THE TORUS EQUIVARIANT COHOMOLOGY RINGS OF SPRINGER VARIETIES

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Abstract. The Springer variety of type A associated to a nilpotent operator on $\mathbb{C}^n$ in Jordan canonical form admits a natural action of the ℓ-dimensional torus T^ℓ where ℓ is the number of the Jordan blocks. We give a presentation of the T^ℓ-equivariant cohomology ring of the Springer variety through an explicit construction of an action of the n-th symmetric group on the T^ℓ-equivariant cohomology group. The T^ℓ-equivariant analogue of so called Tanisaki's ideal will appear in the presentation.

Contents

1. Introduction

The Springer variety of type A associated to a nilpotent operator $N : \mathbb{C}^n \to \mathbb{C}^n$ is a closed subvariety of the flag variety of $\mathbb{C}^n$ defined by

$$\{V_\bullet \in Flags(\mathbb{C}^n) \mid NV_i \subseteq V_{i-1} \text{ for all } 1 \leq i \leq n\}.$$

When the operator N is in Jordan canonical form with Jordan blocks of weakly decreasing size $\lambda = (\lambda_1, \cdots, \lambda_\ell)$, we denote the Springer variety by $\mathcal{S}_\lambda$. In 1970's, Springer constructed a representation of the n-th symmetric group S_n on the cohomology group $H^*(\mathcal{S}_\lambda; \mathbb{C})$, and this representation on the top degree part is the irreducible representation of type λ ([7], [8]). DeConcini-Procesi [] used this representation to give a presentation of the cohomology ring $H^*(\mathcal{S}_\lambda; \mathbb{C})$ as a quotient of a polynomial ring by an ideal. Tanisaki [9] gave another set of generators of this ideal which simplifies their presentation; this ideal is now called Tanisaki's ideal. We remark that his argument in [9] works also over $\mathbb{Z}$-coefficient. Our goal in this paper is to give an explicit presentation of the T^ℓ-equivariant cohomology ring $H^*_{T^\ell}(\mathcal{S}_\lambda; \mathbb{Z})$ where we will explain the ℓ-dimensional torus T^ℓ below. In more detail, we will give a presentation as the quotient of a polynomial

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ring by an ideal whose generators are generalizations of the generators of Tanisaki's ideal given in [9]. Through the the forgetful map $H^*_{T^\ell}(\mathcal{S}_\lambda;\mathbb{Z}) \to H^*(\mathcal{S}_\lambda;\mathbb{Z})$, our presentation naturally induces the presentation of $H^*(\mathcal{S}_\lambda;\mathbb{Z})$ given in [9].

We organize this paper as follows. In Section 2, we introduce a natural action of the ℓ-dimensional torus T^ℓ on the Springer variety $\mathcal{S}_\lambda$ for $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ and give the T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$ where T^ℓ is defined by the following diagonal unitary matrices:

$$\left\{ \begin{pmatrix} h_1 E_{\lambda_1} & & & \\ & h_2 E_{\lambda_2} & & \\ & & \ddots & \\ & & & h_\ell E_{\lambda_\ell} \end{pmatrix} \mid h_i \in \mathbb{C}, |h_i| = 1 \ (1 \leq i \leq \ell) \right\}.$$

Here, E_i is the identity matrix of size i. We construct an S_n-action on the equivariant cohomology group $H^*_{T^\ell}(\mathcal{S}_\lambda;\mathbb{Z})$ in Section 3 by using the localization technique which involves the equivariant cohomology of the T^ℓ-fixed points. We state the main theorem in Section 4, and prove it in Section 5 by using this S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda;\mathbb{Z})$. Our method of the proof is the T^ℓ-equivariant analogue of [9].

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2. Nilpotent Springer varieties and T^ℓ-fixed points

We begin with a definition of type A nilpotent Springer varieties. We work with type A through out this paper and hence omit it in the following. We first recall that a flag variety $Flags(\mathbb{C}^n)$ consists of nested subspaces of $\mathbb{C}^n$:

$$V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n)$$

where $\dim_{\mathbb{C}} V_i = i$ for all i.

Definition. Let $N\colon \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator. The **(nilpotent) Springer variety** $\mathcal{S}_N$ associated to N is the set of flags $V_\bullet$ satisfying $NV_i \subseteq V_{i-1}$ for all $1 \leq i \leq n$.

Since $\mathcal{S}_{gNg^{-1}}$ is homeomorphic (in fact, isomorphic as algebraic varieties) to $\mathcal{S}_N$ for any invertible matrix $g \in GL_n(\mathbb{C})$, we may assume that N is a Jordan canonical form. In this paper, we consider the Springer variety

$$\mathcal{S}_\lambda := \{V_\bullet \in Flags(\mathbb{C}^n) \mid N_0 V_i \subseteq V_{i-1} \text{ for all } 1 \leq i \leq n\}$$

where N_0 is in Jordan canonical form with Jordan blocks of weakly decreasing sizes $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

Let T^n be an n-dimensional torus consisting of diagonal unitary matrices:

$$T^n = \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} \mid g_i \in \mathbb{C}, |g_i| = 1 \ (1 \leq i \leq n) \right\}. \tag{2.1}$$

Then the n-dimensional torus T^n naturally acts on the flag variety $Flags(\mathbb{C}^n)$, but T^n does not preserve the Springer variety $\mathcal{S}_\lambda$ in general. So we introduce the following ℓ-dimensional torus:

$$(2.2)\qquad T^\ell = \left\{ \begin{pmatrix} h_1 E_{\lambda_1} & & & \\ & h_2 E_{\lambda_2} & & \\ & & \ddots & \\ & & & h_\ell E_{\lambda_\ell} \end{pmatrix} \in T^n \mid h_i \in \mathbb{C}, |h_i| = 1 \ (1 \le i \le \ell) \right\}$$

where E_i is the identity matrix of size i and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. Then the torus T^ℓ preserves the Springer variety $\mathcal{S}_\lambda$. Our goal in this section is to give the T^ℓ-fixed point set $\mathcal{S}_\lambda^{T^\ell}$.

The T^n-fixed point set $Flags(\mathbb{C}^n)^{T^n}$ of the flag variety $Flags(\mathbb{C}^n)$ is given by

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \dots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{C}^n$ and S_n is the symmetric group on n letters $\{1, 2, \dots, n\}$, so we may identify $Flags(\mathbb{C}^n)^{T^n}$ with S_n.

Let w be an element of S_n satisfying the following property:

(2.3) for each $1 \le k \le \ell$, the numbers between $\lambda_1 + \cdots + \lambda_{k-1} + 1$ and $\lambda_1 + \cdots + \lambda_k$ appear in the one-line notation of w as a subsequence in the increasing order.

Here, we write $\lambda_1 + \cdots + \lambda_{k-1} + 1 = 1$ when $k = 1$.

Example. We consider the case $n = 6$, $\ell = 3$, and $\lambda = (3, 2, 1)$. Using one-line notation, the following permutations

$$w_1 = 124365, \quad w_2 = 416253, \quad w_3 = 612435$$

satisfy the condition (2.3). In fact, each of the sequences $(1, 2, 3)$, $(4, 5)$, and (6) appears in the one-line notations as a subsequence in the increasing order.

Lemma 2.1. *The T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$ is the set*

$$\{w \in S_n \mid w \text{ satisfy the condition (2.3)}\}.$$

Proof. Let $w = V_\bullet$ be a permutation satisfying the condition (2.3). Since $w(1)$ is equal to one of the numbers 1, $\lambda_1 + 1$, $\lambda_1 + \lambda_2 + 1$, $\dots$, $\lambda_1 + \cdots + \lambda_{\ell-1} + 1$, we have $N_0 V_1 \subseteq \{0\}$. If $w(1) = \lambda_1 + \cdots + \lambda_{k-1} + 1$, then $w(2)$ is equal to one of the numbers 1 ,$\lambda_1 + 1$, $\dots, \lambda_1 + \cdots + \lambda_{k-1} + 2$, $\dots$, $\lambda_1 + \cdots + \lambda_{\ell-1} + 1$. So we also have $N_0 V_2 \subseteq V_1$. Continuing this argument, we have $N_0 V_i \subseteq V_{i-1}$ for all $1 \le i \le n$, and it follows that the w is an element of $\mathcal{S}_\lambda$. On the other hand, the w is clearly fixed by T^ℓ, so the w is an element of $\mathcal{S}_\lambda^{T^\ell}$.

Conversely, let $V_\bullet$ be an element of $\mathcal{S}_\lambda^{T^\ell}$. Let v_1,v_2,$\dots$,v_j be generators for V_j where $v_j = (x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)})^t$ in $\mathbb{C}^n$ for all j. Since we have

$$N_0 v_1 = (\underbrace{x_2^{(1)}, \cdots, x_{\lambda_1}^{(1)}, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+2}^{(1)}, \cdots, x_{\lambda_1+\lambda_2}^{(1)}, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+2}^{(1)}, \cdots, x_n^{(1)}, 0}_{\lambda_\ell})^t,$$

the condition $N_0V_1 \subseteq V_0 = \{0\}$ implies that

(2.4) $$v_1 = (\underbrace{x_1^{(1)}, 0, \cdots, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+1}^{(1)}, 0, \cdots, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(1)}, 0, \cdots, 0}_{\lambda_\ell})^t.$$

It follows that exactly one of $x_i^{(1)}$ $(i = 1, \lambda_1+1, \lambda_1+\lambda_2+1, \ldots, \lambda_1+\cdots+\lambda_{\ell-1}+1)$ which appear in (2.4) is nonzero. In fact, $V_\bullet$ is fixed by the T^ℓ-action and hence we have $\langle h\cdot v_1\rangle = \langle v_1\rangle$ for arbitrary $h \in T^\ell$ where

$$h\cdot v_1 = (\underbrace{h_1x_1^{(1)}, 0, \cdots, 0}_{\lambda_1}, \underbrace{h_2x_{\lambda_1+1}^{(1)}, 0, \cdots, 0}_{\lambda_2}, \cdots\cdots, \underbrace{h_\ell x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(1)}, 0, \cdots, 0}_{\lambda_\ell})^t.$$

Since each h_i runs over all complex numbers whose absolute values are 1, only one of $x_i^{(1)}$ in (2.4) must be nonzero.

If $x_{\lambda_1+\cdots+\lambda_{k-1}+1}^{(1)}$ is nonzero for some $1 \leq k \leq \ell$, then we may assume that

$$v_1 = (0, \cdots, 0, 1, 0, \cdots, 0)^t,$$
$$v_j = (x_1^{(j)}, \cdots, x_{\lambda_1+\cdots+\lambda_{k-1}}^{(j)}, 0, x_{\lambda_1+\cdots+\lambda_{k-1}+2}^{(j)}, \cdots, x_n^{(j)})^t$$

for $2 \leq j \leq n$ where the $(\lambda_1+\cdots+\lambda_{k-1}+1)$-th component of v_1 is one. Since we have

$$N_0v_2 = (\underbrace{x_2^{(2)}, \cdots, x_{\lambda_1}^{(2)}, 0}_{\lambda_1}, \underbrace{x_{\lambda_1+2}^{(2)}, \cdots, x_{\lambda_1+\lambda_2}^{(2)}, 0}_{\lambda_2}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+2}^{(2)}, \cdots, x_n^{(2)}, 0}_{\lambda_\ell})^t,$$

the condition $N_0V_2 \subseteq V_1$ implies that

(2.5)
$$v_2 = (\underbrace{x_1^{(2)}, 0, \cdots, 0}_{\lambda_1}, \cdots\cdots, \underbrace{0, x_{\lambda_1+\cdots+\lambda_{k-1}+2}^{(2)}, 0, \cdots, 0}_{\lambda_k}, \cdots\cdots, \underbrace{x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}^{(2)}, 0, \cdots, 0}_{\lambda_\ell})^t.$$

Therefore, we see that the only one of $x_i^{(2)}$ $(i = 1, \lambda_1+1, \ldots, \lambda_1+\cdots+\lambda_{k-1}+2, \ldots, \lambda_1+\cdots+\lambda_{\ell-1}+1)$ which appear in (2.5) is nonzero by an argument similar to that used above. Continuing this procedure, we conclude that $V_\bullet = w$ for some $w \in S_n$ satisfying the condition (2.3). In fact, $w(1)$ is equal to one of the numbers 1, λ_1+1, $\lambda_1+\lambda_2+1$, $\ldots$, $\lambda_1+\cdots+\lambda_{\ell-1}+1$. If $w(1) = \lambda_1+\cdots+\lambda_{k-1}+1$, then $w(2)$ is equal to one of the numbers 1 ,λ_1+1, $\ldots$, $\lambda_1+\cdots+\lambda_{k-1}+2$, $\ldots$, $\lambda_1+\cdots+\lambda_{\ell-1}+1$ and so on. This means that for each $k = 1, \ldots, \ell$ the numbers between $\lambda_1+\cdots+\lambda_{k-1}+1$ and $\lambda_1+\cdots+\lambda_k$ appear in the one-line notation of w as a subsequence in the increasing order. □

Regarding a product of symmetric groups $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$ as a subgroup of the symmetric group S_n, it follows from Lemma 2.1 that the T^ℓ-fixed points $\mathcal{S}_\lambda^{T^\ell}$ of the Springer variety $\mathcal{S}_\lambda$ is identified with the right cosets $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$ where each $w \in \mathcal{S}_\lambda^{T^\ell}$ corresponds to the right coset $[w]$. In fact, the condition (2.3) provides a unique representative for each right coset.

3. An action of the symmetric group S_n on $H^*_{T^\ell}(\mathcal{S}_\lambda)$

In this section, we introduce an action of the symmetric group S_n on the equivariant cohomology group $H^*_{T^\ell}(\mathcal{S}_\lambda)$ over $\mathbb{Z}$-coefficient by using the localization technique. We will see that the projection map

$$\rho_\lambda : H^*_{T^n}(Flags(\mathbb{C}^n)) \to H^*_{T^\ell}(\mathcal{S}_\lambda)$$

induced from the inclusions of $\mathcal{S}_\lambda$ into $Flags(\mathbb{C}^n)$ and T^ℓ into T^n is an S_n-equivariant map. In particular, we consider the following commutative diagram

$$\begin{array}{ccc} H^*_{T^n}(Flags(\mathbb{C}^n)) & \xrightarrow{\iota_1} & H^*_{T^n}(Flags(\mathbb{C}^n)^{T^n}) = \bigoplus_{w \in S_n} H^*(BT^n) \\ \rho_\lambda \big\downarrow & & \pi \big\downarrow \\ H^*_{T^\ell}(\mathcal{S}_\lambda) & \xrightarrow{\iota_2} & H^*_{T^\ell}(\mathcal{S}_\lambda^{T^\ell}) = \bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} H^*(BT^\ell) \end{array} \tag{3.1}$$

where all the maps are induced from inclusion maps, and construct S_n-actions on the three modules $H^*_{T^n}(Flags(\mathbb{C}^n))$, $\bigoplus_{w\in S_n} H^*(BT^n)$, and $\bigoplus_{w\in \mathcal{S}_\lambda^{T^\ell}} H^*(BT^\ell)$ to construct an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$. All (equivariant) cohomology rings are assumed to be over $\mathbb{Z}$-coefficient unless otherwise specified.

First, we introduce the left action of the symmetric group S_n on the cohomology group $H^*(Flags(\mathbb{C}^n))$. To do that, we consider the right S_n-action on the flag variety $Flags(\mathbb{C}^n)$ as follows.

For any $V_\bullet \in Flags(\mathbb{C}^n)$, there exists $g \in U(n)$ so that $V_i = \bigoplus_{j=1}^i \mathbb{C}g(e_j)$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Then the right action of $w \in S_n$ on $Flags(\mathbb{C}^n)$ can be defined by

$$V_\bullet \cdot w = V'_\bullet \tag{3.2}$$

where $V'_i = \bigoplus_{j=1}^i \mathbb{C}g(e_{w(j)})$.

We recall an explicit presentation of the T^n-equivariant cohomology ring of the flag variety $Flags(\mathbb{C}^n)$. Let E_i be the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T^n-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\bar{x}_i \in H^2_{T^n}(Flags(\mathbb{C}^n))$. Let $\mathbb{C}_i$ be the one dimensional representation of T^n through a map $T^n \to S^1$ given by $diag(g_1, \dots, g_n) \mapsto g_i$. We denote the first Chern class of the line bundle $ET^n \times_{T^n} \mathbb{C}_i$ over BT^n by $t_i \in H^2(BT^n)$. Since $t_1, \dots, t_n$ generate $H^*(BT^n)$ as a ring and they are algebraically independent, we identify $H^*(BT^n)$ with a polynomial ring;

$$H^*(BT^n) = \mathbb{Z}[t_1, \dots, t_n].$$

Then the equivariant cohomology $H^*_{T^n}(Flags(\mathbb{C}^n))$ is generated by $\bar{x}_1, \dots, \bar{x}_n, t_1, \dots, t_n$ as a ring. Defining a surjective ring homomorphism from $\mathbb{Z}[x_1, \dots, x_n, t_1, \dots, t_n]$ to $H^*_{T^n}(Flags(\mathbb{C}^n))$ by sending x_i to $\bar{x}_i$ and t_i to t_i, its kernel $\tilde{I}$ is generated as an ideal by $e_i(x_1, \dots, x_n) - e_i(t_1, \dots, t_n)$ for all $1 \le i \le n$, where e_i is the i-th elementary symmetric polynomial. Thus, we have an isomorphism

$$H^*_{T^n}(Flags(\mathbb{C}^n)) \cong \mathbb{Z}[x_1,\dots,x_n,t_1,\dots,t_n]/\tilde{I}. \tag{3.3}$$

The right action in (3.2) induces the following left action of the symmetric group S_n on $H^*_{T^n}(Flags(\mathbb{C}^n))$:

$$w\cdot\bar{x}_i=\bar{x}_{w(i)},\ w\cdot t_i=t_i \tag{3.4}$$

for $w\in S_n$. In fact, the pullback of the line bundle E_i/E_{i-1} under the right action in (3.2) is exactly the line bundle $E_{w(i)}/E_{w(i)-1}$, and the right action in (3.2) is T^n-equivariant.

Second, we define a left action of $v\in S_n$ on the direct sum $\bigoplus_{w\in S_n}\mathbb{Z}[t_1,\dots,t_n]$ of the polynomial ring as follows:

$$(v\cdot f)|_w=f|_{wv} \tag{3.5}$$

where $w\in S_n$ and $f\in\bigoplus_{w\in S_n}\mathbb{Z}[t_1,\dots,t_n]$. Observe that the map ι_1 in (3.1) is the following mapping

$$\iota_1(\bar{x}_i)|_w=t_{w(i)},\ \iota_1(t_i)|_w=t_i. \tag{3.6}$$

Note that it follows from (3.4), (3.5), and (3.6) that the map ι_1 is S_n-equivariant map, i.e. $w\cdot(\iota_1(f))=\iota_1(w\cdot f)$ for any $f\in H^*_{T^n}(Flags(\mathbb{C}^n))$ and $w\in S_n$.

To construct an S_n-action on $\bigoplus_{w\in \mathcal{S}_\lambda^{T^\ell}}H^*(BT^\ell)$, we need some preparations. We identify $H^*(BT^\ell)$ with a polynomial ring with ℓ variables. That is,

$$H^*(BT^\ell)=\mathbb{Z}[u_1,\dots,u_\ell]$$

where $u_i\in H^2(BT^\ell)$ is the first Chern class of the line bundle $ET^\ell\times_{T^\ell}\mathbb{C}_i$ over BT^ℓ. Here, $\mathbb{C}_i$ is the one dimensional representation of T^ℓ through a map $T^\ell\to S^1$ given by $\mathrm{diag}(h_1,\cdots,h_1,h_2,\cdots,h_2,\cdots\cdots,h_\ell,\cdots,h_\ell)\mapsto h_i$.

It is known that $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ admit a cellular decomposition ([6]), so the odd degree cohomology groups of $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ vanish. The path-connectedness of $Flags(\mathbb{C}^n)$ and $\mathcal{S}_\lambda$ together with this fact implies that the maps ι_1 and ι_2 in (3.1) are injective (cf.[5]) and that the map ρ_λ in (3.1) is surjective (cf.[2]). The map π in (3.1) is clearly surjective. Therefore, we obtain the following lemma. Let $\bar{y}_i$ be the image $\rho_\lambda(\bar{x}_i)$ of $\bar{x}_i$ for each i.

Lemma 3.1. *The T^ℓ-equivariant cohomology ring $H^*_{T^\ell}(\mathcal{S}_\lambda)$ is generated by $\bar{y}_1,\dots,\bar{y}_n$, $u_1,\dots,u_\ell$ as a ring where $\bar{y}_i$ is as above and $H^*(BT^\ell)=\mathbb{Z}[u_1,\dots,u_\ell]$.* □

Let $\phi:[n]\to[\ell]$ ($[n]:=\{1,2,\dots,n\}$) be a map defined by

$$\phi(i)=k \tag{3.7}$$

if $\lambda_1+\cdots+\lambda_{k-1}+1\le i\le\lambda_1+\cdots+\lambda_k$ where $\lambda_1+\cdots+\lambda_{k-1}=0$ when $k=1$. Observe that the map π in (3.1) is the following mapping

$$\pi(f|_w(t_1,\dots,t_n))=f|_w(u_{\phi(1)},\dots,u_{\phi(n)}), \tag{3.8}$$

where $f|_w$ denotes w-component of f. It follows from (3.6), (3.8) and the commutative diagram in (3.1) that

$$\iota_2(\bar{y}_i)|_w = u_{\phi(w(i))} \text{ and } \iota_2(u_i)|_w = u_i. \tag{3.9}$$

Third, we define the left action of $v \in S_n$ on the direct sum $\bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \ldots, u_\ell]$ of the polynomial ring as follows:

$$(v \cdot f)|_w = f|_{w'} \tag{3.10}$$

for $w \in \mathcal{S}_\lambda^{T^\ell}$ and $f \in \bigoplus_{w \in \mathcal{S}_\lambda^{T^\ell}} \mathbb{Z}[u_1, \ldots, u_\ell]$ where w' is the element of $\mathcal{S}_\lambda^{T^\ell}$ whose right coset agrees with the right coset $[wv]$ of $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$. Note that the map π in (3.1) is not S_n-equivariant in general.

Lemma 3.2. *For any $v \in S_n$ and $1 \leq i \leq n$, it follows that*

$$v \cdot (\iota_2(\bar{y}_i)) = \iota_2(\bar{y}_{v(i)}) \text{ and } v \cdot (\iota_2(u_i)) = \iota_2(u_i) \tag{3.11}$$

where the map ι_2 is in (3.1) *and $\bar{y}_i$ is the image of $\bar{x}_i$ under the map ρ_λ in* (3.1).

Proof. From (3.9) and (3.10), we have

$$(v \cdot (\iota_2(u_i)))|_w = \iota_2(u_i)|_{w'} = u_i = \iota_2(u_i)|_w$$

for all $w \in S_n$. So the second equation holds. From (3.9) and (3.10) again, we have

$$\begin{aligned} &(v \cdot (\iota_2(\bar{y}_i)))|_w = \iota_2(\bar{y}_i)|_{w'} = u_{\phi(w'(i))}, \\ &\iota_2(\bar{y}_{v(i)})|_w = u_{\phi(w(v(i)))}. \end{aligned}$$

Therefore, it is enough to prove $\phi(w'(i)) = \phi(wv(i))$. Since $[w'] = [wv]$ in $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell} \backslash S_n$, we have

$$\begin{aligned} &\lambda_1 + \cdots + \lambda_{r-1} + 1 \leq w'(i) \leq \lambda_1 + \cdots + \lambda_r, \\ &\lambda_1 + \cdots + \lambda_{r-1} + 1 \leq wv(i) \leq \lambda_1 + \cdots + \lambda_r \end{aligned}$$

for some r. From the definition (3.7) of the map ϕ, we have $\phi(w'(i)) = \phi(wv(i))$, and the first equation holds. We are done. □

Since the map ι_2 is injective, we obtain an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ satisfying

$$w \cdot \bar{y}_i = \bar{y}_{w(i)} \text{ and } w \cdot u_i = u_i \tag{3.12}$$

for $w \in S_n$ from Lemma 3.1 and Lemma 3.2. Moreover, one can see that the map ρ_λ in (3.1) is S_n-equivariant homomorphism by (3.4) and (3.12). We summarize the results in this section as follows.

Proposition 3.3. *There exists an S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ such that the map ρ_λ in* (3.1) *is S_n-equivariant homomorphism where the S_n-action on $H^*_{T^n}(Flags(\mathbb{C}^n))$ is given by* (3.4).

4. Main theorem

In this section, we state our main theorem. For this purpose, let us clarify our notations. We set $p_\lambda(s) := \lambda_{n-s+1} + \lambda_{n-s+2} + \cdots + \lambda_\ell$ for $s = 1, \cdots, n$. We denote by $\check{\lambda}$

the transpose of λ. That is, $\check{\lambda} = (\eta_1, \cdots, \eta_k)$ where $k = \lambda_1$ and $\eta_i = |\{j \mid \lambda_j \geq i\}|$ for $1 \leq i \leq k$. For indeterminates $y_1, \cdots, y_s$ and $a_1, a_2, \cdots$, let

$$e_d(y_1, \cdots, y_s | a_1, a_2, \cdots) := \sum_{r=0}^{d} (-1)^{d-r} e_r(y_1, \cdots, y_s) h_{d-r}(a_1, \cdots, a_{s+1-d}) \tag{4.1}$$

for $d \geq 0$ where e_i and h_i denote the i-th elementary symmetric polynomial and the i-th complete symmetric polynomial, respectively. In fact, this is the factorial Schur function corresponding to the Young diagram consisting of the unique column of length d as shown in the next section (see Lemma 5.1). We also define a map $\phi_\lambda : [n] \to [\ell]$ by the condition

$$\begin{aligned} &(u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) \\ &\quad = (\underbrace{u_1, \cdots, u_1}_{\lambda_1 - \lambda_2}, \underbrace{u_1, u_2, \cdots, u_1, u_2}_{2(\lambda_2 - \lambda_3)}, \cdots\cdots, \underbrace{u_1, u_2, \cdots, u_\ell, \cdots\cdots, u_1, u_2, \cdots, u_\ell}_{\ell(\lambda_\ell - \lambda_{\ell+1})}) \end{aligned} \tag{4.2}$$

as ordered sequences where for each $1 \leq r \leq \ell$ the r-th sector of the right-hand-side consists of $(u_1, u_2, \cdots, u_r)$ repeated $(\lambda_r - \lambda_{r+1})$-times. Here, we denote $\lambda_{\ell+1} = 0$.

Let us define a ring homomorphism

$$\psi : \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell] \to H^*_{T^\ell}(\mathcal{S}_\lambda) \tag{4.3}$$

by sending y_i to $\bar{y}_i$ and u_i to u_i where $H^*(BT^\ell) = \mathbb{Z}[u_1, \cdots, u_\ell]$. Recall that $\bar{y}_i$ is the equivariant first Chern class of the tautological line bundle E_i/E_{i-1} over $Flags(\mathbb{C}^n)$ (see Section 3) restricted to $\mathcal{S}_\lambda$. This homomorphism ψ is a surjection by Lemma 3.1.

Theorem 4.1. *The map ψ in* (4.3) *induces a ring isomorphism*

$$H^*_{T^\ell}(\mathcal{S}_\lambda) \cong \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\widetilde{I}_\lambda$$

where $\widetilde{I}_\lambda$ is the ideal of the polynomial ring $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]$ generated by the polynomials $e_d(y_{i_1}, \cdots, y_{i_s} | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)})$ defined in (4.1) *with ϕ_λ described in* (4.2) *for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$.*

Remark. The ideal $\widetilde{I}_\lambda$ is the T^ℓ-equivariant analogue of so-called Tanisaki's ideal (it is written as $K_{\check{\lambda}}$ in [9]). Each generator of $\widetilde{I}_\lambda$ given above specializes to a generator of Tanisaki's ideal given in [9] after the evaluation $u_i = 0$ for all i.

5. Proof of the main theorem

In this section, we prove Theorem 4.1. Our argument is the T^ℓ-equivariant version of [9]. We first show that $e_d(\bar{y}_{i_1}, \cdots, \bar{y}_{i_s} | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$ in $H^*_{T^\ell}(\mathcal{S}_\lambda)$ for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$. By the S_n-action on $H^*_{T^\ell}(\mathcal{S}_\lambda)$ constructed in Section 3, we may assume that $i_1 = 1, \cdots, i_s = s$.

Let us first consider the cases for $s < n$, and prove that for $d \geq s + 1 - p_{\check{\lambda}}(s)$ we have $e_d(\bar{y}_1, \cdots, \bar{y}_s | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$ in $H^*_{T^\ell}(\mathcal{S}_\lambda)$. Take a T^n-invariant complete flag $U_\bullet$ by refining the flag $(\cdots \subset N_0^2\mathbb{C}^n \subset N_0\mathbb{C}^n \subset \mathbb{C}^n)$. This is possible since N_0 is in Jordan canonical form. We denote by $\bar{w}$ the element of S_n corresponding to $U_\bullet$, i.e. $U_\bullet = \bar{w}F_\bullet$ where $F_\bullet$ is the standard flag defined by $F_i = \langle e_1, \cdots, e_i \rangle$ for all $1 \leq i \leq n$. For a Young diagram μ with at most s rows and $n-s$ columns, the Schubert variety corresponding to μ with respect to the reference flag $U_\bullet$ is

$$X_\mu(U_\bullet) = \{V \in Gr_s(\mathbb{C}^n) \mid \dim(V \cap U_{n-s+i-\mu_i}) \geq i \text{ for all } 1 \leq i \leq s\}$$

where $Gr_s(\mathbb{C}^n)$ denotes the set of s dimensional complex linear subspaces in $\mathbb{C}^n$. It is known that $X_\mu(\tilde{F}_\bullet) \cap X_\nu(F_\bullet) = \emptyset$ unless $\mu \subset \nu^\dagger$ (cf. [1] § 9.4, Lemma 3). Here, $\nu^\dagger = (n-s-\nu_s, \cdots, n-s-\nu_1)$ and $\tilde{F}_\bullet$ is the opposite flag of $F_\bullet$ defined by $\tilde{F}_i = \langle e_{n+1-i}, \cdots, e_n\rangle$. By multiplying both sides of this equality by $\bar{w}$, we get

$$X_\mu(\bar{w}\tilde{F}_\bullet) \cap X_\nu(U_\bullet) = \emptyset \quad \text{unless } \mu \subset \nu^\dagger. \tag{5.1}$$

Since the flag $\bar{w}\tilde{F}_\bullet$ is T^n-invariant, the Schubert variety $X_\mu(\bar{w}\tilde{F}_\bullet)$ is a T^n-invariant irreducible subvariety of $Gr_s(\mathbb{C}^n)$. Let $\tilde{S}_\mu := [X_\mu(\bar{w}\tilde{F}_\bullet)] \in H^*_{T^n}(Gr_s(\mathbb{C}^n))$ be the associated T^n-equivariant cohomology class.

Let $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ be the projection defined by $p(V_\bullet) = V_s$. Then it follows that

$$p(\mathcal{S}_\lambda) \subset X_{\mu_0}(U_\bullet)$$

where $\mu_0 = (n-s, \cdots, n-s, 0, \cdots, 0)$ with $n-s$ repeated $p_{\check{\lambda}}(s)$-times and 0 repeated $(s-p_{\check{\lambda}}(s))$-times (cf. [9] § 3, Proposition 3). Hence, we obtain the following commutative diagram

$$\begin{array}{ccc} H^*_{T^n}(Flags(\mathbb{C}^n)) & \xleftarrow{p^*} & H^*_{T^n}(Gr_s(\mathbb{C}^n)) \\ {\scriptstyle\rho_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle i^*} \\ H^*_{T^\ell}(\mathcal{S}_\lambda) & \xleftarrow{k^*} & H^*_{T^n}(X_{\mu_0}(U_\bullet)) \end{array} \tag{5.2}$$

where i^* is the map induced by the inclusion and k is the restriction of the projection map p. Let $\mu_{s,d} = (1, \cdots, 1, 0, \cdots, 0)$ with 1 repeated d-times and 0 repeated $(s-d)$-times. This Young diagram has at most s rows and $n-s$ columns since we are assuming that $s < n$. Recall that the T^n-equivariant Schubert class $\tilde{S}_\mu = [X_\mu(\bar{w}\tilde{F}_\bullet)]$ comes from the relative cohomology $H^*_{T^n}(Gr_s(\mathbb{C}^n), Gr_s(\mathbb{C}^n)\backslash X_\mu(\bar{w}\tilde{F}_\bullet))$. So it follows that $i^*\tilde{S}_{\mu_{s,d}} = 0$ for $d \geq s+1-p_{\check{\lambda}}(s)$ since $\mu_{s,d} \not\subset \mu_0^\dagger$ and (5.1) show that any cycle in $X_{\mu_0}(U_\bullet)$ does not intersect with $X_{\mu_{s,d}}(\bar{w}\tilde{F}_\bullet)$. Thus, we obtain $\rho_\lambda(p^*\tilde{S}_{\mu_{s,d}}) = 0$ by the commutativity of the diagram (5.2).

To give a polynomial representative of $\rho_\lambda(p^*\tilde{S}_{\mu_{s,d}})$, let us first describe $p^*\tilde{S}_{\mu_{s,d}}$ in terms of $\bar{x}_1, \cdots, \bar{x}_n$ and $t_1, \cdots, t_n$. Observe that $w \in S_n$ acts on $\mathbb{C}^n$ from the left by

$$w \cdot (x_1, \cdots, x_n) = (x_{w^{-1}(1)}, \cdots, x_{w^{-1}(n)})$$

for $(x_1, \cdots, x_n) \in \mathbb{C}^n$, and this naturally induces S_n-action on $Flags(\mathbb{C}^n)$. For each $w \in S_n$, the induced map on $Flags(\mathbb{C}^n)$ is equivariant with respect to a group homomorphism $\psi_w : T^n \to T^n$ defined by $(g_1, \cdots, g_n) \mapsto (g_{w^{-1}(1)}, \cdots, g_{w^{-1}(n)})$. This ψ_w induces a ring homomorphism on $H^*(BT^n) = \mathbb{Z}[t_1, \cdots, t_n]$:

$$\psi_w^* : \mathbb{Z}[t_1, \cdots, t_n] \to \mathbb{Z}[t_1, \cdots, t_n] \quad ; \quad t_i \mapsto t_{w^{-1}(i)},$$

and the induced map w^* on $H^*_{T^n}(Flags(\mathbb{C}^n))$ is a ring homomorphism satisfying $w^*(t_i\alpha) = \psi^*_w(t_i)w^*(\alpha)$ for any $\alpha \in H^*_{T^n}(Flags(\mathbb{C}^n))$ and $i = 1, \cdots, n$ where the products are taken by the cup products via the canonical homomorphism $H^*(BT^n) \to H^*_{T^n}(Flags(\mathbb{C}^n))$. Similarly, S_n acts on $Gr_s(\mathbb{C}^n)$ from the left, and the projection $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ is S_n-equivariant. Observe that $w^*\bar{x}_i = \bar{x}_i$ for any $w \in S_n$ since w naturally induces a map $E_i/E_{i-1} \to E_i/E_{i-1}$ which is a fiber-wise isomorphism.

Recall from [3] that the T^n-equivariant Schubert class $[X_\mu(F_\bullet)] \in H^*_{T^n}(Gr_s(\mathbb{C}^n))$ with respect to the standard reference flag $F_\bullet$ is represented by the factorial Schur function (see [4]) in the T^n-equivariant cohomology of $Flags(\mathbb{C}^n)$:

$$p^*[X_\mu(F_\bullet)] = s_\mu(-\bar{x}_1, \cdots, -\bar{x}_s | -t_n, \cdots, -t_1).$$

For the convenience of the reader, we here recall the definition of factorial Schur functions from [4]: for a Young diagram μ with at most s rows, the factorial Schur function associated to μ is defined to be

$$s_\mu(x_1, \cdots, x_s | a_1, a_2, \cdots) = \sum_T \prod_{\alpha\in\mu} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)})$$

as a polynomial in $\mathbb{Z}[x_1, \cdots, x_s] \otimes \mathbb{Z}[a_1, a_2, \cdots]$ where T runs over all semistandard tableaux of shape μ with entries in $\{1, \cdots, s\}$, $T(\alpha)$ is the entry of T in the cell $\alpha \in \mu$, and $c(\alpha) = j - i$ is the content of $\alpha = (i, j)$. This polynomial is symmetric in x-variables.

From the definition, we have that $X_\mu(\bar{w}\tilde{F}_\bullet) = \bar{w}w_0X_\mu(F_\bullet)$ where $w_0 \in S_n$ is the longest element with respect to the Bruhat order. So it follows that

$$\begin{aligned} p^*\tilde{S}_\mu &= p^*((\bar{w}w_0)^{-1})^*[X_\mu(F_\bullet)] = ((\bar{w}w_0)^{-1})^*p^*[X_\mu(F_\bullet)] \\ &= s_\mu(-\bar{x}_1, \cdots, -\bar{x}_s | -t_{\bar{w}(1)}, \cdots, -t_{\bar{w}(n)}) \end{aligned}$$

since the projection $p : Flags(\mathbb{C}^n) \to Gr_s(\mathbb{C}^n)$ is equivariant with respect to the left S_n-actions. In particular, the following lemma with the definition (4.1) shows that

$$p^*\tilde{S}_{\mu_{s,d}} = (-1)^d e_d(\bar{x}_1, \cdots, \bar{x}_s | t_{\bar{w}(1)}, \cdots, t_{\bar{w}(n)}). \tag{5.3}$$

Lemma 5.1. *For indeterminates $x_1, \cdots, x_s, a_1, a_2, \cdots$, we have*

$$s_{\mu_{s,k}}(x_1, \cdots, x_s | a_1, a_2, \cdots) = \sum_{r=0}^{k} (-1)^{k-r} e_r(x_1, \cdots, x_s) h_{k-r}(a_1, \cdots, a_{s+1-k})$$

for $k \geq 0$ where $\mu_{s,k} = (1, \cdots, 1, 0, \cdots, 0)$ with 1 repeated k-times and 0 repeated $(s-k)$-times.

Proof. We first find the coefficient of the monomial $x_1 \cdots x_r$ in $s_{\mu_{s,k}}(x|a)$. For each $I = (i_1, i_2, \cdots, i_{k-r})$ satisfying $r + 1 \leq i_1 < i_2 < \cdots < i_{k-r} \leq s$, there is a summand in $s_{\mu_{s,k}}(x|a)$ corresponding to the standard tableau T_I of shape $\mu_{s,k}$ whose $(j, 1)$-th entry is

$$\begin{cases} j & \text{if } 1 \leq j \leq r, \\ i_{j-r} & \text{if } r+1 \leq j \leq k. \end{cases}$$

The summand is of the form

$$(x_1-a_1)(x_2-a_1)\cdots(x_r-a_1)(x_{i_1}-a_{i_1-r})(x_{i_2}-a_{i_2-r-1})\cdots(x_{i_{k-r}}-a_{i_{k-r}-k+1}),$$

and the contribution of the monomial $x_1\cdots x_r$ from this polynomial is

$$(-1)^{k-r}(a_{i_1-r}a_{i_2-r-1}\cdots a_{i_{k-r}-k+1})x_1\cdots x_r.$$

Since the condition on I is equivalent to

$$1\le i_1-r\le i_2-r-1\le\cdots\le i_{k-r}-k+1\le s-k+1,$$

we see that the coefficient of $x_1\cdots x_r$ in $s_{\mu_{s,k}}(x_1,\cdots,x_s|a_1,a_2,\cdots)$ is

$$(-1)^{k-r}h_{k-r}(a_1,\cdots,a_{s-k+1}).$$

Recalling that $s_{\mu_{s,k}}(x_1,\cdots,x_s|a_1,a_2,\cdots)$ is symmetric in x-variables, we conclude that the coefficient of $x_{j_1}\cdots x_{j_r}$ is $(-1)^{k-r}h_{k-r}(a_1,\cdots,a_{s-k+1})$ for any $1\le j_1<\cdots<j_r\le s$. Thus, the polynomial

$$(-1)^{k-r}e_r(x_1,\cdots,x_s)h_{k-r}(a_1,\cdots,a_{s-k+1})$$

gives the summand in $s_{\mu_{s,k}}(x_1,\cdots,x_s|a_1,a_2,\cdots)$ whose degree in x-variables is r. □

From now on, we take a specific choice of $\bar{w}$ as follows, and we study the image of the Schubert classes $p^*\tilde{S}_\mu$ under ρ_λ. We choose $\bar{w}$ so that its one-line notation is given by

$$\bar{w}=J_1\cdots J_\ell$$

where each sector J_r is a sequence of subsectors

$$J_r=j_r^{(1)}\cdots j_r^{(\lambda_r-\lambda_{r+1})}$$

consisted by sequences of the form

$$j_r^{(m)}=(\lambda_1-\lambda_r)+m\ ,\ (\lambda_1-\lambda_r)+\lambda_2+m\ ,\ \ldots\ldots\ ,\ (\lambda_1-\lambda_r)+\lambda_2+\cdots+\lambda_r+m.$$

Note that $j_r^{(m)}$ is a sequence of length r, and J_r is a sequence of length $r(\lambda_r-\lambda_{r+1})$. We define J_r to be the empty sequence if $\lambda_r=\lambda_{r+1}$. Writing down J_r for some small r, the reader can see how the complete flag $\bar{w}F_\bullet$ refines the flag $(\cdots\subset N_0^2\mathbb{C}^n\subset N_0\mathbb{C}^n\subset\mathbb{C}^n)$.

Example. If $n=16$ and $\lambda=(7,5,2,2)$, then

$$\bar{w}=1\ 2\ 3\ 8\ 4\ 9\ 5\ 10\ 6\ 11\ 13\ 15\ 7\ 12\ 14\ 16$$

where $J_1=j_1^{(1)}j_1^{(2)}=1\ 2$, $J_2=j_2^{(1)}j_2^{(2)}j_2^{(3)}=3\ 8\ 4\ 9\ 5\ 10$, J_3 is the empty sequence, and $J_4=j_4^{(1)}j_4^{(2)}=6\ 11\ 13\ 15\ 7\ 12\ 14\ 16$. The reader should check that $\bar{w}F_\bullet$ refines the flag $(\cdots\subset N_0^2\mathbb{C}^n\subset N_0\mathbb{C}^n\subset\mathbb{C}^n)$.

The map $\phi:[n]\to[\ell]$ defined in (3.7) takes each sequence $j_r^{(m)}$ to the sequence $1,\cdots,r$ since k-th number of $j_r^{(m)}$ satisfies

$$\lambda_1+\cdots+\lambda_{k-1}+1\le(\lambda_1-\lambda_r)+\lambda_2+\cdots+\lambda_k+m\le\lambda_1+\cdots+\lambda_k.$$

This shows that $\phi \circ \bar{w}$ coincides with the map ϕ_λ defined in (4.2). Applying ρ_λ to (5.3), we obtain

$$\rho_\lambda \circ p^*(\tilde{S}_{\mu_{s,d}}) = (-1)^d e_d(\bar{y}_1, \cdots, \bar{y}_s | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)})$$

in $H^*_{T^\ell}(\mathcal{S}_\lambda)$. Since $i^*(\tilde{S}_{\mu_{s,j}}) = 0$, the commutative diagram (5.2) shows that the left-hand-side of this equality vanishes. That is, we proved that $e_d(\bar{y}_1, \cdots, \bar{y}_s | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$ for the cases $s < n$.

We are left with the case $s = n$. In this case, we have that $d \geq n + 1 - p_{\check{\lambda}}(n) = 1$. Observe that in $H^*_{T^n}(Flags(\mathbb{C}^n))$ we have

$$\begin{aligned} & e_d(\bar{x}_1, \cdots, \bar{x}_n | t_1, \cdots, t_n) \\ & = \sum_{r=0}^{d} (-1)^{d-r} e_r(\bar{x}_1, \cdots, \bar{x}_n) h_{d-r}(t_1, \cdots, t_{n+1-d}) \\ & = \sum_{r=0}^{d} (-1)^{d-r} e_r(t_1, \cdots, t_n) h_{d-r}(t_1, \cdots, t_{n+1-d}) \end{aligned}$$

by the presentation given in (3.3). It is straightforward to check that this is equal to $e_d(t_{n+2-d}, \cdots, t_n)$ (which is zero since the number of variables is greater than d) by considering the generating functions with a formal variable z for elementary and complete symmetric polynomials :

$$\prod_{i=1}^{n} (1 - t_i z) = \sum_{r=0}^{n} (-1)^r e_r(t_1, \cdots, t_n) z^r,$$

$$\prod_{i=1}^{n} \frac{1}{1 - t_i z} = \sum_{r \geq 0} h_r(t_1, \cdots, t_n) z^r.$$

That is, the polynomial $e_d(\bar{x}_1, \cdots, \bar{x}_n | t_1, \cdots, t_n)$ vanishes in $H^*_{T^n}(Flags(\mathbb{C}^n))$, and hence we see that $e_d(\bar{y}_1, \cdots, \bar{y}_n | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)}) = 0$.

Now, the homomorphism (4.3) induces a surjective ring homomorphism

$$\bar{\psi} : \mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots u_\ell] / \widetilde{I}_\lambda \longrightarrow H^*_{T^\ell}(\mathcal{S}_\lambda).$$

In what follows, we prove that this is an isomorphism by thinking of both sides as $\mathbb{Z}[u_1, \cdots u_\ell]$-algebras. Namely, the ring on the left-hand-side admits the obvious multiplication by $u_1, \cdots, u_n$, and the ring on the right-hand-side has the canonical ring homomorphism $H^*(BT^\ell) \to H^*_{T^\ell}(\mathcal{S}_\lambda)$ with the identification $H^*(BT^\ell) = \mathbb{Z}[u_1, \cdots u_\ell]$.

Recall that $\mathcal{S}_\lambda$ admits a cellular decomposition by even dimensional cells constructed by [6] (c.f. [2]). So the spectral sequence for the fiber bundle $ET^\ell \times_{T^\ell} \mathcal{S}_\lambda \to BT^\ell$ shows that $H^*_{T^\ell}(\mathcal{S}_\lambda)$ is a free $\mathbb{Z}[u_1, \cdots, u_\ell]$-module and that its rank over $\mathbb{Z}[u_1, \cdots, u_\ell]$ coincides with the rank of the non-equivariant cohomology:

$$\text{rank}_{\mathbb{Z}[u_1, \cdots, u_\ell]} H^*_{T^\ell}(\mathcal{S}_\lambda) = \text{rank}_{\mathbb{Z}} H^*(\mathcal{S}_\lambda) = \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_\ell!} =: \binom{n}{\lambda}.$$

Hence, to prove that the map $\bar{\psi}$ is an isomorphism, it is sufficient to show that the module $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots u_\ell]/\tilde{I}_\lambda$ is generated by $\binom{n}{\lambda}$ elements as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-module. To do that, let us consider a graded ring[1] $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ where I_λ is Tanisaki's ideal, namely this is generated by $e_d(y_{i_1}, \cdots, y_{i_s})$ for $1 \leq s \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, and $d \geq s + 1 - p_{\check{\lambda}}(s)$. In [9], it is shown that this is a free $\mathbb{Z}$-module of rank $\binom{n}{\lambda}$.

Lemma 5.2. *Let $\Phi_1(y), \cdots, \Phi_k(y)$ be homogeneous polynomials in $\mathbb{Z}[y_1, \cdots, y_n]$ which give an additive basis of $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ where $k = \binom{n}{\lambda}$. If we think of $\Phi_1(y), \cdots, \Phi_k(y)$ as elements of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$, then they generate $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-module.*

Proof. It suffices to show that any monomial m of $y_1, \cdots, y_n$ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ can be written as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$. We prove this by induction on the degree d of m. The base case $d = 0$ is clear, i.e. $\Phi_i(y) = 1$ for some i. We assume that $d \geq 1$ and the claim holds for $d - 1$. Let θ be a homomorphism from $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ to $\mathbb{Z}[y_1, \cdots, y_n]/I_\lambda$ sending y_i to y_i and u_i to 0. This is well-defined since each generator $e_d(\bar{y}_1, \cdots, \bar{y}_s | u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(n)})$ of $\tilde{I}_\lambda$ is mapped to the corresponding generator $e_d(y_{i_1}, \cdots, y_{i_s})$ of I_λ. By the assumption, $\theta(m)$ can be written as a $\mathbb{Z}$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$, that is, we have

$$m - \sum_i a_i \Phi_i(y) \in \ker \theta$$

for some $a_i \in \mathbb{Z}$. Here, $\ker \theta$ is the ideal of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ generated by $u_1, \cdots, u_\ell$. In fact, it follows that the image of I_λ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ is included in the ideal $(u_1, \cdots, u_\ell)$ of $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$ from the following equation in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$:

$$e_d(y_{i_1}, \cdots, y_{i_s}) = -\sum_{0 \leq r < d} (-1)^{d-r} e_r(y_{i_1}, \cdots, y_{i_s}) h_{d-r}(u_{\phi_\lambda(1)}, \cdots, u_{\phi_\lambda(s+1-d)}).$$

Therefore, the monomial m can be written as

$$m = \sum_i a_i \Phi_i(y) + \sum_{j=1}^{\ell} f_j(y, u) u_j \tag{5.4}$$

for some polynomials $f_1(y, u), \cdots, f_\ell(y, u)$. Since m has degree d, we can replace the polynomials in the right-hand-side by their homogeneous components of degree d. Namely, we can assume that $\deg \Phi_i(y) = \deg f_j(y, u) + 1 = d$. Now, the induction assumption shows that each $f_j(y, u)$ is written as a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$ since the degree of each monomial in y contained in $f_j(y, u)$ is less than d. Hence, the element m is written by a $\mathbb{Z}[u_1, \cdots, u_\ell]$-linear combination of $\Phi_1(y), \cdots, \Phi_k(y)$ in $\mathbb{Z}[y_1, \cdots, y_n, u_1, \cdots, u_\ell]/\tilde{I}_\lambda$, as desired. □

From Lemma 5.2, the surjection $\bar{\psi}$ has to be an isomorphism as discussed above.

[1]The argument in [9] to give a presentation of the ring $H^*(\mathcal{S}_\lambda; \mathbb{C})$ works also over $\mathbb{Z}$-coefficient, and in that sense this ring is the presentation given in [9].

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