A SIMPLE PROOF OF HARDY'S INEQUALITY IN A LIMITING CASE

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Abstract. In this short note, we provide a simple proof of Hardy's inequality in a limiting case. In the proof we do not need any rearrangement technique or the one-dimensional argument.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$. The classical Hardy's inequality is of the form

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \le \int_\Omega |\nabla u|^p dx \tag{1.1}$$

for all $u \in W_0^{1,p}(\Omega)$, where $N \ge 3$ and $1 < p < N$. See [3] for its simple proof which uses the Fundamental Theorem of Calculus and Hölder's inequality only. It is well known that the constant $\left(\frac{N-p}{p}\right)^p$ is optimal and never attained in $W_0^{1,p}(\Omega)$.

For $p = N$, the inequality (1.1) loses its sense and instead of (1.1) the inequality

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N \left(\log \frac{Re}{|x|}\right)^N} dx \le \int_\Omega |\nabla u|^N dx \tag{1.2}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x\in\Omega} |x|$. Again, the constant $\left(\frac{N-1}{N}\right)^N$ is known to be optimal; see for example, [1], [2]. We call (1.2) as Hardy's inequality in a limiting case. Main aim of this short note is to provide a simple proof of Hardy's inequality in a limiting case. We do not need any rearrangement technique such as Polya-Szegö inequality for the spherical decreasing rearrangement, or a technical one-dimensional argument. Also our method can provide the sharper inequality treated in [8], [2], [9] and [7].

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Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$ with $0 \in \Omega$. Let $g : (1, +\infty) \to \mathbb{R}$ be a C^2 function with the properties that*

$$g'(s) < 0, \quad g''(s) > 0 \quad \text{for any } s > 1 \tag{1.3}$$

and there exists $C > 0$ such that

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} \le C \quad \text{for any } s > 1. \tag{1.4}$$

Put $R = \sup_{x\in\Omega} |x|$. Then the inequality

$$\begin{aligned}\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N} \left(-g'\left(\log \frac{Re}{|x|}\right)\right)^{N-2} g''\left(\log \frac{Re}{|x|}\right) dx \\ \le \int_\Omega \frac{\left(-g'\left(\log \frac{Re}{|x|}\right)\right)^{2(N-1)}}{\left(g''\left(\log \frac{Re}{|x|}\right)\right)^{N-1}} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx\end{aligned} \tag{1.5}$$

holds true for any $u \in W_0^{1,N}(\Omega)$.

Corollary 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$ with $0 \in \Omega$. Then the inequality*

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u(x)|^N}{|x|^N \left(\log \frac{R}{|x|}\right)^N} dx \le \int_\Omega \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \tag{1.6}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x\in\Omega} |x|$.

Note that, different from the function $\frac{1}{|x|^N \left(\log \frac{Re}{|x|}\right)^N}$ appeared in (1.2), the function $\frac{1}{|x|^N \left(\log \frac{R}{|x|}\right)^N}$ in (1.6) becomes unbounded when $|x| \sim 0$ and also $|x| \sim R$.

Corollary 1.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$ with $0 \in \Omega$. Let $\alpha > 0$. Then the inequality*

$$\left(\frac{N-1}{N}\right)^N \alpha^N \int_\Omega \frac{|u(x)|^N}{|x|^{N-\alpha(N-1)}} dx \le \int_\Omega |x|^{\alpha(N-1)} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \tag{1.7}$$

holds for all $u \in W_0^{1,N}(\Omega)$.

If we take $\alpha = \frac{N}{N-1}$ in (1.7), we have

$$\int_\Omega |u(x)|^N dx \le \int_\Omega |x|^N \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \left(\le R^N \int_\Omega |\nabla u|^N dx\right),$$

thus (1.7) can be seen as a generalization of Poincaré's inequality for $u \in W_0^{1,N}(\Omega)$.

The proof of Theorem 1.1 relies on the divergence theorem and Hölder's inequality. Similar "simple" approaches have been proposed in [10], [4], and [5], mainly to derive (1.1). Their proof uses the identity

$$\operatorname{div}\left(\frac{x}{|x|^\lambda}\right)=\frac{N-\lambda}{|x|^\lambda}\quad \text{for } |x|\neq 0, \lambda\in\mathbb{R}.$$

On the other hand, the identity

$$\operatorname{div}\left(\frac{x}{|x|^N\left(\log\frac{R}{|x|}\right)^{N-1}}\right)=\frac{N-1}{|x|^N\left(\log\frac{R}{|x|}\right)^{N}}\quad \text{for } |x|\neq 0, R,$$

will be the base of our proof. See the next section.

Another approach to the limiting case of Hardy's inequality, using the mean integral of a Schwarz symmetrization, has been done by Ioku; see [6]:Remark 1.4.

2. Proof of Theorem.

In this section, we prove Theorem 1.1. For $\varepsilon>0$ small, put

$$R_\varepsilon=\sup_{x\in\Omega}\left(|x|^2+2\varepsilon^2\right)^{1/2},\quad X_\varepsilon(x)=\log\frac{R_\varepsilon e}{\left(|x|^2+\varepsilon^2\right)^{1/2}}$$

and

$$\psi_\varepsilon(x)=g(X_\varepsilon(x))\in C^2(\overline{\Omega}). \tag{2.1}$$

We calculate

$$\begin{aligned}
&|\nabla\psi_\varepsilon(x)|^{N-2}\nabla\psi_\varepsilon(x)=\left(-g'(X_\varepsilon)\right)^{N-1}\left(\frac{|x|^{N-2}x}{\left(|x|^2+\varepsilon^2\right)^{N-1}}\right),\\
&\Delta_N\psi_\varepsilon(x)=\operatorname{div}\left(|\nabla\psi_\varepsilon(x)|^{N-2}\nabla\psi_\varepsilon(x)\right)\\
&=(N-1)(-g'(X_\varepsilon))^{N-2}g''(X_\varepsilon)\frac{|x|^N}{\left(|x|^2+\varepsilon^2\right)^N}+(-g'(X_\varepsilon))^{N-1}\frac{2(N-1)\varepsilon^2|x|^{N-2}}{\left(|x|^2+\varepsilon^2\right)^N},
\end{aligned}$$

where we have used the assumption that $g'(s)<0$. For $u\in W_0^{1,N}(\Omega)$, divergence theorem assures that

$$\int_\Omega |u|^N\Delta_N\psi_\varepsilon dx=-\int_\Omega\nabla\left(|u|^N\right)\cdot|\nabla\psi_\varepsilon|^{N-2}\nabla\psi_\varepsilon dx. \tag{2.2}$$

The RHS of (2.2) is estimated from above as

$$
\begin{aligned}
|\text{RHS of (2.2)}| &= \left| N \int_\Omega |u|^{N-2} u \nabla u \cdot |\nabla \psi_\varepsilon|^{N-2} \nabla \psi_\varepsilon dx \right| \\
&= \left| N \int_\Omega |u|^{N-2} u (-g'(X_\varepsilon))^{N-1} \left(\frac{|x|^{N-2} x \cdot \nabla u}{(|x|^2+\varepsilon^2)^{N-1}} \right) dx \right| \\
&\le N \int_\Omega |u|^{N-1} (-g'(X_\varepsilon))^{N-1} \left(\frac{|x|^{N-2} |x \cdot \nabla u|}{(|x|^2+\varepsilon^2)^{N-1}} \right) dx \\
&\le N \left(\int_\Omega \frac{|u|^N |x|^N}{(|x|^2+\varepsilon^2)^N} \left(-g'(X_\varepsilon)\right)^{N-2} g''(X_\varepsilon) dx \right)^{\frac{N-1}{N}} \times \\
&\times \left(\int_\Omega (-g'(X_\varepsilon))^{2(N-1)} (g''(X_\varepsilon))^{-(N-1)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \right)^{\frac{1}{N}},
\end{aligned}
$$

where we have used Hölder's inequality. On the other hand, the LHS of (2.2) is estimated from below as

$$
\begin{aligned}
&|\text{LHS of (2.2)}| \\
&= (N-1) \int_\Omega |u|^N \left\{ (-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) \frac{|x|^N}{(|x|^2+\varepsilon^2)^N} + (-g'(X_\varepsilon))^{N-1} \frac{2\varepsilon^2 |x|^{N-2}}{(|x|^2+\varepsilon^2)^N} \right\} dx \\
&\ge (N-1) \int_\Omega |u|^N (-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) \frac{|x|^N}{(|x|^2+\varepsilon^2)^N} dx.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&(N-1) \int_\Omega \frac{|u|^N |x|^N}{(|x|^2+\varepsilon^2)^N} (-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) dx \\
&\quad \le N \left(\int_\Omega \frac{|u|^N |x|^N}{(|x|^2+\varepsilon^2)^N} (-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) dx \right)^{\frac{N-1}{N}} \times \\
&\quad \times \left(\int_\Omega (-g'(X_\varepsilon))^{2(N-1)} (g''(X_\varepsilon))^{-(N-1)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \right)^{\frac{1}{N}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left(\frac{N-1}{N} \right)^N \int_\Omega |u|^N (-g'(X_\varepsilon))^{N-2} g''(X_\varepsilon) \frac{|x|^N}{(|x|^2+\varepsilon^2)^N} dx \\
&\qquad \le \int_\Omega \frac{(-g'(X_\varepsilon))^{2(N-1)}}{(g''(X_\varepsilon))^{N-1}} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx.
\end{aligned}
$$

Finally, we let $\varepsilon \to 0$ in the both sides of the above inequality. Note that $R_\varepsilon \to R$ and $X_\varepsilon \to \log\left(\frac{Re}{|x|}\right)$ a.e. $x \in \Omega$ as $\varepsilon \to 0$. By using Fatou's lemma in the LHS and the Lebesgue dominated convergence theorem with the assumption (1.4) in the RHS, we conclude (1.5) holds. □

Proof of Corollary 1.2, 1.3.

Proof. For the proof of Corollary 1.2, let $g(s) = -\log(s-1)$ for $s > 1$. Then we see $g'(s) = -\frac{1}{s-1} < 0$, $g''(s) = \frac{1}{(s-1)^2} > 0$, and

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} = \frac{(\frac{1}{(s-1)})^{2(N-1)}}{(\frac{1}{(s-1)^2})^{N-1}} = 1 \quad \text{for any } s > 1.$$

Thus the assumptions (1.3),(1.4) are satisfied. Inserting

$$g'\left(\log\frac{Re}{|x|}\right) = \frac{-1}{\log\frac{R}{|x|}}, \quad g''\left(\log\frac{Re}{|x|}\right) = \frac{1}{(\log\frac{R}{|x|})^2}$$

into (1.5), we obtain (1.6).

For Corollary 1.3, take $g(s) = e^{-\alpha s}$ for $s > 1$. We easily see that

$$\frac{(-g'(s))^{2(N-1)}}{(g''(s))^{N-1}} = e^{-(N-1)\alpha s} \leq 1 \quad \text{for any } s > 1$$

and

$$g'\left(\log\frac{Re}{|x|}\right) = -\alpha\left(\frac{|x|}{Re}\right)^{\alpha}, \quad g''\left(\log\frac{Re}{|x|}\right) = \alpha^2\left(\frac{|x|}{Re}\right)^{\alpha}.$$

Using these and (1.5), we have (1.7). □

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