SIMPLICIAL 2-SPHERES OBTAINED FROM NON-SINGULAR COMPLETE FANS

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ABSTRACT. We prove that a simplicial 2-sphere satisfying a certain condition is the underlying simplicial complex of a 3-dimensional non-singular complete fan. In particular, this implies that any simplicial 2-sphere with ≤ 18 vertices is the underlying simplicial complex of such a fan.

1. INTRODUCTION

A *rational strongly convex polyhedral cone* in $\mathbb{R}^n$ is a cone σ spanned by finitely many vectors in $\mathbb{Z}^n$ which does not contain any non-zero linear subspace of $\mathbb{R}^n$. A *fan* in $\mathbb{R}^n$ is a non-empty collection Δ of such cones satisfying the following conditions:

(1) If $\sigma \in \Delta$, then each face of σ is in Δ;
(2) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of each.

A fan Δ is *non-singular* if any cone in Δ is spanned by a part of a basis of $\mathbb{Z}^n$, and *complete* if $\bigcup_{\sigma\in\Delta} \sigma = \mathbb{R}^n$.

A *toric variety* of complex dimension n is a normal algebraic variety X over $\mathbb{C}$ containing $(\mathbb{C}^*)^n$ as an open dense subset, such that the natural action of $(\mathbb{C}^*)^n$ on itself extends to an action on X. The category of toric varieties is equivalent to the category of fans (see [3]). A toric variety is smooth if and only if the corresponding fan is non-singular, and compact if and only if the fan is complete.

Given a non-singular fan Δ with m edges spanned by $v_1, \ldots, v_m \in \mathbb{Z}^n$, we define its *underlying simplicial complex* as

$$\{I \subset \{1, \ldots, m\} \mid \{v_i \mid i \in I\} \text{ spans a cone in } \Delta\}.$$

The underlying simplicial complex of an n-dimensional complete fan is a *simplicial* $(n-1)$*-sphere*, that is, a triangulation of the $(n-1)$-sphere.

For $n \geq 4$, a simplicial $(n-1)$-sphere is not always the underlying simplicial complex of an n-dimensional non-singular complete fan (see [2, Corollary 1.23]). On the other hand, successive equivariant blow-ups of $\mathbb{C}P^2$ produce non-singular complete fans whose underlying simplicial complexes are all simplicial 1-spheres. We consider the following problem:

Problem 1. *Is any simplicial 2-sphere the underlying simplicial complex of a 3-dimensional non-singular complete fan?*

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No counterexamples to Problem 1 are currently known. In this paper we give a partial affirmative answer to Problem 1. The *degree* of a vertex of a simplicial 2-sphere is the number of incident edges.

Theorem 2. *Let K be a simplicial 2-sphere with m_K vertices. We denote the number of vertices of K with degree k by $p_K(k)$. If $p_K(3)+p_K(4)+18 \geq m_K$, then K is the underlying simplicial complex of a 3-dimensional non-singular complete fan. In particular, if $m_K \leq 18$, then K is the underlying simplicial complex of such a fan.*

The proof is done by reducing a given simplicial 2-sphere to another one in a collection of certain simplicial 2-spheres with minimum degree 5. For each such simplicial 2-sphere, we use a computer to find a non-singular complete fan whose underlying simplicial complex is the simplicial 2-sphere.

The structure of the paper is as follows: In Section 2, we give a complete list of the simplicial 2-spheres with minimum degree 5 up to 18 vertices. In Section 3, we prove Theorem 2.

2. The simplicial 2-spheres with minimum degree 5 up to 18 vertices

G. Brinkmann and B. D. McKay calculated the number of combinatorially different simplicial 2-spheres with minimum degree 5 [1]:

vertices	simplicial 2-spheres	simplicial 2-spheres with min. deg. 5
4	1	0
5	1	0
6	2	0
7	5	0
8	14	0
9	50	0
10	233	0
11	1,249	0
12	7,595	1
13	49,566	0
14	339,722	1
15	2,406,841	1
16	17,490,241	3
17	129,664,753	4
18	977,526,957	12

Table 1. The number of simplicial 2-spheres.

Remark 3. An n-dimensional *small cover* of a simple n-polytope is a closed n-manifold M with a locally standard $(\mathbb{Z}_2)^n$-action such that the orbit space $M/(\mathbb{Z}_2)^n$ is the simple polytope. It follows from Steinitz's theorem that any simplicial 2-sphere is the boundary of a simplicial 3-polytope. The dual of the simplicial 3-polytope is a simple 3-polytope P. It follows from the four color theorem that P is the orbit space of a 3-dimensional small cover. A 3-dimensional small cover of P admits a hyperbolic structure if and only if P has no triangles or squares as facets, that is, the original simplicial 2-sphere has no vertices with degree 3 or 4 [2]. Table 1 shows that "most" 3-dimensional small covers do not admit any hyperbolic structure.

We give a complete list of such simplicial 2-spheres up to 18 vertices (see Tables 2 and 3). They are labeled as $\prod_{k\geq 5} k^{p(k)}$. If there are more than one simplicial 2-spheres with the same label, then we add (i), (ii), ... to the label. Letters and $\star$ on vertices in Tables 2 and 3 are used in Section 3.

For each simplicial 2-sphere, we consider the subcomplex consisting of the vertices with degree greater than or equal to 6 and the edges whose both endpoints have degree greater than or equal to 6 (red vertices and edges in Tables 2 and 3). These show that all simplicial 2-spheres in Tables 2 and 3 are distinct except $5^{12}6^6$ (ii) and $5^{12}6^6$ (iii) (they have the same subcomplex).

Since the subcomplexes of $5^{12}6^6$ (ii) and $5^{12}6^6$ (iii) are cycles, each cycle determines two subcomplexes surrounded by the cycle (see Figures 1 and 2). These are clearly distinct.

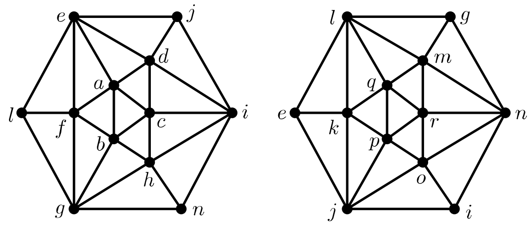


FIGURE 1. Subcomplexes of $5^{12}6^6$ (ii).

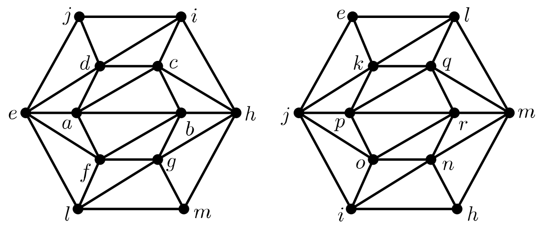


FIGURE 2. Subcomplexes of $5^{12}6^6$ (iii).

So all simplicial 2-spheres in Tables 2 and 3 are distinct.

For $m \leq 18$, the number of the simplicial 2-spheres with m vertices in Tables 2 and 3 agrees with the number in Table 1. So this is a complete list of the simplicial 2-spheres with minimum degree 5 up to 18 vertices.

3. Proof of the Theorem 2

Let K be a simplicial 2-sphere with m_K vertices.

Lemma 4. *If K is the underlying simplicial complex of a non-singular complete fan, then a simplicial 2-sphere obtained from K by an operation* (i), (ii) *or C_k ($k \geq 5$) is also the underlying simplicial complex of such a fan (see Figure 3).*

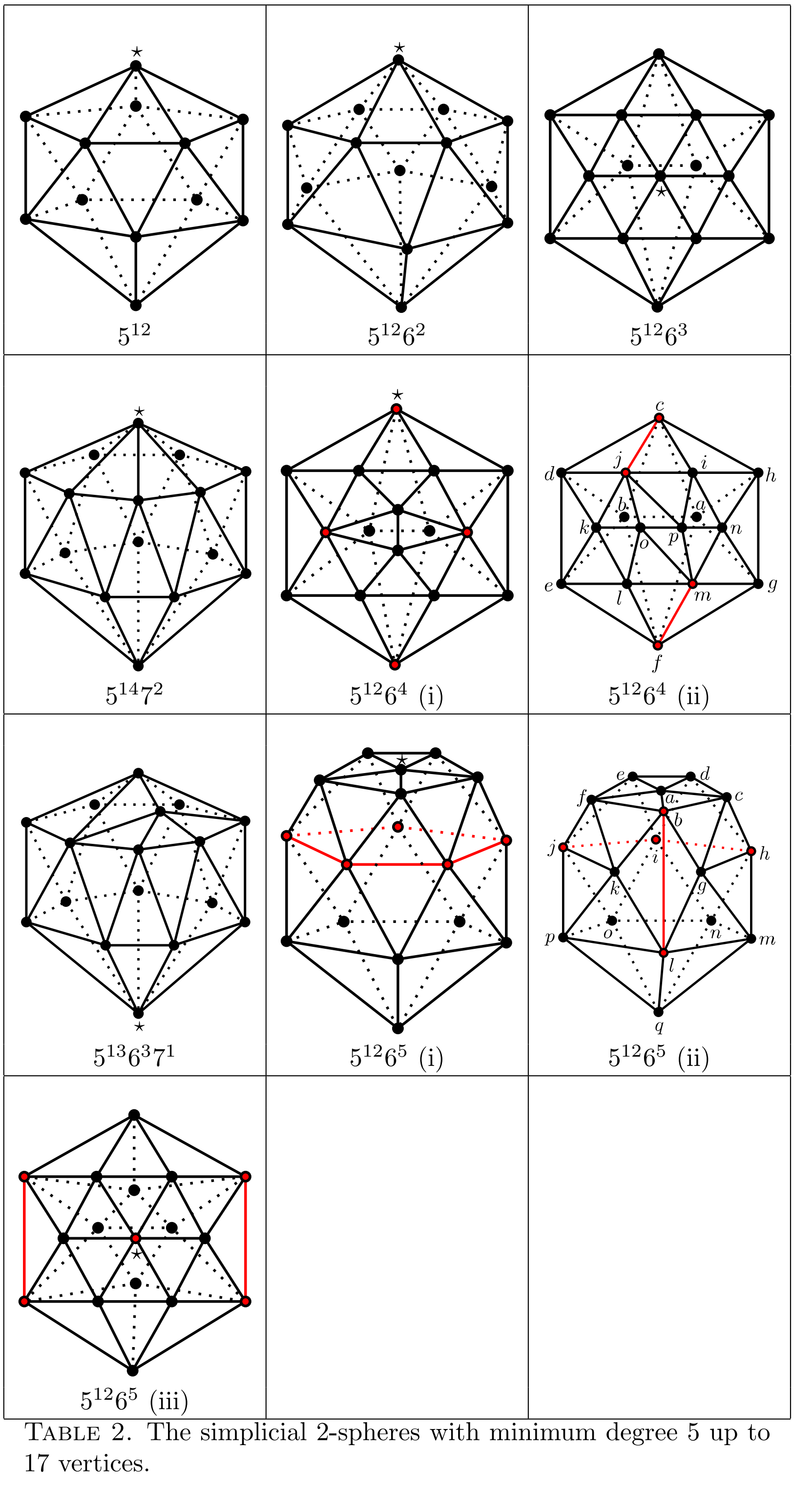


TABLE 2. The simplicial 2-spheres with minimum degree 5 up to 17 vertices.

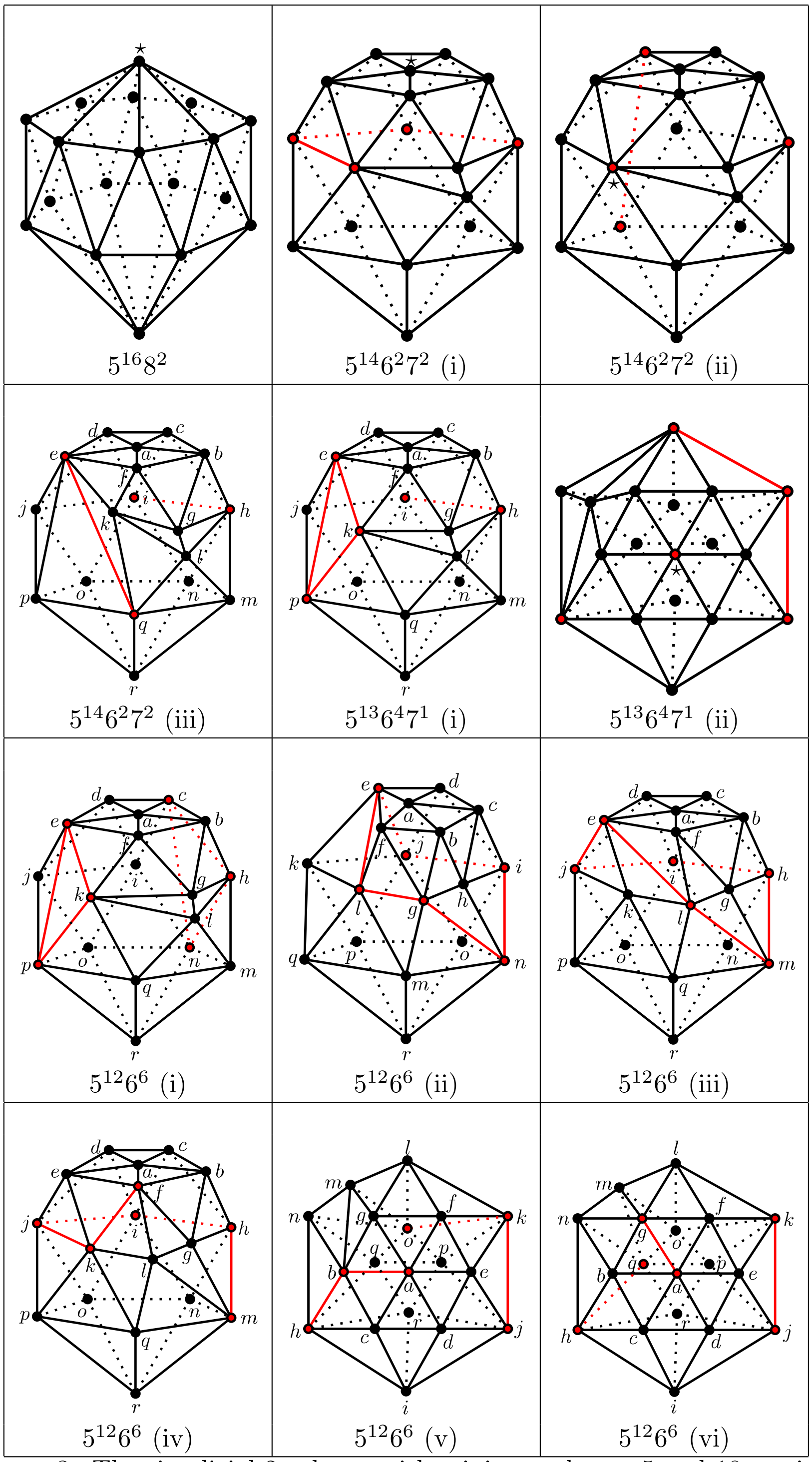
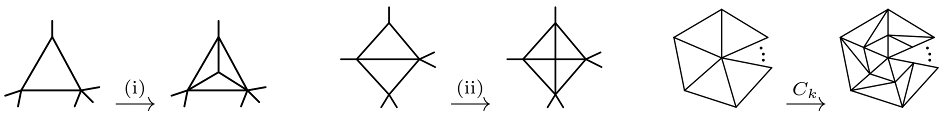


TABLE 3. The simplicial 2-spheres with minimum degree 5 and 18 vertices.



For the operation C_k, the degree of the vertex in the center of the diagram is k.

FIGURE 3. Operations (i), (ii) and C_k.

Proof. Suppose that the three vertices of a 2-face of K correspond to edge vectors $v_1, v_2, v_3 \in \mathbb{Z}^3$. Then we have $\det(v_1, v_2, v_3) = 1$. We assign $v_1 + v_2 + v_3$ to the new vertex made by the operation (i). The corresponding fan is non-singular and complete since $\det(v_1, v_2, v_1 + v_2 + v_3) = \det(v_2, v_3, v_1 + v_2 + v_3) = \det(v_3, v_1, v_1 + v_2 + v_3) = 1$. Thus the lemma holds for an operation (i) (see Figure 4).

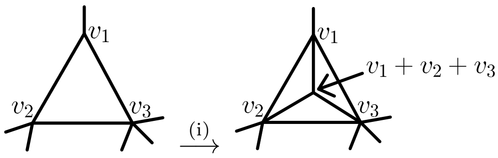


FIGURE 4. An operation (i).

Suppose that K contains a subcomplex in Figure 5 and the vertices correspond to edge vectors $v_1, v_2, v_3, v_4 \in \mathbb{Z}^3$ as in Figure 5. Then we have $\det(v_1, v_2, v_3) = \det(v_4, v_3, v_2) = 1$. We assign $v_2 + v_3$ to the new vertex made by the operation (ii). The corresponding fan is non-singular and complete since $\det(v_1, v_2, v_2 + v_3) = \det(v_3, v_1, v_2 + v_3) = \det(v_2, v_4, v_2 + v_3) = \det(v_4, v_3, v_2 + v_3) = 1$. Thus the lemma holds for an operation (ii).

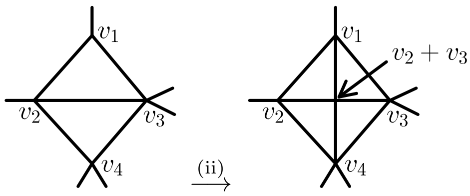


FIGURE 5. An operation (ii).

Suppose that K contains a subcomplex in Figure 6 and the vertices correspond to edge vectors $v, v_1, \ldots, v_k \in \mathbb{Z}^3$ as in Figure 6. Then we have $\det(v, v_i, v_{i+1}) = 1$ for any $i = 1, \ldots, k$, where $v_{k+1} = v_1$. For each $i = 1, \ldots, k$, we assign $v + v_i$ to the new vertex between v and v_i, which is made by the operation C_k. The corresponding fan is non-singular and complete since $\det(v, v + v_i, v + v_{i+1}) = \det(v_i, v + v_{i+1}, v + v_i) = \det(v_i, v_{i+1}, v + v_{i+1}) = 1$ for any $i = 1, \ldots, k$. Thus the lemma holds for an operation C_k. This completes the proof. $\square$

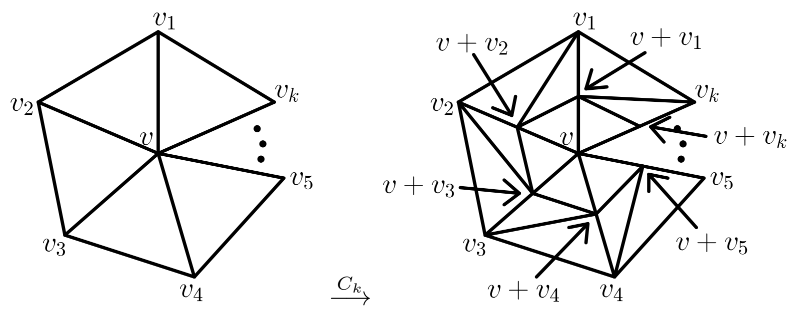


FIGURE 6. An operation C_k.

Now we prove Theorem 2 by induction on m_K. The tetrahedron is the only simplicial 2-sphere with 4 vertices, which is the underlying simplicial complex of the fan of $\mathbb{C}P^3$. Assume that $m_K \geq 5$.

(1) The case where there exists a vertex with degree 3. All adjacent vertices have degree greater than or equal to 4, since, if two vertices with degree 3 are adjacent, then K must be the tetrahedron, which contradicts $m_K \geq 5$. Thus we can perform an inverse operation of (i) and we get a simplicial 2-sphere K'. We see that $p_{K'}(3) + p_{K'}(4) \geq p_K(3) + p_K(4) - 1$. So we have $p_{K'}(3) + p_{K'}(4) + 18 \geq p_K(3) + p_K(4) + 18 - 1 \geq m_K - 1 = m_{K'}$. K' is the underlying simplicial complex of a non-singular complete fan by the induction hypothesis. Hence K is also the underlying simplicial complex of such a fan by Lemma 4.

(2) The case where there does not exist a vertex with degree 3 and there exists a vertex with degree 4. Since all adjacent vertices have degree greater than or equal to 4, we can perform an inverse operation of (ii) and we get a simplicial 2-sphere K'. We see that $p_{K'}(3) + p_{K'}(4) \geq p_K(3) + p_K(4) - 1$. The same argument as (1) implies that K is the underlying simplicial complex of a non-singular complete fan.

(3) The case where there does not exist a vertex with degree 3 or 4. The Euler relation implies that $\sum_{k\geq 3}(6 - k)p_K(k) = 12$ (see [3, p.190]). This shows that K must have a vertex with degree 5. Since $m_K \leq p_K(3) + p_K(4) + 18 = 18$ by assumption, K falls into 22 types in Tables 2 and 3.

Suppose that K has a vertex v with degree $k \geq 5$ such that any vertex adjacent to v has degree 5, and any vertex adjacent to a vertex adjacent to v has degree greater than or equal to 5. Then we can perform an inverse operation of C_k and we get a simplicial 2-sphere K'. Since $m_{K'} = m_K - k < 18 \leq p_{K'}(3) + p_{K'}(4) + 18$, K' is the underlying simplicial complex of a non-singular complete fan by the induction hypothesis. Hence K is also the underlying simplicial complex of such a fan by Lemma 4.

Each of 5^{12}, $5^{12}6^5$ (i) and $5^{14}6^27^2$ (i) has such a vertex for $k = 5$; each of $5^{12}6^2$, $5^{12}6^3$, $5^{12}6^4$ (i), $5^{12}6^5$ (iii) and $5^{13}6^47^1$ (ii) has such a vertex for $k = 6$; each of $5^{14}7^2$, $5^{13}6^37^1$ and $5^{14}6^27^2$ (ii) has such a vertex for $k = 7$; $5^{16}8^2$ has such a vertex for $k = 8$ (these vertices are indicated by $\star$ in Tables 2 and 3). So they are the underlying simplicial complexes of non-singular complete fans.

We show that the rest of simplicial 2-spheres $5^{12}6^4$ (ii), $5^{12}6^5$ (ii), $5^{14}6^27^2$ (iii), $5^{13}6^47^1$ (i) and $5^{12}6^6$ (i)–(vi) are the underlying simplicial complexes of non-singular complete fans with a computer aid. We assign vectors to the vertices as in Table 4.

They determine complete fans and it can be checked that all fans are non-singular by calculation.

vertex	$5^{12}6^4$ (ii)	$5^{12}6^5$ (ii)	$5^{14}6^27^2$ (iii), $5^{13}6^47^1$ (i), $5^{12}6^6$ (i)
a	$(1,0,0)$	$(1,0,0)$	$(0,-1,0)$
b	$(0,1,0)$	$(1,0,1)$	$(1,-1,0)$
c	$(0,0,1)$	$(2,-1,1)$	$(0,-1,1)$
d	$(-1,2,-1)$	$(3,0,-1)$	$(-1,-1,1)$
e	$(0,-1,-1)$	$(2,1,-1)$	$(-1,-1,0)$
f	$(1,0,-1)$	$(1,1,0)$	$(-1,-1,-1)$
g	$(1,-1,0)$	$(1,-1,1)$	$(0,-1,-1)$
h	$(1,-1,1)$	$(2,0,-1)$	$(1,0,0)$
i	$(-1,0,1)$	$(1,1,-1)$	$(0,0,1)$
j	$(-1,1,0)$	$(0,1,0)$	$(-1,0,1)$
k	$(-1,1,-1)$	$(0,0,1)$	$(-1,0,-1)$
l	$(0,-2,-1)$	$(0,-1,1)$	$(0,0,-1)$
m	$(1,-1,-1)$	$(2,-1,0)$	$(0,1,-1)$
n	$(0,-1,1)$	$(1,0,-1)$	$(1,1,0)$
o	$(0,-1,0)$	$(0,1,-1)$	$(0,1,1)$
p	$(0,-2,1)$	$(-1,1,0)$	$(-1,0,0)$
q		$(-1,0,0)$	$(-1,1,-1)$
r			$(0,1,0)$

vertex	$5^{12}6^6$ (ii)	$5^{12}6^6$ (iii)	$5^{12}6^6$ (iv)	$5^{12}6^6$ (v)	$5^{12}6^6$ (vi)
a	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(0,-1,0)$	$(0,-1,0)$
b	$(3,0,-1)$	$(3,0,-1)$	$(3,0,-1)$	$(-1,1,-1)$	$(-1,0,-1)$
c	$(2,1,-1)$	$(2,1,-1)$	$(2,1,-1)$	$(0,-2,-1)$	$(0,-2,-1)$
d	$(1,1,0)$	$(1,1,0)$	$(1,1,0)$	$(1,-1,-1)$	$(1,-1,-1)$
e	$(3,0,1)$	$(1,0,1)$	$(1,0,1)$	$(0,-1,1)$	$(0,-1,1)$
f	$(3,-1,1)$	$(3,-1,1)$	$(2,-1,1)$	$(-1,0,1)$	$(-1,0,1)$
g	$(2,0,-1)$	$(2,0,-1)$	$(2,0,-1)$	$(-1,1,0)$	$(-1,1,0)$
h	$(1,1,-1)$	$(1,1,-1)$	$(1,1,-1)$	$(0,-1,-1)$	$(0,-1,-1)$
i	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(1,0,-1)$	$(1,0,-1)$
j	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(1,-1,0)$	$(1,-1,0)$
k	$(1,-1,1)$	$(1,-1,1)$	$(1,-1,1)$	$(1,-1,1)$	$(1,-1,1)$
l	$(2,-1,1)$	$(2,-1,1)$	$(3,-1,0)$	$(0,0,1)$	$(0,0,1)$
m	$(1,0,-1)$	$(1,0,-1)$	$(1,0,-1)$	$(-1,2,0)$	$(-1,2,2)$
n	$(-1,1,0)$	$(0,1,-1)$	$(0,1,-1)$	$(-1,2,-1)$	$(-2,2,-1)$
o	$(0,0,1)$	$(-1,1,0)$	$(-1,1,0)$	$(0,1,2)$	$(0,1,2)$
p	$(0,-1,1)$	$(0,-1,1)$	$(0,-1,1)$	$(0,1,1)$	$(0,1,1)$
q	$(2,-1,0)$	$(2,-1,0)$	$(2,-1,0)$	$(-1,2,-2)$	$(-1,1,-1)$
r	$(-1,0,0)$	$(-1,0,0)$	$(-1,0,0)$	$(0,1,0)$	$(0,1,0)$

TABLE 4. Assigning vectors to the vertices.

For example, we show that $5^{14}6^27^2$ (iii) is the underlying simplicial complex of a non-singular complete fan. Vectors in Table 4 determine a 3-dimensional complete fan. Its underlying simplicial complex is illustrated in Figure 7, which confirms that there are no overlaps among the 3-dimensional cones. Calculating determinants, say $\det(a, b, c) = 1$, we see that every cone is non-singular.

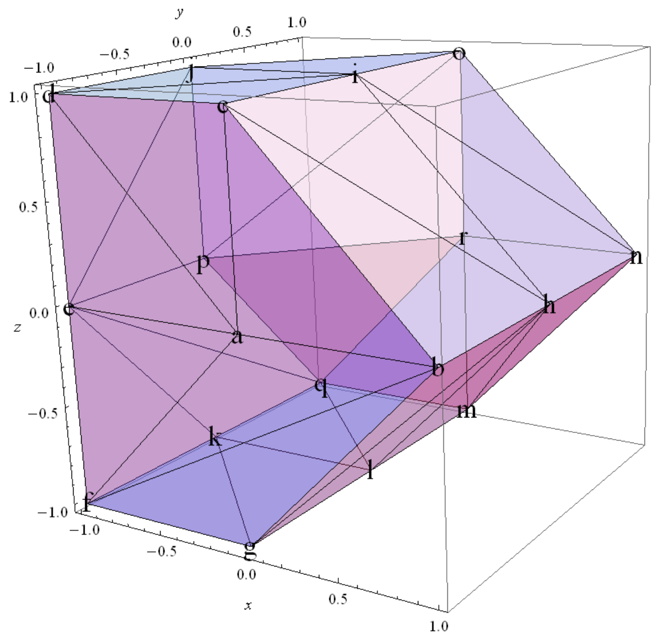


FIGURE 7. $5^{14}6^{2}7^{2}$ (iii).

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