Minimization problem related with Lyapunov inequality

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Abstract

We consider a minimization problem on bounded smooth domain Ω in $\mathbb{R}^N$

$$S' := \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \;\middle|\; u \in H^1(\Omega) \setminus \{0\}\,,\ \int_\Omega |u|^{2^*-2}u = 0 \right\}.$$

This minimization problem plays a crucial role related with L^p Lyapunov-type inequalities ($1 \leq p \leq \infty$) for linear partial differential equations with Neumann boundary conditions (on bounded smooth domains in $\mathbb{R}^N$). In this paper, we prove that existence of the minimizer of S' and L^p Lyapunov-type inequalities in critical case.

Keywords: Minimization problem, Critical, Sign changing, Lyapunov inequalities, Neumann, Neumann boundary value problem

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1. Introduction

Let $N \geq 3$ and Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary. We consider the linear elliptic equation

$$\begin{cases} -\Delta u(x) = a(x)u(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ defined as

$$\Lambda := \left\{ a \in L^{N/2}(\Omega) \setminus \{0\} \;\middle|\; \int_\Omega a(x)dx \geq 0 \text{ and (1) has nontrivial solutions} \right\}.$$

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We define β_p as

$$\beta_p = \inf\left\{\|a^+\|_{L^p(\Omega)}\middle| a \in \Lambda \cap L^p(\Omega)\right\}.$$

The eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega \end{cases}$$

belong to Λ. Thus Λ is not empty therefore Λ is well defined. Cañada, Montero and Villegas [4] proved that β_p is attained in the case $N/2 < p \leq \infty$, $\beta_p = 0$ and it is not attained in the case $1 \leq p < N/2$. But the case $p = N/2$ has not been studied so far. In this paper we prove the case $p = N/2$ for $N \geq 4$. As result, $\beta_{N/2}$ is attained and the minimizer $a(x)$ is represented by the form

$$a(x) = |u(x)|^{\frac{4}{N-2}}$$

where $u(x)$ is solutions of some quasilinear elliptic equation. Timoshin[10] considered similar problem with Dirichlet boundary conditions, that is,

$$\begin{cases} -\Delta u(x) = a(x)u(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

$$\tilde{\Lambda} := \left\{a \in L^{N/2}(\Omega) \setminus \{0\} \middle| (2) \text{ has nontrivial solutions}\right\}.$$

$$\tilde{\beta}_p = \inf\left\{\|a\|_{L^p(\Omega)}\middle| a \in \tilde{\Lambda} \cap L^p(\Omega)\right\}.$$

About this problem, he proved that $\tilde{\beta}_p$ is not attained in the case $p = N/2$ by using not attainability of Sobolev best constant on the bounded domains. The result is $\tilde{\beta}_p = S$ is not attained where S is Sobolev best constant. This result is different from with Neumann boundary conditions.

2. Main Theorem

Theorem 2.1.
Let $N \geq 4$, Ω be bounded with smooth boundary. Then $\beta_{N/2}$ is attained. Furthermore $\beta_{N/2} = S'$ where

$$S' := \inf\left\{\frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \middle| u \in H^1(\Omega) \setminus \{0\}, \int_\Omega |u|^{2^*-2}u = 0\right\}.$$

From this, the minimizer of $\beta_{N/2}$ is represented that

$$a(x) = |u(x)|^{\frac{4}{N-2}}$$

where $u(x)$ is a solution of

$$\begin{cases} -\Delta u(x) = |u(x)|^{\frac{4}{N-2}} u(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

3. Preliminaries

Lemma 3.1. We have

$$S' < \frac{S}{2^{\frac{2}{N}}}$$

where S is Sobolev best constant.

Without loss of generality, we may assume that $0 \in \partial\Omega$, and that the mean curvature of $\partial\Omega$ at 0 is strictly positive.
For all $\varepsilon > 0$, $u_\varepsilon(x) \in H^1(\Omega)$ is defined by

$$u_\varepsilon(x) := \frac{(N(N-2)\varepsilon^2)^{\frac{N-2}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}} = \varepsilon^{\frac{2-N}{2}} U\Big(\frac{x}{\varepsilon}\Big)$$

where

$$U(x) = \frac{(N(N-2))^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

In addition, we define $\tilde{u}_\varepsilon(x)$ as follows.

$$\tilde{u}_\varepsilon(x) := \phi(x) u_\varepsilon(x)$$

where $\phi(x)$ is a suitable cut off function. Then, we have the following estimates due to Adimurthi and Mancini(see [1]) as $\varepsilon \to 0$:

$$\frac{\|\nabla \tilde{u}_\varepsilon\|_2^2}{\|\tilde{u}_\varepsilon\|_{2^*}^2} = \begin{cases} \frac{S}{2^{\frac{2}{N}}}(1 - c_0\varepsilon|\log\varepsilon| + O(\varepsilon)) & N = 3 \\ \frac{S}{2^{\frac{2}{N}}}(1 - c_1\varepsilon + O(\varepsilon^2|\log\varepsilon|)) & N = 4 \\ \frac{S}{2^{\frac{2}{N}}}(1 - c_2\varepsilon + O(\varepsilon^2)) & N \geq 5 \end{cases}$$

where c_0, c_1, c_2 are positive constants which depend only on N.

For each $\tilde{u}_\varepsilon$ there exist a constant $a_\varepsilon > 0$ such that

$$\tilde{u}_\varepsilon - a_\varepsilon \in X := \left\{ u \in H^1(\Omega) \middle| \int_\Omega |u|^{2^*-2} u = 0 \right\}.$$

Proposition 3.2. We obtain

$$a_\varepsilon = O\Big(\varepsilon^{\frac{(N-2)^2}{2(N+2)}}\Big).$$

Proof of Proposition 3.2. For $s \geq 1 (s \neq N/(N-2))$ we have

$$\|\tilde{u}_\varepsilon\|_s^s = O\Big(\varepsilon^{\min\left\{s\frac{2-N}{2}+N,\ s\frac{N-2}{2}\right\}}\Big).$$

In particular,

$$\begin{aligned} \|\tilde{u}_\varepsilon\|_1 &= O\big(\varepsilon^{\frac{N-2}{2}}\big) \\ \|\tilde{u}_\varepsilon\|_{\frac{N+2}{N-2}}^{\frac{N+2}{N-2}} &= O\big(\varepsilon^{\frac{N-2}{2}}\big) \\ \|\tilde{u}_\varepsilon\|_{2^*}^{2^*} &= O(1). \end{aligned}$$

Recall that

$$2^{p-1}(a^p + b^p) \geq (a+b)^p \ (a, b \geq 0, p \geq 1).$$

a, b and p are replaced by $a = |a_\varepsilon - \tilde{u}_\varepsilon|,\ b = \tilde{u}_\varepsilon,\ p = (N+2)/(N-2)$ in each, we obtain

$$\begin{aligned} 2^{\frac{4}{N-2}}\big(|a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}\big) &\geq \big(|a_\varepsilon - \tilde{u}_\varepsilon| + \tilde{u}_\varepsilon\big)^{\frac{N+2}{N-2}} \\ &\geq a_\varepsilon^{\frac{N+2}{N-2}}. \end{aligned}$$

We integrate above inequality over Ω and we have

$$2^{\frac{4}{N-2}} \int_\Omega \big(|a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}\big) \geq \int_\Omega a_\varepsilon^{\frac{N+2}{N-2}}$$

hence

$$2^{\frac{4}{N-2}} \int_\Omega |a_\varepsilon - \tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} \geq \int_\Omega a_\varepsilon^{\frac{N+2}{N-2}} - 2^{\frac{4}{N-2}} \int_\Omega \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}. \tag{4}$$

Since

$$\int_\Omega |\tilde{u}_\varepsilon - a_\varepsilon|^{2^*-2}(\tilde{u}_\varepsilon - a_\varepsilon) = 0$$

we calculate using (4)

$$\begin{aligned}
0 &= \int_\Omega |\tilde{u}_\varepsilon - a_\varepsilon|^{2^*-2}(\tilde{u}_\varepsilon - a_\varepsilon) \\
&= \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - \int_{[a_\varepsilon > \tilde{u}_\varepsilon]} (a_\varepsilon - \tilde{u}_\varepsilon)^{\frac{N+2}{N-2}} \\
&= 2^{\frac{N+2}{N-2}} \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - 2^{\frac{4}{N-2}} \int_\Omega |a_\varepsilon - u_\varepsilon|^{\frac{N+2}{N-2}} \\
&\leq 2^{\frac{N+2}{N-2}} \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} - \left\{ \int_\Omega a_\varepsilon^{\frac{N+2}{N-2}} - 2^{\frac{4}{N-2}} \int_\Omega \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_\Omega a_\varepsilon^{\frac{N+2}{N-2}} &\leq 2^{\frac{4}{N-2}} \left\{ 2 \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} (\tilde{u}_\varepsilon - a_\varepsilon)^{\frac{N+2}{N-2}} + \int_\Omega \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\} \\
&\leq 2^{\frac{4}{N-2}} \left\{ 2 \int_{[\tilde{u}_\varepsilon > a_\varepsilon]} \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} + \int_\Omega \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \right\} \\
&\leq 2^{\frac{4}{N-2}} 3 \int_\Omega u_\varepsilon^{\frac{N+2}{N-2}}.
\end{aligned}$$

Therefore

$$a_\varepsilon^{\frac{N+2}{N-2}} \leq C_0 \int_\Omega u_\varepsilon^{\frac{N+2}{N-2}} = C_0 \|u_\varepsilon\|_{\frac{N+2}{N-2}}^{\frac{N+2}{N-2}} = O(\varepsilon^{\frac{N-2}{2}}).$$

Hence we obtain

$$a_\varepsilon = O\Big(\varepsilon^{\frac{(N-2)^2}{2(N+2)}}\Big).$$ □

Proof. We estimate $\|\tilde{u}_\varepsilon - a_\varepsilon\|_{2^*}^2$ similarly to Girão and Weth(see [7]) using Proposition.

$$\begin{aligned}
\int_\Omega |\tilde{u}_\varepsilon - a_\varepsilon|^{2^*} &\geq \int_\Omega |\tilde{u}_\varepsilon|^{2^*} + |\Omega| a_\varepsilon^{2^*} - C\bigg(a_\varepsilon \int_\Omega |\tilde{u}_\varepsilon|^{\frac{N+2}{N-2}} + a_\varepsilon^{\frac{N+2}{N-2}} \int_\Omega |\tilde{u}_\varepsilon| \bigg) \\
&= \int_\Omega |\tilde{u}_\varepsilon|^{2^*} + O(\varepsilon^{\frac{N(N-2)}{N+2}}).
\end{aligned}$$

Consequently

$$\|\tilde{u}_\varepsilon - a_\varepsilon\|_{2^*}^2 \geq \|\tilde{u}_\varepsilon\|_{2^*}^2 + O(\varepsilon^{\frac{N(N-2)}{N+2}})$$

and therefore

$$\begin{aligned}\frac{\|\nabla(\tilde{u}_\varepsilon - a_\varepsilon)\|_2^2}{\|\tilde{u}_\varepsilon - a_\varepsilon\|_{2^*}^2} &\leq \frac{\|\nabla\tilde{u}_\varepsilon\|_2^2}{\|\tilde{u}_\varepsilon\|_{2^*}^2 + O(\varepsilon^{\frac{N(N-2)}{N+2}})} \\ &= \frac{\|\nabla\tilde{u}_\varepsilon\|_2^2}{\|\tilde{u}_\varepsilon\|_{2^*}^2} + O(\varepsilon^{\frac{N(N-2)}{N+2}}).\end{aligned}$$

We recall that the value of Sobolev quotient of $\tilde{u}_\varepsilon(x)$ in the case $N = 3$, $N = 4$ and $N \geq 5$ and taking account of the fact that $N \geq 4$ we obtain

$$\frac{\|\nabla(\tilde{u}_\varepsilon - a_\varepsilon)\|_2^2}{\|\tilde{u}_\varepsilon - a_\varepsilon\|_{2^*}^2} < \frac{S}{2^{\frac{2}{N}}} \text{ for } \varepsilon \text{ small enough,}$$

and hence

$$S' < \frac{S}{2^{\frac{2}{N}}}.$$

□

Lemma 3.3. If $S' < S/2^{N/2}$ then S' is attained.

Proof. We consider a minimizing sequence $\{u_n\} \in X$ for S'. Then u_n is bounded in $H^1(\Omega)$. So we can suppose, up to a subsequence,

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } H^1(\Omega) \quad (n \to \infty)\\ &u_n \to u \text{ in } L^p(\Omega) \quad (n \to \infty) \quad (1 \leq p < 2^*)\\ &u_n \to u \text{ a.e.} \quad (n \to \infty)\end{aligned}$$

In addition, since $H^1(\Omega) \hookrightarrow L^{2^*-1}(\Omega)$ is a compact embedding, we have

$$\int_\Omega |u_n|^{2^*-2}u_n \to \int_\Omega |u|^{2^*-2}u \quad (n \to \infty).$$

Furthermore, we may assume that

$$\begin{aligned}&\|u_n\|_{2^*} = 1 \quad (n \in \mathbb{N}),\\ &\|\nabla u_n\|_2^2 = S' + o(1) \ (n \to \infty).\end{aligned}$$

For each u_n there exist a constant a_n such that

$$u_n - u - a_n \in X.$$

We calculate similarly to the proof of Proposition 3.2. We obtain that

$$a_n = o(1) \quad (n \to \infty).$$

Since $\|u_n\|_{2^*}^{2^*} = 1$ for all $n \in \mathbb{N}$ by Brezis-Lieb lemma(see [2]) we have

$$\|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|u_n - u\|_{2^*}^{2^*} + o(1) \quad (n \to \infty).$$

Thus

$$1 = \|u_n\|_{2^*}^2 = (\|u\|_{2^*}^{2^*} + \|u_n - u\|_{2^*}^{2^*})^{\frac{2}{2^*}} + o(1) \le \|u\|_{2^*}^2 + \|u_n - u\|_{2^*}^2 + o(1).$$

On the other hand, we have

$$\begin{aligned} \|u\|_{2^*}^2 + (\|u_n - u - a_n\|_{2^*} + \|a_n\|_{2^*})^2 &\le \frac{\|\nabla u\|_2^2}{S'} + \frac{\|\nabla(u_n - u)\|_2^2}{S'} + o(1) \\ &= \frac{\|\nabla u_n\|_2^2}{S'} = 1 + o(1) \end{aligned}$$

and

$$\|u\|_{2^*}^2 + (\|u_n - u - a_n\|_{2^*} + \|a_n\|_{2^*})^2 \ge \|u\|_{2^*}^2 + \|u_n - u\|_{2^*}^2.$$

Thus

$$\|u\|_{2^*}^2 + \|u_n - u\|_{2^*}^2 \le 1 + o(1).$$

Hence there exists a limit and we have the equality.

$$\lim_{n\to\infty} (\|u_n - u\|_{2^*}^{2^*} + \|u\|_{2^*}^{2^*})^{\frac{2}{2^*}} = \lim_{n\to\infty} (\|u\|_{2^*}^2 + \|u_n - u\|_{2^*}^2) = 1.$$

Above equality holds if and only if $u \equiv 0$ or $u_n \to u$ in $L^{2^*}(\Omega)$. Suppose that $u \equiv 0$ a.e. By Cherrier's inequality(see [5][6]) we obtain

$$\frac{S}{2^{\frac{2}{N}}} \|u_n\|_{2^*}^2 \le (1 + \varepsilon)\|\nabla u_n\|_2^2 + C_\varepsilon \|u_n\|_2^2 \quad (\varepsilon > 0, n \in \mathbb{N}).$$

Replacing ε by $S/(S' 2^{(2+N)/N}) - 1/2 > 0$ and tending n to ∞, taking account to $u_n \to 0$ in $L^2(\Omega)$ we obtain

$$\lim_{n\to\infty} \frac{S}{2^{\frac{2}{N}}} \|u_n\|_{2^*}^2 \le \lim_{n\to\infty} \left(1 + \frac{1}{2S'} \frac{S}{2^{\frac{2}{N}}} - \frac{1}{2}\right) \|\nabla u_n\|_2^2.$$

Therefore
$$\frac{S}{2^{\frac{2}{N}}} \leq \left(\frac{1}{2S'} \frac{S}{2^{\frac{2}{N}}} + \frac{1}{2} \right) \lim_{n\to\infty} \|\nabla u_n\|_2^2.$$
Consequently
$$\frac{S}{2^{\frac{2}{N}}} \leq S'.$$
It is contradict $S/2^{N/2} > S'$. Hence $u \neq 0$ and $u_n \to u$ in L^{2^*}. Thus u is the minimizer of S'. □

4. Proof of Main theorem

We prove $\beta_{N/2} = S'$ and attainability of $\beta_{N/2}$ similar to Cañada, Montero and Villegas (see [4] the supercritical case). Since

$$X := \left\{ u \in H^1(\Omega) \middle| \phi(u) = 0 \right\}, \ \phi(u) := \int_\Omega |u|^{2^*-2} u$$

if $u_0 \in X \setminus \{0\}$ is any minimizer of S', Lagrange multiplier theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$F'(u_0) = \lambda \phi'(u_0)$$

where $F : H^1(\Omega) \to \mathbb{R}$ is defined by

$$F(u) = \|\nabla u\|_2^2 - S' \|u\|_{2^*}^2.$$

Also, since $u_0 \in X$ we have $\langle F'(u_0), 1 \rangle = 0$. Moreover, $\langle F'(u_0), v \rangle = 0$, $\forall v \in H^1(\Omega)$ satisfying $\langle \phi'(u_0), v \rangle = 0$. As any $v \in H^1(\Omega)$ may be written in the form $v = a + w$, $a \in \mathbb{R}$, and w satisfying $\langle \phi'(u_0), w \rangle = 0$, we conclude $\langle F'(u_0), v \rangle = 0$, $\forall v \in H^1(\Omega)$, i.e. $F'(u_0) \equiv 0$. Hence u_0 satisfies

$$\begin{cases} -\Delta u_0 = A(u_0)|u_0|^{\frac{4}{N-2}} u_0 & \text{in } \Omega \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

where

$$A(u) = S' \left(\int_\Omega |u|^{\frac{2N}{N-2}} \right)^{-\frac{2}{N}}.$$

If $a \in \Lambda \cap L^{\frac{N}{2}}(\Omega)$ and $u \in H^1(\Omega)$ is a nontrivial solution in (1), then for each $k \in \mathbb{R}$ we have

$$\begin{aligned} \|\nabla(u+k)\|_2^2 &= \|\nabla u\|_2^2 = \int_\Omega au^2 \le \int_\Omega au^2 + k^2 \int_\Omega a \\ &= \int_\Omega au^2 + k^2 \int_\Omega a + 2k \int_\Omega au = \int_\Omega a(u+k)^2 \le \|a^+\|_{\frac{N}{2}} \|u+k\|_{2^*}^2. \end{aligned}$$

Since u is a nontrivial solution of (1), $u+k$ is a nontrivial function. Consequently

$$\|a^+\|_{\frac{N}{2}} \ge \frac{\|\nabla(u+k)\|_2^2}{\|u+k\|_{2^*}^2}.$$

By choosing $k_0 \in \mathbb{R}$ such that $u + k_0 \in X$, we obtain

$$\beta_{\frac{N}{2}} \ge S'.$$

Conversely, if $u_0 \in X \setminus \{0\}$ is any minimizer of S', then u_0 satisfies (5). Therefore $A(u_0)|u_0|^{\frac{4}{N-2}} \in \Lambda \cap L^{N/2}(\Omega)$ and

$$\|A(u_0)|u_0|^{\frac{4}{N-2}}\|_{\frac{N}{2}} = S'\left(\int_\Omega |u_0|^{\frac{2N}{N-2}}\right)^{-\frac{2}{N}} \left(\int_\Omega |u_0|^{\frac{2N}{N-2}}\right)^{\frac{2}{N}} = S'.$$

Hence $\beta_{N/2} = S'$ and $\beta_{N/2}$ is attained.

On the other hand, let $a \in \Lambda \cap L^{\frac{N}{2}}$ be any minimizer of $\beta_{N/2}$. Then

$$\|a^+\|_{\frac{N}{2}} \|u+k_0\|_{2^*}^2 = \|\nabla(u+k_0)\|_2^2.$$

Hence $a(x) \equiv M|u(x)+k_0|^{\frac{4}{N-2}}(M > 0 : \text{constant})$. Furthermore, since $a(x) > 0$ we have $\int_\Omega a(x) \ge 0$. In addition, since

$$\int_\Omega au^2 = \int_\Omega a(u+k_0)^2$$

we obtain $k_0 \equiv 0$. Finally, we define $w(x) = M^{\frac{N-2}{4}}|u(x)|$ we have that

$$|w(x)|^{\frac{4}{N-2}} = M|u(x)|^{\frac{4}{N-2}} = a(x).$$

Moreover, since $u(x)$ is a solution of (1) and $w(x)$ is multiple of $u(x)$, then $w(x)$ is a solution of (1) and consequently a solution of (3). □

5. Corollary

Corollary 5.1. Let Ω be a ball $B := B(0,1)$ and u be a minimizer for S' on B. Then u is foliated Schwarz symmetric, i.e. there exists a unit vector $e \in \mathbb{R}^N$, $|e| = 1$ such that $u(x)$ only depends on $r = |x|$ and $\theta := \arccos(x/|x| \cdot e)$, and u is nonincreasing in θ. Moreover, either u does not depend on θ(hence it is a radial function), or $(\partial u/\partial\theta)(r, \theta) < 0$ for $0 < r \leq 1$, $0 < \theta < \pi$.

Proof. We can prove the Corollary 5.1. similar to Girão-Weth(see [7] Proposition 4.1.) □

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