LAGRANGIAN INTERSECTION OF THE GAUSS IMAGES OF ISOPARAMETRIC HYPERSURFACES (A PRELIMINARY REPORT ON JOINT WORK WITH H. IRIYEH, H. MA AND R. MIYAOKA)

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Introduction

It is a quite interesting problem in geometry to study Lagrangian submanifolds L of various Kähler manifolds (M, ω, g, J). From about 2005, I am working on Lagrangian submanifolds of complex hyperquadrics obtained as the Gauss images of isoparametric hypersurfaces jointly with Professor Hui Ma of Tsinghua University in Beijing. I am reporting on progress of our joint work at this meeting every time. In this note we shall mention recent results on Lagrangian intersection of the Gauss images of isoparametric hypersurfaces in my new joint work with Hiroshi Iriyeh, Hui Ma and Reiko Miyaoka.

1. Gauss images of isoparametric hypersurfaces

Let $Q_n(\mathbb{C})$ be a complex hyperquadrics of $\mathbb{C}P^{n+1}$ defined by the homogeneous quadratic equation $z_0^2+z_1^2+\cdots+z_{n+1}^2 = 0$. Let $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ be the real Grassmann manifold of all oriented 2-dimensional vector subspaces of $\mathbb{R}^{n+2}$ and $Gr_2(\mathbb{R}^{n+2})$ the real Grassmann manifold of all 2-dimensional vector subspaces of $\mathbb{R}^{n+2}$. Then we have the identification

$$Q_n(\mathbb{C}) \ni [\mathbf{a} + \sqrt{-1}\mathbf{b}] \longleftrightarrow [W] = \mathbf{a} \wedge \mathbf{b} \in \widetilde{Gr}_2(\mathbb{R}^{n+2}),$$

where $\{\mathbf{a}, \mathbf{b}\}$ denotes an orthonormal basis of W compatible with the orientation of $[W]$.

Let N^n be an oriented hypersurface of the unit standard hypersphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Denote by $\mathbf{x}(p)$ the position vector of a point $p \in N^n$

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and by $\mathbf{n}(p)$ the unit normal vector at a point $p \in N^n$ to N^n in $S^{n+1}(1)$ compatible to the orientation. The *Gauss map* $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ of N^n is defined by

$$\mathcal{G} : N^n \ni p \longrightarrow [\mathbf{x}(p)+\sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p)\wedge\mathbf{n}(p) \in Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}).$$

Then we know that

Proposition 1.1. *The Gauss map $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ is always a Lagrangian immersion.*

Suppose that N^n is an oriented hypersurface with constant principal curvatures in $S^{n+1}(1)$, the so-called *isoparametric hypersurface*. From Palmer's results ([23]) we see that

Proposition 1.2. *The Gauss map $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.*

The fundamental structures of isoparametric hypersurfaces were first investigated by E. Cartan and Münzner ([14]). Denote by g the number of distinct principal curvatures of N^n. It is known that their multiplicities satisfy $m_1 = m_3 = \cdots = m_{2i-1} = \cdots$ and $m_2 = m_4 = \cdots = m_{2i} = \cdots$. Thus $\frac{2n}{g}$ must be an integer given as

$$\frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \text{ is even,} \\ 2m_1 & \text{if } g \text{ is odd.} \end{cases}$$

The famous and surprising Münzner's result ([15]) is that g must be $1, 2, 3, 4$ or 6. The cohomology groups of isoparametric hypersurfaces N^n and their focal manifolds $N_\pm$ were determined by Münzner (II [15]).

The Lagrangian immersion $\mathcal{G}$ and the Gauss image $\mathcal{G}(N^n)$ of an isoparametric hypersurface have the following properties ([9], [19], [11]):

Proposition 1.3. (1) *The Gauss image $L^n = \mathcal{G}(N^n)$ is a compact smooth minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$.*

(2) *The Gauss map $\mathcal{G}$ gives a covering map $\mathcal{G} : N^n \to \mathcal{G}(N^n)$ over the Gauss image with the deck transformation group $\mathbb{Z}_g$. Note that the $\mathbb{Z}_g$-action does not preserve the induced metric on N^n from $S^{n+1}(1)$ if $g \geq 3$.*

(3) *$\mathcal{G}(N^n)$ is invariant under the deck transformation group $\mathbb{Z}_2$ of the universal covering $Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \to Gr_2(\mathbb{R}^{n+2})$.*

(4) *$\frac{2n}{g}$ is even (resp. odd) if and only if $\mathcal{G}(N^n)$ is orientable (resp. non-orientable).*

(5) $L^n = \mathcal{G}(N^n)$ *is a monotone and cyclic Lagrangian submanifold in* $Q_n(\mathbb{C})$ *with minimal Maslov number* Σ_L *equal to* $\dfrac{2n}{g}$.

We observe that $L = \mathcal{G}(N)$ has minimal Maslov number $\Sigma_L = 2$ if and only if N is one of the following examples:
$g = 1$: $m_1 = 1$, $n = 1$ N^1 is a great or small circle of S^2.
$g = 2$: $m_1 = m_2 = 1$, $n = 2$ N^2 is a Clifford torus of S^3.
$g = 3$: $m_1 = m_2 = 1$, $n = 3$ $N^3 \cong SO(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \subset S^4$.
$g = 4$: $m_1 = m_2 = 1$, $n = 4$ $N^4 \cong (SO(2) \times SO(3))/\mathbb{Z}_2 \subset S^5$.
$g = 6$: $m_1 = m_2 = 1$, $n = 6$ $N^6 \cong SO(4)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \subset S^7$.

Hence we see that the Lagrangian intersection Floer cohomology for the Gauss images of isoparametric hypersurfaces is well-defined by Y. G. Oh's works ([16], [17], [18]).

By taking the quotient space of N^n by $\mathbb{Z}_g$ the topology can be drastically changed. We should notice that

Theorem 1.1 (IMMO [8])**.** *The Gauss image* $L = \mathcal{G}(N^n)$ *of each isoparametric hypersurface of* $g = 3$*, i.e. Cartan hypersurface, is a* $\mathbb{Z}_2$*-homology sphere.*

A submanifold of a Riemannian manifold is said to be *homogeneous* if it is obtained as an orbit of a connected Lie subgroup of its isometry group. In the classification theory of isoparametric hypersurfaces, it is well-known that any homogeneous isoparametric hypersurface in the standard sphere is obtained as a principal orbit of the isotropy representation of a Riemannian symmetric pair (U, K) of rank 2 (Hsiang-Lawson [5], Takagi-Takahashi [24]). By Elie Cartan, Dorfmeister-Nehr and R. Miyaoka ([13]), it is known that for $g = 1, 2, 3, 6$ isoparametric hypersurfaces are homogeneous. *Non-homogeneous* isoparametric hypersurfaces appear only in the case of $g = 4$. The Clifford system construction of non-homogeneous isoparametric hypersurfaces was discovered first by Ozeki-Takeuchi ([21], [22]) and generalized by Ferus-Karcher-Münzner ([4]). Isoparametric hypersurfaces with $g = 4$ were classified except for the case $(m_1, m_2) = (7, 8)$ by Cecil - Q. S. Chi - Jensen [1], Immervoll [6], Q. S. Chi [2], [3].

Note that $g = 1$ or 2 if and only if $\mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold of $Q_n(\mathbb{C})$, that is, a real form (real hyperquadric) of a complex hyperquadric.

In the joint works of the author and Hui Ma, we have done

(1) Classification of all compact homogeneous Lagrangian submanifolds in complex hyperquadrics ([9]).

(2) Determination of Hamiltonian stability, Hamiltonian rigidity and strict Hamiltonian stability for the Guass images of all homogeneous isoparametric hypersurfaces:
   (a) $g = 1, 2, 3$ ([9]).
   (b) $g = 4$, (U, K) is of classical type ([11]).
   (c) $g = 6$ and $g = 4$, (U, K) is of exceptional type ([12]).
(3) Lower bound of the number of transversal intersection points of Guass images of isoparametric hypersurfaces (under holomorphic isometries) ([20]).

2. Hamiltonian non-displaceability of Lagrangian submanifolds in symplectic manifolds

Let (M, ω) be a symplectic manifold. A diffeomorphism $\phi : M \to M$ is called a *Hamiltonian diffeomorphism* of (M, ω) if there are time-dependent Hamiltonians $\{H_t\}$ and diffeomorphisms $\{\phi_t\}$ of $\phi_0 = \mathrm{Id}_M$ and $\phi_1 = \phi$ satisfying

$$\frac{\partial \phi_t(x)}{\partial t} = (X_{H_t})_{\phi_t(x)} \quad (\forall x \in M),$$

where X_{H_t} is the Hamiltonian vector field corresponding to the Hamiltonian H_t defined by

$$dH_t = \omega(X_{H_t}, \cdot\,).$$

We know that $\phi_t^* \omega = \omega$, that is, ϕ_t is a symplectic diffeomorphism of (M, ω) for each t.

Let $\mathrm{Ham}(M, \omega)$ be the set of all Hamiltonian diffeomorphisms of (M, ω). Then we know that $\mathrm{Hamil}(M, \omega)$ is a group. A Lagrangian submanifold L of a symplectic manifold (M, ω) is called *Hamiltonianly non-displaceable* if $L \cap \phi(L) \neq \emptyset$ for each $\phi \in \mathrm{Ham}(M, \omega)$. By definition of Lagrangian intersection Floer homology $HF(L)$, if L is Hamiltonianly displaceable, then we have $HF(L) = \{0\}$. Equivalently, if $HF(L) \neq \{0\}$, then L is Hamiltonianly non-displaceable.

Not so many examples of Hamiltonianly non-displaceable Lagrangian submanifolds are known now.

3. Hamiltonian non-displaceability of Gauss images of isoparametric hypersurfaces

At RIMS Joint Research in June, 2014, we have obtained

Theorem 3.1 (IMMO[8]). *Assume that N^n is an isoparametric hypersurface of $S^{n+1}(1)$ with $g = 3$ and $m = m_1 = m_2 = 2, 4,$ or 8.*

Then the Lagrangian intersection Floer homology of the Gauss image $L^n = \mathcal{G}(N^n)$ is non-zero, that is,

$$HF(L; \Lambda) \neq \{0\}$$

where $\Lambda = \mathbb{Z}_2[T, T^{-1}]$. Hence L is Hamiltonian non-displaceable in $Q_n(\mathbb{C})$.

More recently, by Research-In-Team at TSIMF in December, 2014, we had progress as follows:

Theorem 3.2 (IMMO[8])**.** *Suppose that N^n is an isoparametric hypersurface of $S^{n+1}(1)$ except for the cases of $(g, (m_1, m_2)) = (3, (1, 1))$, $(4, (1, k))$ $(k \geq 1)$, $(6, (1, 1))$. Then the Gauss image $L^n = \mathcal{G}(N^n)$ is Hamiltonian non-displaceable in $Q_n(\mathbb{C})$,*

Remark. The cases of $g = 1$ or 2 are already well-investigated by [7].

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