LYAPUNOV INEQUALITY FOR AN ELLIPTIC PROBLEM WITH THE ROBIN BOUNDARY CONDITION

MASATO HASHIZUME AND FUTOSHI TAKAHASHI

Abstract. In this paper, we study the L^q-Lyapunov type inequalities for quasi-linear elliptic problems with the Robin boundary conditions, in which the principal part of the equation is p-Laplacian operator. Similar problems have been considered by several authors for linear elliptic problems with the Dirichlet or Neumann boundary conditions. We show that the critical value of the problem is $(N-1)/(p-1)$, where N is the dimension of the domain. We reveal the relation between this critical value and the critical exponent p_* of the trace Sobolev embedding. $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$.

1. Introduction.

The famous Lyapunov inequality in the theory of ordinary differential equations states the following: For a given function $a = a(x) \in C([b,c])$ on the interval $[b,c] \subset \mathbb{R}$, consider the problem

$$y''(x) + a(x)y(x) = 0, \quad x \in (b,c), \quad y(b) = y(c) = 0, \tag{1.1}$$

and put $\Lambda_0 = \{a \in C([b,c]) : \text{(1.1) has a nontrivial solution}\}$. Then it holds that

$$\inf_{a \in \Lambda_0} \int_b^c |a(x)| dx = \frac{4}{c-b}$$

and the infimum is never attained by a function in Λ_0; see for example, [1], [2].

In [3], Cañada, Montero, and Villegas extend the notion of Lyapunov inequality to partial differential equations. Namely, they consider the following linear elliptic problem

$$-\Delta u = a(x)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$ and the function $a : \Omega \to \mathbb{R}$ belongs to the set

$\Lambda_D = \{a \in L^{N/2}(\Omega) \ : \ \text{(1.2) has a nontrivial solution}\}, \quad \text{if } N \geq 3,$
$\Lambda_D = \{a : a \in L^r(\Omega) \text{ for some } r \in (1, +\infty] \text{ and (1.2) has a nontrivial solution}\},$
if $N = 2$.

Date: November 29, 2014.
2010 *Mathematics Subject Classification.* Primary 35J25, Secondary 35J20, 46E35.

Define

$$\beta_q(D) := \inf_{a \in \Lambda_D \cap L^q(\Omega)} \|a\|_{L^q(\Omega)}, \quad 1 \le q \le +\infty.$$

In [3], the authors initiated the qualitative study on the value $\beta_q(D)$ and proved several results. Later, Timoshin [9] treated the same problem and provided an additional information to the results in [3]. Their results can be summarized as follows.

Theorem 1.1. *([3], [9]) The following statements hold true:*

(i) *If $N = 2$ and $q = 1$, or $N \ge 3$ and $1 \le q < \frac{N}{2}$, then $\beta_q(D) = 0$ and $\beta_q(D)$ is not attained.*

(ii) *If $\frac{N}{2} < q \le \infty$, then $\beta_q(D)$ is attained.*

(iii) *If $N \ge 3$ and $q = \frac{N}{2}$, then $\beta_{\frac{N}{2}}(D) > 0$.*

([9]) More precisely, $\beta_{\frac{N}{2}}(D) = S_N$, where S_N is the best constant of the Sobolev inequality in $\mathbb{R}^N$: $S_N = \pi N(N-2)\left[\frac{\Gamma(N/2)}{\Gamma(N)}\right]^{2/N}$ and $\beta_{\frac{N}{2}}(D)$ is not attained.

In [3], it was left open whether $\beta_{\frac{N}{2}}(D)$ is attained or not. The non-attainability of $\beta_{\frac{N}{2}}(D)$ claimed above is first proved in [9].

The authors in [3] treated also the problem with the Neumann boundary condition

$$-\Delta u = a(x)u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \ge 2$. As before, set

$$\tilde{\Lambda} = \{a \in L^{N/2}(\Omega) \setminus \{0\} : \int_\Omega a(x)dx \ge 0 \text{ and (1.3) has a nontrivial solution}\},$$

if $N \ge 3$,

$$\tilde{\Lambda} = \{a : a \in L^r(\Omega) \setminus \{0\} \text{ for some } r \in (1, +\infty], \int_\Omega a(x)dx \ge 0,$$
$$\text{and (1.3) has a nontrivial solution}\}, \quad \text{if } N = 2.$$

Define

$$\tilde{\beta}_q := \inf_{a \in \tilde{\Lambda} \cap L^q(\Omega)} \|a\|_{L^q(\Omega)}, \quad 1 \le q \le +\infty.$$

Then the authors in [3] proved that the same statements in Theorem 1.1 hold true even for the value $\tilde{\beta}_q$, except for the attainability of $\tilde{\beta}_{N/2}$ in the critical case. Recently, the first author of the present paper proves that the value $\tilde{\beta}_{N/2}$ is attained for $N \ge 4$ [7], which is quite different from the fact in Theorem 1.1 (iii).

The aim of this paper is to extend the above results to a more general situation. Namely, we consider the following quasi-linear elliptic equation with the Robin boundary conditions

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = a(x)|u|^{p-2}u & \text{on } \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$, $a = a(x)$ is a given nonnegative function on the boundary $\partial\Omega$, $1 < p \leq N$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. By a term *solution*, we mean a weak solution in $W^{1,p}(\Omega)$, that is, a function $u \in W^{1,p}(\Omega)$ satisfying the weak form of (1.4)

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla\psi + |u|^{p-2}u\psi)dx = \int_{\partial\Omega} a(x)|u|^{p-2}\psi ds_x$$

for any $\psi \in W^{1,p}(\Omega)$. We assume that the function a belongs to the set Λ where

$\Lambda = \{a \in L^{\frac{N-1}{p-1}}(\partial\Omega) : (1.4) \text{ has a nontrivial solution}\}$, if $N > p$,

$\Lambda = \{a : a \in L^r(\partial\Omega) \text{ for some } r \in (1, +\infty] \text{ and } (1.4) \text{ has a nontrivial solution}\}$, if $N = p$.

Note that if $a \equiv 0$ on $\partial\Omega$, then (1.4) admits $u \equiv 0$ as the unique weak solution. This is because the solutions for $a \equiv 0$ correspond to the critical points of the convex functional

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx, \quad u \in W^{1,p}(\Omega),$$

which has $u \equiv 0$ as the unique critical point. Thus $0 \notin \Lambda$. As before, we define the value

$$\beta_q = \inf_{a \in \Lambda \cap L^q(\partial\Omega)} \|a\|_{L^q(\partial\Omega)}, \quad \text{for } 1 \leq q \leq +\infty. \tag{1.5}$$

Motivated by the former works, we study the qualitative properties of the value β_q, especially, the attainability of it according to the values p and q. Our main result in this paper reads as follows:

Theorem 1.2. *For $N \geq 2$ and $1 < p \leq N$, set $q_c = \frac{N-1}{p-1}$. The following statements hold true:*

(I) *If $1 \leq q < q_c$ when $1 < p < N$, or if $q = 1$ when $p = N = 2$, then $\beta_q = 0$ and β_q is not attained.*

(II) *If $q_c < q \leq \infty$ ($1 < p \leq N$), then β_q is attained. More precisely, $\beta_q = K_q$ if $q_c < q < \infty$, and $\beta_\infty = \lambda_1$, where*

$$K_q = \inf\left\{ \frac{\int_\Omega (|\nabla u|^p + |u|^p)dx}{(\int_{\partial\Omega} |u|^{\frac{pq}{q-1}} ds_x)^{\frac{q-1}{q}}} \Big| u \in W^{1,p}(\Omega) \setminus \{0\} \right\}, \tag{1.6}$$

and λ_1 is the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Any function $a \in \Lambda \cap L^q(\partial\Omega)$ which achieves β_q is of the form

(i) $a(x) \equiv \lambda_1$ *if* $q = \infty$.

(ii) $a(x) = |u(x)|^{\frac{p}{q-1}}$ *if* $q_c < q < \infty$, *where* u *is a solution of the problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{\frac{pq}{q-1}-2}u & \text{on } \Omega. \end{cases} \tag{1.8}$$

(III) *If* $q = q_c$, *then* $\beta_{q_c} = S_{p_*}(\Omega)$ *where* $p_* = \frac{(N-1)p}{N-p}$ *is the critical exponent and*

$$S_{p_*}(\Omega) = \inf\left\{ \frac{\int_\Omega(|\nabla u|^p + |u|^p)dx}{(\int_{\partial\Omega}|u|^{p_*}ds_x)^{\frac{p}{p_*}}} \Big| u \in W^{1,p}(\Omega) \setminus \{0\} \right\} \tag{1.9}$$

is the best constant of the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$, *respectively.*

Furthermore, there exists a constant $\gamma(\Omega) > 0$ *such that if* $1 < p < \frac{N+1}{2} + \gamma(\Omega)$, *then* β_{q_c} *is attained. The minimizer of* β_{q_c} *is written by* $a(x) = |u(x)|^{p_*-p}$ *where* $u(x)$ *is any solution of*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{p_*-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Remark 1.3. Here, we make some remarks.

(1) Different from Theorem 1.1:case (iii), in which elliptic problems with the Dirichlet boundary conditions are considered, β_q is attained in some cases even when q is the critical value $q_c = \frac{N-1}{p-1}$. Especially, β_{q_c} is always achieved when the equation in (1.4) is linear, i.e., when $p = 2$. This difference occurs due to the facts that the best constant in the Sobolev inequality cannot be attained on bounded domains, while the best constant in the trace Sobolev inequality is attained in some cases, see Theorem 4.1.

(2) It is plausible that $\beta_1 = 0$ and β_1 is not attained also for $p = N \geq 3$, but this is left open. One of the way to settle the problem affirmatively is to establish asymptotic estimates for the least energy solutions $\{u_r\}$ of the problem

$$\begin{cases} -\text{div}(|\nabla u|^{N-2}\nabla u) + u^{N-1} = 0 & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \overline{\Omega}, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} = u^r & \text{on } \partial\Omega \end{cases}$$

as $r \to \infty$, see Lemma 2.2.

2. Proof of Theorem 1.2 (I).

In this section, we prove the first part of Theorem 1.2. We divide the proof into several Lemmas according to the cases in Theorem 1.2 (I).

Lemma 2.1. *If $1 < p < N$ and $1 \le q < q_c = \frac{N-1}{p-1}$, then $\beta_q = 0$ and β_q is not attained.*

Proof. Given a nonnegative weight function V on $\partial\Omega$, $V \not\equiv 0$, let us consider the eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda V|u|^{p-2}u \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

For each $V \in L^s(\partial\Omega)$ with $s > (N-1)/(p-1)$, it is known that the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p_V(\partial\Omega)$ is compact, where

$$L^p_V(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R} \ : \ \int_{\partial\Omega} |u|^p V(x) ds_x < +\infty\}$$

denotes a weighted Lebesgue space on the boundary. Thus the existence, simplicity, and the variational characterization of the first eigenvalue λ_V of the problem (2.1) is well-known: see for example [6], [5], [4] and the references therein. Variational characterization of λ_V leads to the formula

$$\lambda_V = \inf\left\{ \frac{\int_\Omega (|\nabla u|^p + |u|^p)dx}{\int_{\partial\Omega} V|u|^p ds_x} \,\middle|\, u \in W^{1,p}(\Omega) \setminus \{0\} \right\}. \tag{2.2}$$

We recall some notations which are used by Nazarov and Reznikov [8]. A point x in $\mathbb{R}^N$ is denoted by $x = (x', x_N)$ where $x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$. Put $r = |x'|$ and define a cylinder $Q_\rho := \{x \in \mathbb{R}^N \ : \ r < \rho, \ 0 < x_N < \rho\}$. We denote by x_ε a point $x_\varepsilon = (0, \cdots, 0, -\varepsilon)$. ω_{N-1} denotes the surface area of the unit ball in $\mathbb{R}^N$. We use letter C to denote various positive constants.

Now, let us consider the least ball B which contains Ω and take a point of contact of Ω with B. By this choice, all principal curvatures, and therefore, the mean curvature $H(x_0)$ at x_0 is positive. We introduce a local coordinate system such that x_0 is the origin, and in some neighborhood of the origin, $\partial\Omega$ is expressed be the graph of a function $x_N = F(x')$. Since $\partial\Omega$ is smooth, we may assume that F is also smooth and $F(x') = (Ax', x') + o(r^2)$ as $r \to 0$, where A is a positive definite matrix since $H(x_0) > 0$. For $\varepsilon > 0$ small we define a cut-off function

$$\phi \in C_0^\infty(\mathbb{R}^N), \quad \phi = 1 \text{ on } Q_{\frac{\varepsilon}{2}}, \quad \phi = 0 \text{ on } \mathbb{R}^N \setminus Q_\varepsilon, \quad |\nabla\phi| \le \frac{C}{\varepsilon},$$

and put $V_\varepsilon := \phi(x)|x - x_\varepsilon|^{-(p-1)}$. Note that $V_\varepsilon \in L^\infty(\partial\Omega)$ since $x_\varepsilon \notin \partial\Omega \cap Q_\varepsilon$.

We claim that there exists a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$\lambda_\varepsilon := \lambda_{V_\varepsilon} \le C \tag{2.3}$$

holds true for any $\varepsilon > 0$ small.

In fact, test (2.2) with $V = V_\varepsilon$ by $\phi \in W^{1,p}(\Omega)$, we have

$$\int_\Omega |\nabla \phi|^p dx \le \int_0^\varepsilon \int_{|x'|<\varepsilon} \left(\frac{C}{\varepsilon}\right)^p dx' dx_N \le C\varepsilon^{N-p},$$
$$\int_\Omega |\phi|^p dx \le \int_0^\varepsilon \int_{|x'|<\varepsilon} dx' dx_N = C\varepsilon^N. \tag{2.4}$$

On the other hand, since

$$ds_x = \sqrt{1+|\nabla_{x'}F(x')|^2}dx', \quad |F(x')| \le C|x'|^2 \quad \text{if } |x'|<\varepsilon,$$
$$V_\varepsilon(x) = \frac{1}{(|x'|^2+(x_N+\varepsilon)^2)^{\frac{p-1}{2}}} \quad \text{on } Q_{\varepsilon/2}\cap\partial\Omega,$$

we have

$$\begin{aligned}\int_{\partial\Omega} V_\varepsilon|\phi|^p ds_x &\ge \int_{|x'|<\frac{\varepsilon}{2}} \frac{\sqrt{1+|\nabla_{x'}F(x')|^2}}{(|x'|^2+(F(x')+\varepsilon)^2)^{\frac{p-1}{2}}}dx' \\ &\ge \omega_{N-2}\int_0^{\frac{\varepsilon}{2}} \frac{r^{N-2}}{(r^2+(Cr^2+\varepsilon)^2)^{\frac{p-1}{2}}}dr \\ &\ge \frac{C\omega_{N-2}}{\left((\frac{\varepsilon}{2})^2+(C(\frac{\varepsilon}{2})^2+\varepsilon)^2\right)^{\frac{p-1}{2}}}\left(\frac{\varepsilon}{2}\right)^{N-1} \\ &= \frac{C}{(\varepsilon^2+o(1))^{\frac{p-1}{2}}}\varepsilon^{N-1} = C\varepsilon^{N-p}. \end{aligned} \tag{2.5}$$

The estimates (2.4) and (2.5) yield that

$$\lambda_\varepsilon \le \frac{\int_\Omega(|\nabla\phi|^p+|\phi|^p)dx}{\int_{\partial\Omega}V_\varepsilon|\phi|^p ds_x} \le C,$$

where C is independent of ε. This proves the claim.

Next, we prove $\beta_q = 0$. Indeed, since there exists a nontrivial first eigenfunction of (2.1) with $V = V_\varepsilon$ and $\lambda = \lambda_\varepsilon$, the above claim implies

$$\begin{aligned}\beta_q^q &= \inf_{a\in\Lambda\cap L^q(\partial\Omega)} \|a\|^q_{L^q(\partial\Omega)} \\ &\le \|\lambda_\varepsilon V_\varepsilon\|^q_{L^q(\partial\Omega)} = \lambda_\varepsilon^q \int_{\partial\Omega}\left(\phi(x)|x-x_\varepsilon|^{-(p-1)}\right)^q ds_x \\ &\le \lambda_\varepsilon^q \int_{|x'|<\varepsilon} \frac{\sqrt{1+|\nabla_{x'}F(x')|^2}}{(|x'|^2+(F(x')+\varepsilon)^2)^{\frac{q(p-1)}{2}}}dx' \\ &\le \lambda_\varepsilon^p C\int_{|x'|<\varepsilon}\frac{1}{|x'|^{q(p-1)}}dx' = \lambda_\varepsilon^q C\int_0^\varepsilon \frac{r^{N-2}}{r^{q(p-1)}}dr \\ &= \lambda_\varepsilon^p C\varepsilon^{N-1-q(p-1)} \to 0 \quad (\varepsilon\to 0),\end{aligned}$$

here we have used the assumption $q < \frac{N-1}{p-1}$ and the fact that $|\nabla_{x'}F(x')| \le C|x'|$ for $|x'|<\varepsilon$. Thus we obtain $\beta_q = 0$ for $q < \frac{N-1}{p-1}$ and β_q is not attained by nontrivial functions. □

Next, we consider the case when $q = 1$ and $p = N = 2$. For this purpose, we recall some result from [10]. Given $r > 1$, we define the quantity $S_r^2(\Omega)$ by

$$S_r^2(\Omega) = \inf \left\{ \frac{\int_\Omega (|\nabla u|^2 + u^2)dx}{(\int_{\partial\Omega} |u|^{r+1} ds_x)^{\frac{2}{r+1}}} \mid u \in W^{1,2}(\Omega) \setminus \{0\} \right\}.$$

Since the trace Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{r+1}(\partial\Omega)$ is compact for any $r > 1$ if $\Omega \subset \mathbb{R}^2$ is a regular bounded domain, we obtain a positive minimizer $\overline{u}_r$ for any $r > 1$ by standard variational methods. Define $u_r = S_r^{2/(r-1)} \overline{u}_r$, then u_r solves the elliptic problem

$$\begin{cases} -\Delta u + u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u > 0 \quad \text{in } \overline{\Omega}, \\ \frac{\partial u}{\partial \nu} = u^r \quad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

and u_r is called a least energy solution. In the sequel, we need the following estimates for the least energy solutions u_r.

Lemma 2.2. ([10]) *For the least energy solutions $\{u_r\}$ to (2.6) obtained as above,*

$$\lim_{r\to\infty} r \int_{\partial\Omega} u_r^{r+1} ds_x = \lim_{r\to\infty} r \int_\Omega (|\nabla u_r|^2 + u_r^2)dx = 2\pi e$$

holds true.

Now, we prove the following:

Lemma 2.3. *If $p = N = 2$, then $\beta_1 = 0$ and β_1 is not attained.*

Proof. Since there exist least energy solutions u_r to (2.6) for any $r > 1$ large, we have

$$\begin{aligned} \beta_1 &= \inf_{a \in \Lambda \cap L^1(\partial\Omega)} \|a\|_{L^1(\partial\Omega)} \leq \|u_r^{r-1}\|_{L^1(\partial\Omega)} \\ &= \int_{\partial\Omega} u_r^{r-1} ds_x \leq \left(\int_{\partial\Omega} u_r^{r+1} ds_x \right)^{\frac{r-1}{r+1}} |\partial\Omega|^{\frac{2}{r+1}} \\ &\leq \left(\frac{2\pi e}{r} + o(1) \right)^{\frac{r-1}{r+1}} |\partial\Omega|^{\frac{2}{r+1}} \to 0 \quad \text{as } r \to \infty, \end{aligned}$$

which proves Lemma 2.3. □

3. Proof of Theorem 1.2 (II).

In this section, we prove the second part of Theorem 1.2.

Proof. Assume that (1.4) admits a nontrivial solution $u \in W^{1,p}(\Omega)$. Multiplying (1.4) by u and integrating by parts, we have

$$0 = \int_\Omega \{(-\Delta_p u)u + |u|^p\}\,dx = \int_\Omega (|\nabla u|^p + |u|^p)\,dx - \int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} u ds_x$$
$$= \int_\Omega (|\nabla u|^p + |u|^p)\,dx - \int_{\partial\Omega} a(x)|u|^p ds_x.$$

First, we treat the case $\frac{N-1}{p-1} < q < \infty$. By using the Hölder inequality, we have

$$\int_\Omega (|\nabla u|^p + |u|^p)\,dx = \int_{\partial\Omega} a|u|^p ds_x \le \left(\int_{\partial\Omega} |a|^q ds_x\right)^{\frac{1}{q}} \left(\int_{\partial\Omega} |u|^{\frac{pq}{q-1}} ds_x\right)^{\frac{q-1}{q}}. \tag{3.1}$$

Thus

$$\|a\|_{L^q(\partial\Omega)} \ge \frac{\int_\Omega(|\nabla u|^p + |u|^p)dx}{(\int_{\partial\Omega} |u|^{\frac{pq}{q-1}} ds_x)^{\frac{q-1}{q}}} \ge \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^p + u^p)dx}{\left(\int_{\partial\Omega} |u|^{\frac{pq}{q-1}} ds_x\right)^{\frac{q-1}{q}}} = K_q.$$

Hence $\beta_q \ge K_q$.

On the othe hand, let $\{u_n\} \in W^{1,p}(\Omega)$ be a minimizing sequence for K_q. We can assume without loss of generality that

$$\int_{\partial\Omega} |u_n|^{\frac{pq}{q-1}} ds_x = 1, \quad \text{and} \quad \int_\Omega (|\nabla u_n|^p + |u_n|^p)dx \to K_q,$$

as $n \to \infty$. Thus $\{u_n\}$ becomes a bounded sequence in $W^{1,p}(\Omega)$. Now, since we assume $q > q_c = \frac{N-1}{p-1}$, we see

$$p < \frac{pq}{q-1} < \frac{pq_c}{q_c - 1} = \frac{(N-1)p}{N-p} = p_*.$$

Thus we can choose a subsequence (denoted again by the same symbol $\{u_n\}$) and a function $u_0 \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ and $u_n \to u_0$ strongly in $L^{pq/(q-1)}(\partial\Omega)$. The strong convergence in $L^{pq/(q-1)}(\partial\Omega)$ gives us

$$\int_{\partial\Omega} |u_0|^{\frac{pq}{q-1}} = 1.$$

Also the weak convergence in $W^{1,p}(\Omega)$ implies

$$\frac{\int_\Omega(|\nabla u_0|^p + |u_0|^p)dx}{\left(\int_{\partial\Omega} |u_0|^{\frac{pq}{q-1}} ds_x\right)^{\frac{q-1}{q}}} \le \liminf_{n\to\infty} \frac{\int_\Omega(|\nabla u_n|^p + |u_n|^p)dx}{(\int_{\partial\Omega} |u_n|^{\frac{pq}{q-1}} ds_x)^{\frac{q-1}{q}}} = K_q.$$

Hence $u_0 \in W^{1,p}(\Omega), u_0 \not\equiv 0$ is a minimizer of K_q. It is easy to see that u_0 is a weak solution to

$$\begin{cases} -\Delta_p u_0 + |u_0|^{p-2}u_0 = 0 & \text{in } \Omega, \\ |\nabla u_0|^{p-2}\frac{\partial u_0}{\partial \nu} = A_q(u_0)|u_0|^{\frac{p}{q-1}+p-2}u_0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

where

$$A_q(u_0) = K_q \left(\int_{\partial\Omega} |u_0|^{\frac{pq}{q-1}} ds_x \right)^{-\frac{1}{q}}.$$

Let $u_q \in W^{1,p}(\Omega)$ be any nontrivial minimizer of K_q, whose existence has been just proved above. Then u_q satisfies (3.2), thus $A_q(u_q)|u_q|^{p/(q-1)} \in \Lambda$, which implies

$$\beta_q^q \leq \| |A_q(u_q)|u_q|^{p/(q-1)} \|_{L^q(\partial\Omega)}^q = A_q(u_q)^q \int_{\partial\Omega} |u_q|^{\frac{pq}{q-1}} ds_x = K_q^q.$$

Hence $\beta_q = K_q$ and β_q is attained by a function $a_q(x) = A_q(u_q)|u_q|^{p/(q-1)} \in \Lambda$.

Conversely, let $a \in \Lambda \cap L^q(\partial\Omega)$ be any minimizer of β_q such that $\|a\|_{L^q(\partial\Omega)} = \beta_q$. Then all inequalities in (3.1) become the equality. Thus by the equality condition for the Hölder inequality, we see there exists $C > 0$ such that $a(x) \equiv C|u(x)|^{p/(q-1)}$. Finally, if we define $w(x) = C^{(q-1)/p}u(x)$, then we have $|w(x)|^{p/(q-1)} = C|u(x)|^{p/(q-1)} = a(x)$. Since u is any nontrivial solution of (1.4) and w is a constant multiple of u, w is also a solution of (1.4). Thus w satisfies (1.8) and we have proved the whole claim of Theorem 1.2 (2) when $q = \infty$.

Next, we consider the case $q = \infty$. For any $a \in L^\infty(\partial\Omega)$ and any nontrivial solution $u \in W^{1,p}(\Omega)$ to (1.4), we have

$$\int_\Omega (|\nabla u|^p + |u|^p) dx = \int_{\partial\Omega} a(x)|u|^p ds_x \leq \|a\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u|^p ds_x.$$

Therefore

$$\|a\|_{L^\infty(\partial\Omega)} \geq \frac{\int_\Omega (|\nabla u|^p + |u|^p) dx}{\int_{\partial\Omega} |u|^p ds_x} \geq \inf_{u \in W^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|^p + |u|^p) dx}{\int_{\partial\Omega} |u|^p ds_x} = \lambda_1,$$

where λ_1 denotes the first eigenvalue of the problem (1.7). Hence $\beta_\infty \geq \lambda_1$. On the other hand, for the constant function $a(x) \equiv \lambda_1$, there exists a nontrivial solution to (1.4), i.e., the first eigenfunction of (1.7) associated with λ_1, hence $\lambda_1 \in \Lambda \cap L^\infty(\partial\Omega)$ and we conclude $\beta_\infty = \lambda_1$. □

4. Proof of Theorem 1.2 (III).

In this final section, we prove the last part of Theorem 1.2 by invoking the recent result proved by Nazarov and Reznikov [8].

Theorem 4.1. ([8]*:Theorem 1.) Let $N \geq 2$. There exists a constant $\gamma(\Omega) > 0$ such that if $1 < p < \frac{N+1}{2} + \gamma(\Omega)$, then $S_{p_*}(\Omega)$ defined in (1.9) is attained.*

Note that $\frac{pq_c}{q_c-1} = p_* = \frac{(N-1)p}{N-p}$ for $q_c = \frac{N-1}{p-1}$. Thus we are aware of the relation $S_{p_*}(\Omega) = K_{q_c}$ where K_q is defined in (1.6). Since $\beta_q = K_q$ for $q_c < q < \infty$ by Theorem 1.2 (II), the continuity of K_q with respect to q implies that $\beta_{q_c} = K_{q_c} = S_{p_*}(\Omega)$.

Now, the proof of the third part of Theorem 1.2 can be carried just as in the former section. In fact, though the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ is not compact, Theorem 4.1 assures the existence of the minimizer for

$$K_{q_c} = S_{p_*}(\Omega) = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^{p_*}(\partial\Omega)}}$$

when $1 < p < \frac{N+1}{2} + \gamma(\Omega)$. The rest of the proof is identical to that of Theorem 1.2 (II). □

Acknowledgement. Part of this work was done under the support by JSPS Grant-in-Aid for Scientific Research (B), No. 23340038, and JSPS Grant-in-Aid for Challenging Exploratory Research, No. 24654043.

References

[1] R.C. Brown, and D.B. Hilton: *Lyapunov inequalities and their applications,* Survey on Classical Inequalities, Math. Appl., **517**, Kluwer Acad. Publ., Dordrecht, (2010), 1–25.

[2] A. Cañada, J.A. Montero, and S. Villegas: *Lyapunov-type inequalities and applications to PDE,* Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl. **63**, Birkhäuser, Basel (2005), 103–110.

[3] A. Cañada, J.A. Montero, and S. Villegas: *Lyapunov inequalities for partial differential equations,* J. Funct. Anal. **237** no.1, (2006), 176–193.

[4] J. Garcia-Azorero, I. Peral, and J. D. Rossi: *A convex-concave problem with a nonlinear boundary condition,* J. Differential Equations. **198**, (2004), 91–128.

[5] B. Fernandez, and J.D. Rossi: *A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding,* Publ. Mat.**46** no. 1, (2002), 221–235.

[6] J. D. Rossi: *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem,* Handbook of differential equations: stationary partial differential equations. Vol. II, (2005), 311–406.

[7] M. Hashizume: *Minimization problem related with Lyapunov inequality,* in preparation

[8] A.I. Nazarov, and A.B. Reznikov: *On the existence of an extremal function in critical Sobolev trace embedding theorem,* J. Funct. Anal. **258** no. 11, (2010), 3906–3921.

[9] S. A. Timoshin: *Lyapunov inequality for elliptic equations involving limiting nonlinearities,* Proc. Japan Acad. Ser. A., **86**, (2010), 139–142.

[10] F. Takahashi: *Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent,* J. Math. Anal. Appl. **411** no.1, (2014), 95–106.

Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail address: m13sa60s23@st.osaka-cu.ac.jp

Department of Mathematics, Osaka City University & OCAMI, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail address: futoshi@sci.osaka-cu.ac.jp