$ and $\frac{t+1}{2t+1} < \frac{2t}{2t+1}$, that is, if $1 < t < 2$, then we have

$$\int_\Omega e^{(2t+1)u}dx \le \left[\left(1-\frac{t}{2}\right)^{-1}\left\{2|\Omega|^{\frac{t}{2t+1}} + \frac{t\lambda^*}{2}\left(\int_\Omega f^{2t+1}dx\right)^{\frac{1}{2t+1}}\right\}\right]^{2t+1} =: C,$$

here $C = C(|\Omega|, f)$ is independent of $\lambda \in (0,\lambda^*)$. Thus we have $\|e^u\|_{L^{2t+1}(\Omega)} \le C$, which implies

$$\|e^{u_\lambda} + \lambda f\|_{L^{2t+1}(\Omega)} \le C$$

when $1 < t < 2$. Now, standard elliptic estimates and Sobolev embedding imply that $\|u_\lambda\|_{L^\infty(\Omega)} \le C$ uniformly in λ if $2(2t+1) > N$. Since we may choose $t \in (1,2)$ very close to 2, we obtain the uniform L^∞ bound for u_λ when $N \le 9$. This proves Theorem 2.1. □

3. Singular extremal solutions when $N \ge 10$.

In this section, we prove the following theorem, which says that the restriction of the dimension in Theorem 2.1 is sharp concerning the boundedness of the extremal solutions.

Theorem 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Assume that $N \ge 10$, $0 \in \Omega$ and*

$$\max_{x\in\partial\Omega} |x|^2 \le 2(N-2) \tag{3.1}$$

holds true. Then there exists $f \in C^\infty(\Omega)$, $f \ge 0, f \not\equiv 0$ such that the extremal solution u^* to (1.1) *with f satisfies*

$$u^* \notin L^\infty(\Omega) \quad \textit{and} \quad \lambda^* = 1.$$

In the proof of Theorem 3.1, we need a characterization of the unbounded extremal solutions in the energy class $H^1(\Omega)$, which is similar to Brezis and Vázquez [3], Theorem 3.1. See also Dávila [5], Lemma 4.

Lemma 3.2. *Let $u \in H_0^1(\Omega)$, $u \notin L^\infty(\Omega)$, be a singular weak solution to* $(1.1)_\lambda$. *Then the followings are equivalent:*

(i) $e^u\delta \in L^1(\Omega)$ *and*

$$\int_\Omega |\nabla\phi|^2 dx \ge \int_\Omega e^u \phi^2 dx$$

holds for every $\phi \in C_0^1(\Omega)$.

(ii) $\lambda = \lambda^*$ *and* $u = u^*$.

Proof. The implication $(ii) \implies (i)$ follows easily by the stability property of the minimal solutions u_λ and Fatou's lemma.

Let us prove $(i) \implies (ii)$. Since no solution exists for $\lambda > \lambda^*$ by Proposition 1.2, we have $\lambda \le \lambda^*$. Assume the contrary that $\lambda < \lambda^*$. By the density argument and the fact that $u, u_\lambda \in H_0^1(\Omega)$, we can take the test function $\phi = u - u_\lambda \in H_0^1(\Omega)$. By the minimality of u_λ, we see $u - u_\lambda \ge 0$ in Ω, and the assumption $u \notin L^\infty(\Omega)$ implies that $u - u_\lambda \not\equiv 0$, since u_λ is bounded for $\lambda < \lambda^*$. Combining the equation satisfied by $u - u_\lambda$ with (i), we obtain

$$\begin{aligned}\int_\Omega \left(e^u + \lambda f - e^{u_\lambda} - \lambda f\right)(u - u_\lambda) dx &= \int_\Omega |\nabla(u-u_\lambda)|^2 dx \\ &\ge \int_\Omega e^u (u-u_\lambda)^2 dx,\end{aligned}$$

which implies

$$\int_\Omega (u-u_\lambda)\left(e^u - e^{u_\lambda} - e^u(u-u_\lambda)\right) dx \ge 0.$$

Since the integrand is non positive by the convexity of $s \mapsto e^s$, we conclude that $e^u = e^{u_\lambda} + e^u(u-u_\lambda)$ a.e. on Ω. Again the strict convexity of $s \mapsto e^s$ implies $u = u_\lambda$ a.e. on Ω, which is a contradiction. Thus we must have $\lambda = \lambda^*$. □

In the following, let v_s denote the explicit singular radial function defined as

$$v_s(x) = -2\log|x| + \log 2(N-2), \quad x \in \mathbb{R}^N. \tag{3.2}$$

Then $v_s \in H^1_{loc}(\mathbb{R}^N)$ if $N \ge 3$ and v_s satisfies the equation $-\Delta v = e^v$ in $\mathbb{R}^N$. Recall we have assumed $0 \in \Omega$ in Theorem 3.1. As in [5], our strategy is to look for a singular solution u to (1.1) (with a suitable f) of the form

$$u = v_s - \psi$$

for some $\psi \in C^\infty(\Omega)$, $\psi \ge 0$. The extremality of u will follow from the fact that $u \in H_0^1(\Omega)$ and Lemma 3.2.

Next simple lemma is well-known and in fact is used in [5].

Lemma 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and ω be a smooth subdomain of Ω with $\overline{\omega} \subset \Omega$. Let ψ satisfy*

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \psi = 0 & \text{on } \partial\omega, \\ \frac{\partial\psi}{\partial\nu} \geq 0 & \text{on } \partial\omega, \end{cases}$$

where ν is the unit normal vector on $\partial\omega$ pointing to the inside of $\Omega \setminus \overline{\omega}$. Then if we put

$$\begin{cases} \overline{\psi} = \psi & \text{on } \Omega \setminus \overline{\omega}, \\ \overline{\psi} = 0 & \text{on } \omega, \end{cases}$$

$\overline{\psi}$ satisfies

$$\Delta\overline{\psi} \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. For any $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$, we have

$$\int_\Omega \overline{\psi}\Delta\phi dx = \int_{\Omega\setminus\overline{\omega}} \psi\Delta\phi dx = \int_{\Omega\setminus\overline{\omega}} \phi\Delta\psi dx + \int_{\partial(\Omega\setminus\overline{\omega})} \frac{\partial\phi}{\partial\nu}\psi dx - \int_{\partial(\Omega\setminus\overline{\omega})} \frac{\partial\psi}{\partial\nu}\phi dx.$$

Now,

$$\int_{\partial(\Omega\setminus\overline{\omega})} \frac{\partial\phi}{\partial\nu}\psi dx = \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}\psi dx - \int_{\partial\omega} \frac{\partial\phi}{\partial\nu}\psi dx = 0$$

since $\psi = 0$ on $\partial\omega$ and $\frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$. On the other hand,

$$-\int_{\partial(\Omega\setminus\overline{\omega})} \frac{\partial\psi}{\partial\nu}\phi dx = -\left(\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\phi dx - \int_{\partial\omega} \frac{\partial\psi}{\partial\nu}\phi dx\right) = \int_{\partial\omega} \frac{\partial\psi}{\partial\nu}\phi dx \geq 0$$

by $\frac{\partial\psi}{\partial\nu} \geq 0$ and $\phi \geq 0$. Thus we obtain

$$\int_\Omega \overline{\psi}\Delta\phi dx = -\int_{\partial(\Omega\setminus\overline{\omega})} \frac{\partial\psi}{\partial\nu}\phi dx \geq 0,$$

which proves the lemma. □

Next is a variant of [5]: Lemma 5.

Lemma 3.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, satisfying the assumption (3.1). Then there exists a function $\psi \in C^\infty(\overline{\Omega})$ such that*

(i) *$\psi \geq 0$ in $\overline{\Omega}$,*
(ii) *$\Delta\psi \geq 0$ in Ω,*
(iii) *$\psi \equiv 0$ in a neighborhood of $0 \in \Omega$,*

(iv) $\psi(x) = v_s(x) = \log \frac{2(N-2)}{|x|^2}$ *on* $\partial\Omega$.

Proof. This lemma is essentially the same one in Dávila [5]. We recall the proof here for the reader's convenience.

Put $r = \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$ and let B_r denote the open ball with center 0 and radius r. Note that the smallness assumption of Ω (3.1) implies that $v_s(x) \geq 0$ for $x \in \partial\Omega$. Now, let ψ_1 be th solution of

$$\begin{cases} \Delta\psi_1 = 0 & \text{in } \Omega \setminus \overline{B}_r, \\ \psi_1 = v_s & \text{on } \partial\Omega, \\ \psi_1 = 0 & \text{on } \partial B_r \end{cases}$$

where v_s is defined in (3.2). Then ψ_1 is smooth and by the maximum principle, $\psi_1 > 0$ on $\Omega \setminus \overline{B}_r$. Thus $\frac{\partial\psi_1}{\partial\nu} > 0$ by the Hopf lemma, where ν is the unit normal vector on ∂B_r pointing to the inside of $\Omega \setminus \overline{B}_r$. Put

$$\begin{cases} \overline{\psi}_1 = \psi_1 & \text{on } \Omega \setminus \overline{B}_r, \\ \overline{\psi}_1 = 0 & \text{on } B_r. \end{cases}$$

Then by Lemma 3.3, we have

$$\Delta\overline{\psi}_1 \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Put

$$\psi = \overline{\psi}_1 * \rho_\varepsilon$$

where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon})$ with ρ satisfying $\rho \in C_0^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\rho(x) = \rho(|x|)$, $\mathrm{supp}(\rho) \subset B_1$, and $\int_{\mathbb{R}^N} \rho dx = 1$. Then we check that ψ is the desired function. □

Proof of Theorem 3.1. Let $u = v_s - \psi$, where v_s is an explicit singular solution (3.2) and $\psi \in C^\infty(\overline{\Omega})$ is as in Lemma 3.4. Since we assume $0 \in \Omega$, we have $u \notin L^\infty(\Omega)$. By Lemma 3.4 (ii) and (iv), we have

$$-\Delta u = -\Delta v_s + \Delta\psi = e^{v_s} + \Delta\psi \geq e^{v_s} > 0$$

on Ω and $u = 0$ on $\partial\Omega$. Thus $u \geq 0$ by the maximum principle. Now, put

$$f(x) = e^{v_s} + \Delta\psi - e^u = e^{v_s} - e^{v_s - \psi} + \Delta\psi.$$

Then $f \geq 0$ in $\overline{\Omega}$ since $v_s \geq u$ by Lemma 3.4 (i) and (ii). Also, we have

$$-\Delta u = e^{v_s} + \Delta\psi = e^u + f(x)$$

in Ω. Furthermore, by Lemma 3.4 (iv),

$$f(x) = e^{v_s(x)}(1 - e^{-\psi(x)}) + \Delta\psi(x) = \Delta\psi(x)$$

for x in a neighborhood of 0. Thus f is smooth on Ω.

Finally, we check that u is stable in the sense of (1.3). Indeed, for any $\phi \in C_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega e^u \phi^2 dx \le \int_\Omega e^{v_s}\phi^2 dx &= 2(N-2)\int_\Omega \frac{\phi^2}{|x|^2}dx \\ &\le 2(N-2)\left(\frac{2}{N-2}\right)^2 \int_\Omega |\nabla \phi|^2 dx \\ &\le \int_\Omega |\nabla\phi|^2 dx,\end{aligned}$$

here we have used the fact $u \le v_s$ for the first inequality, the Hardy inequality

$$\left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\phi^2}{|x|^2}dx \le \int_\Omega |\nabla\phi|^2 dx \quad \forall \phi \in C_0^1(\Omega)$$

for the second inequality. Note that the assumption $N \ge 10$ is equivalent to $2(N-2)\left(\frac{2}{N-2}\right)^2 \le 1$ for the third inequality.

Thus u is an unbounded, stable, H_0^1-solution of (1.1) (with $\lambda = 1$). By the characterization of the singular energy extremal solutions Lemma 3.2, we conclude that $u = u^*$ and $\lambda^* = 1$. □

4. The ball case.

In this section, we treat the case where the domain is a ball. Note that in this case, the minimal solution u_λ of $(1.1)_\lambda$ is radially symmetric if f is assumed to be radial. More generally, we prove the lemma below, which is a slight modification of Proposition 1.3.4 in [7].

Lemma 4.1. *Let $g \in C^1(\mathbb{R})$. Let Ω be a smooth bounded, radially symmetric domain with the symmetric center the origin (ball or annulus) in $\mathbb{R}^N, N \ge 2$, and $f = f(x)$ be a smooth radial function. If $u \in C^2(\Omega)$ is a stable solution of*

$$\begin{cases} -\Delta u = g(u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then u is radially symmetric.

Proof. We show that any tangential derivative $h = x_i u_{x_j} - x_j u_{x_i}$, $(i, j \in \{1, \cdots, N\})$ must satisfy $h \equiv 0$. First, by integrating by parts and using the boundary condition, we have

$$\int_\Omega h dx = \int_{\partial\Omega} (x_i \nu_j - x_j \nu_i) u ds_x = 0,$$

here ν_i denotes the i-th component of the unit normal vector ν to $\partial\Omega$. Next, by differentiating the equation, we have

$$-\Delta h = g'(u)h + x_i f_{x_j} - x_j f_{x_i} = g'(u)h \quad \text{in } \Omega$$

since $\Delta(x_i u_{x_j}) = x_i \Delta u_{x_j} + 2u_{x_i x_j}$ and f is radially symmetric. Also we have $h = 0$ on $\partial\Omega$ since $\nabla u \perp \partial\Omega$ and thus $x \wedge \nabla u = 0$ on $\partial\Omega$, where $\wedge$ denotes the exterior product. Then, multiplying h and integrating by parts, we obtain

$$\int_\Omega |\nabla h|^2 dx - \int_\Omega g'(u)h^2 dx = 0.$$

Since u is stable, this means that h is a minimizer of

$$\inf_{\phi \in H_0^1(\Omega), \phi \not\equiv 0} \frac{\int_\Omega |\nabla \phi|^2 dx - \int_\Omega g'(u)\phi^2 dx}{\int_\Omega \phi^2 dx}$$

if $h \not\equiv 0$. Thus the linearized operator $-\Delta - g'(u)\cdot$ (acting on $H_0^1(\Omega)$) has the smallest eigenvalue $\lambda_1(-\Delta - g'(u)\cdot) = 0$, and $h \not\equiv 0$ is the first eigenfunction corresponding to $\lambda_1(-\Delta - g'(u)\cdot)$. But in this case, h must be of constant sign on Ω, which contradicts the fact $\int_\Omega h dx = 0$. Thus we obtain $h \equiv 0$, which in turn implies u is radial. □

If the domain is a ball, we obtain the following result.

Theorem 4.2. *Let B denote the open unit ball in $\mathbb{R}^N$ and assume that $f \geq 0$, $f \not\equiv 0$ be any smooth radially symmetric function. If $N \geq 10$, then the extremal solution u^* of the problem*

$$\begin{cases} -\Delta u = e^u + \lambda f & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

satisfies $u^* \notin L^\infty(B)$.

Proof. First, we recall the improved Hardy inequality by Brezis and Vázquez [3]: For any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and for any $\phi \in H_0^1(\Omega)$, it holds that

$$\int_\Omega |\nabla \phi|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\phi^2}{|x|^2} dx + H_2 \left(\frac{\omega_N}{|\Omega|}\right)^{2/N} \int_\Omega \phi^2 dx,$$

where H_2 is the first Dirichlet eigenvalue of the Laplacian on the unit ball in $\mathbb{R}^2$ and ω_N is the measure of the unit ball in $\mathbb{R}^N$. By this inequality, we derive that the linearized operator $-\Delta - e^{v_s}\cdot = -\Delta - \frac{2(N-2)}{|x|^2}\cdot$ (acting on $H_0^1(\Omega)$), where v_s is a function as in (3.2), has a strict positive first eigenvalue. This fact in turn implies that the maximum principle is valid for the operator $-\Delta - e^{v_s}\cdot$; see, for example, [1].

Next, we claim that $u_\lambda < v_s$ holds for the minimal solution u_λ for any $\lambda \in (0, \lambda^*)$. Indeed, u_λ is radial by Lemma 4.1. Assume the contrary that there exists $r \in (0, 1)$ such that $u_\lambda(r) \geq v_s(r)$ for some $\lambda \in (0, \lambda^*)$, where $r = |x|$. Then $u_\lambda - v_s \geq 0$ on ∂B_r and

$$-\Delta(u_\lambda - v_s) = e^{u_\lambda} - e^{v_s} + \lambda f \geq e^{v_s}(u_\lambda - v_s) + \lambda f$$

by the convexity of $s \mapsto e^s$. Thus

$$-\Delta(u_\lambda - v_s) - e^{v_s}(u_\lambda - v_s) \geq 0 \quad \text{on } B_r$$

and we have $u_\lambda - v_s \geq 0$ on B_r by the maximum principle for the operator $-\Delta - e^{v_s}\cdot$. But this is impossible since $0 \in B_r$, $u_\lambda \in L^\infty(B_r)$ and $v_s \notin L^\infty(B_r)$. Thus we obtain the claim. By letting $\lambda \to \lambda^*$, we also get that $u^* \leq v_s$ on B.

By the above claim, we obtain that

$$\int_B |\nabla\phi|^2 dx - \int_B e^{u^*}\phi^2 dx \geq \inf_{\|\phi\|_{L^2(B)}=1} \left\{ \int_B |\nabla\phi|^2 dx - \int_B e^{v_s}\phi^2 dx \right\}$$

for any $\phi \in H_0^1(B)$ with $\|\phi\|_{L^2(B)} = 1$. The right hand side is strictly positive by the improved Hardy inequality and the assumption $N \geq 10$. On the other hand, if u^* is the classical solution to $(1.1)_{\lambda^*}$, the first eigenvalue of the operator $-\Delta - e^{u^*}\cdot$ (acting on $H_0^1(B)$)

$$\lambda_1(-\Delta - e^{u^*}) = \inf_{\phi \in H_0^1(B), \phi \not\equiv 0} \frac{\int_B |\nabla\phi|^2 dx - \int_B e^{u^*}\phi^2 dx}{\int_B \phi^2 dx}$$

must be 0 by the Implicit Function Theorem. Thus u^* cannot be bounded. This proves Theorem 4.2. □

Acknowledgement.

Part of this work was supported by JSPS Grant-in-Aid for Scientific Research (B), No. 23340038, and JSPS Grant-in-Aid for Challenging Exploratory Research, No. 26610030.

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