THE EQUIVARIANT COHOMOLOGY RINGS OF REGULAR NILPOTENT HESSENBERG VARIETIES IN LIE TYPE A : RESEARCH ANNOUNCEMENT

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Dedicated to the memory of Samuel Gitler (1933-2014)

ABSTRACT. Let n be a fixed positive integer and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function. The main result of this manuscript is to give a systematic method for producing an explicit presentation by generators and relations of the equivariant and ordinary cohomology rings (with $\mathbb{Q}$ coefficients) of any regular nilpotent Hessenberg variety $\mathrm{Hess}(h)$ in type A. Specifically, we give an explicit algorithm, depending only on the Hessenberg function h, which produces the n defining relations $\{f_{h(j),j}\}_{j=1}^n$ in the equivariant cohomology ring. Our result generalizes known results: for the case $h = (2, 3, 4, \ldots, n, n)$, which corresponds to the Peterson variety Pet_n, we recover the presentation of $H^*_S(Pet_n)$ given previously by Fukukawa, Harada, and Masuda. Moreover, in the case $h = (n, n, \ldots, n)$, for which the corresponding regular nilpotent Hessenberg variety is the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$, we can explicitly relate the generators of our ideal with those in the usual Borel presentation of the cohomology ring of $\mathcal{F}\ell ags(\mathbb{C}^n)$. The proof of our main theorem includes an argument that the restriction homomorphism $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_S(\mathrm{Hess}(h))$ is surjective. In this research announcement, we briefly recount the context and state our results; we also give a sketch of our proofs and conclude with a brief discussion of open questions. A manuscript containing more details and full proofs is forthcoming.

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INTRODUCTION

This paper is a research announcement and is a contribution to the volume dedicated to the illustrious career of Samuel Gitler. A manuscript containing full details is in preparation [1].

Hessenberg varieties (in type A) are subvarieties of the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$ of nested sequences of subspaces in $\mathbb{C}^n$. Their geometry and (equivariant) topology have been studied extensively since the late 1980s [6, 7, 8]. This subject lies at the intersection of, and makes connections between, many research areas such as: geometric representation theory [26, 14], combinatorics [12, 23], and algebraic geometry and topology [5, 20]. Hessenberg varieties also arise in the study of the quantum cohomology of the flag variety [22, 25].

The (equivariant) cohomology rings of Hessenberg varieties has been actively studied in recent years. For instance, Brion and Carrell showed an isomorphism between the equivariant cohomology ring of a regular nilpotent Hessenberg variety with the affine coordinate ring of a certain affine curve [5]. In the special case of Peterson varieties Pet_n (in type A), the second author and Tymoczko provided an explicit set of generators for $H^*_S(Pet_n)$ and also proved a Schubert-calculus-type "Monk formula", thus giving a

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presentation of $H^*_S(Pet_n)$ via generators and relations [16]. Using this Monk formula, Bayegan and the second author derived a "Giambelli formula" [3] for $H^*_S(Pet_n)$ which then yields a simplification of the original presentation given in [16]. Drellich has generalized the results in [16] and [3] to Peterson varieties in all Lie types [10]. In another direction, descriptions of the equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties in type A have been studied by Dewitt and the second author [9], the third author [18], the first and third authors [2], and Bayegan and the second author [4]. However, it has been an open question to give a general and systematic description of the equivariant cohomology rings of all regular nilpotent Hessenberg varieties [19, Introduction, page 2], to which our results provide an answer (in Lie type A).

Finally, we mention that, as a stepping stone to our main result, we can additionally prove a fact (cf. Section 3) which seems to be well-known by experts but for which we did not find an explicit proof in the literature: namely, that the natural restriction homomorphism $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_S(\mathrm{Hess}(h))$ is surjective when $\mathrm{Hess}(h)$ is a regular nilpotent Hessenberg variety (of type A).

1. Background on Hessenberg varieties

In this section we briefly recall the terminology required to understand the statements of our main results; in particular we recall the definition of a regular nilpotent Hessenberg variety, denoted $\mathrm{Hess}(h)$, along with a natural S^1-action on it. In this manuscript we only discuss the Lie type A case (i.e. the $GL(n,\mathbb{C})$ case). We also record some observations regarding the S^1-fixed points of $\mathrm{Hess}(h)$, which will be important in later sections.

By the **flag variety** we mean the homogeneous space $GL(n,\mathbb{C})/B$ which may also be identified with

$$\mathcal{F}\ell ags(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}.$$

A **Hessenberg function** is a function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$. Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. The **Hessenberg variety (associated to N and h)** $\mathrm{Hess}(N,h)$ is defined as the following subvariety of $\mathcal{F}\ell ags(\mathbb{C}^n)$:

$$\mathrm{Hess}(N,h) := \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } i = 1, \ldots, n\} \subseteq \mathcal{F}\ell ags(\mathbb{C}^n). \tag{1.1}$$

If N is nilpotent, we say $\mathrm{Hess}(N,h)$ is a **nilpotent Hessenberg variety**, and if N is a principal nilpotent operator then $\mathrm{Hess}(N,h)$ is called a **regular nilpotent Hessenberg variety**. In this manuscript we restrict to the regular nilpotent case, and as such we denote $\mathrm{Hess}(N,h)$ simply as $\mathrm{Hess}(h)$ where N is understood to be the standard principal nilpotent operator, i.e. N has one Jordan block with eigenvalue 0.

Next recall that the following standard torus

$$T = \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} \mid g_i \in \mathbb{C}^* \ (i = 1, 2, \ldots n) \right\} \tag{1.2}$$

acts on the flag variety $Flags(\mathbb{C}^n)$ by left multiplication. However, this T-action does not preserve the subvariety $\mathrm{Hess}(h)$ in general. This problem can be rectified by considering instead the action of the following circle subgroup S of T, which does preserve $\mathrm{Hess}(h)$ ([17, Lemma 5.1]):

$$S := \left\{ \begin{pmatrix} g & & & \\ & g^2 & & \\ & & \ddots & \\ & & & g^n \end{pmatrix} \mid g \in \mathbb{C}^* \right\}. \tag{1.3}$$

(Indeed it can be checked that $S^{-1}NS = gN$ which implies that S preserves $\mathrm{Hess}(h)$.) Recall that the T-fixed points $Flags(\mathbb{C}^n)^T$ of the flag variety $Flags(\mathbb{C}^n)$ can be identified with the permutation group S_n

on n letters. More concretely, it is straightforward to see that the T-fixed points are the set

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$.

It is known that for a regular nilpotent Hessenberg variety $\mathrm{Hess}(h)$ we have

$$\mathrm{Hess}(h)^S = \mathrm{Hess}(h) \cap (Flags(\mathbb{C}^n))^T$$

so we may view $\mathrm{Hess}(h)^S$ as a subset of S_n.

2. Statement of the main theorem

In this section we state the main result of this paper. We first recall some notation and terminology. Let E_i denote the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\tilde{\tau}_i \in H^2_T(Flags(\mathbb{C}^n))$. Let $\mathbb{C}_i$ denote the one dimensional representation of T through the map $T \to \mathbb{C}^*$ given by $diag(g_1, \ldots, g_n) \mapsto g_i$. In addition we denote the first Chern class of the line bundle $ET \times_T \mathbb{C}_i$ over BT by $t_i \in H^2(BT)$. It is well-known that the $t_1, \ldots, t_n$ generate $H^*(BT)$ as a ring and are algebraically independent, so we may identify $H^*(BT)$ with the polynomial ring $\mathbb{Q}[t_1, \ldots, t_n]$ as rings. Furthermore, it is known that $H^*_T(Flags(\mathbb{C}^n))$ is generated as a ring by the elements $\tilde{\tau}_1, \ldots, \tilde{\tau}_n, t_1, \ldots, t_n$. Indeed, by sending x_i to $\tilde{\tau}_i$ and the t_i to t_i we obtain the following isomorphism:

$$H^*_T(Flags(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n, t_1, \ldots, t_n]/(e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n) \mid 1 \leq i \leq n).$$

Here the e_i denote the degree-i elementary symmetric polynomials in the relevant variables. In particular, since the odd cohomology of the flag variety $Flags(\mathbb{C}^n)$ vanishes, we additionally obtain the following:

$$H^*(Flags(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n]/(e_i(x_1, \ldots, x_n) \mid 1 \leq i \leq n). \tag{2.1}$$

As mentioned in Section 1, in this manuscript we focus on a particular circle subgroup S of the usual maximal torus T. For this subgroup S, we denote the first Chern class of the line bundle $ES \times_S \mathbb{C}$ over BS by $t \in H^2(BS)$, where by $\mathbb{C}$ we mean the standard one-dimensional representation of S through the map $S \to \mathbb{C}^*$ given by $diag(g, g^2, \ldots, g^n) \mapsto g$. Analogous to the identification $H^*(BT) \cong \mathbb{Q}[t_1, \ldots, t_n]$, we may also identify $H^*(BS)$ with $\mathbb{Q}[t]$ as rings.

Consider the restricion homomorphism

$$H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_S(\mathrm{Hess}(h)). \tag{2.2}$$

Let τ_i denote the image of $\tilde{\tau}_i$ under (2.2). We next analyze some algebraic relations satisfied by the τ_i. For this purpose, we now introduce some polynomials $f_{i,j} = f_{i,j}(x_1, \ldots, x_n, t) \in \mathbb{Q}[x_1, \ldots, x_n, t]$.

First we define

$$p_i := \sum_{k=1}^{i} (x_k - kt) \quad (1 \leq i \leq n). \tag{2.3}$$

For convenience we also set $p_0 := 0$ by definition. Let (i,j) be a pair of natural numbers satisfying $n \geq i \geq j \geq 1$. These polynomials should be visualized as being associated to the (i,j)-th spot in an $n \times n$ matrix. Note that by assumption on the indices, we only define the $f_{i,j}$ for entries in the lower-triangular part of the matrix, i.e. the part at or below the diagonal. The definition of the $f_{i,j}$ is inductive, beginning with the case when $i = j$, i.e. the two indices are equal. In this case we make the following definition:

$$f_{j,j} := p_j \quad (1 \leq j \leq n). \tag{2.4}$$

Now we proceed inductively for the rest of the $f_{i,j}$ as follows: for (i,j) with $n \geq i > j \geq 1$ we define:

$$f_{i,j} := f_{i-1,j-1} + \big(x_j - x_i - t\big) f_{i-1,j}. \tag{2.5}$$

Again for convenience we define $f_{*,0} := 0$ for any $*$. Informally, we may visualize each $f_{i,j}$ as being associated to the lower-triangular (i,j)-th entry in an $n \times n$ matrix, as follows:

$$\begin{pmatrix} f_{1,1} & 0 & \cdots & \cdots & 0 \\ f_{2,1} & f_{2,2} & 0 & \cdots & \\ f_{3,1} & f_{3,2} & f_{3,3} & \ddots & \\ \vdots & & & & \\ f_{n,1} & f_{n,2} & \cdots & & f_{n,n} \end{pmatrix} \tag{2.6}$$

To make the discussion more concrete, we present an explicit example.

Example 1. Suppose $n = 4$. Then the $f_{i,j}$ have the following form.
$f_{i,i} = p_i \quad (1 \leq i \leq 4)$
$f_{2,1} = (x_1 - x_2 - t)p_1$
$f_{3,2} = (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2$
$f_{4,3} = (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2 + (x_3 - x_4 - t)p_3$
$f_{3,1} = (x_1 - x_3 - t)(x_1 - x_2 - t)p_1$
$f_{4,2} = (x_1 - x_3 - t)(x_1 - x_2 - t)p_1 + (x_2 - x_4 - t)\{(x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2\}$
$f_{4,1} = (x_1 - x_4 - t)(x_1 - x_3 - t)(x_1 - x_2 - t)p_1$

For general n, the polynomials $f_{i,j}$ for each (i,j)-th entry in the matrix (2.6) above can also be expressed in a closed formula in terms of certain polynomials $\Delta_{i,j}$ for $i \geq j$ which are determined inductively, starting on the main diagonal. As for the $f_{i,j}$, we think of $\Delta_{i,j}$ for $i \geq j$ as being associated to the (i,j)-th box in an $n \times n$ matrix. In what follows, for $0 < k \leq n-1$, we refer to the lower-triangular matrix entries in the (i,j)-th spots where $i - j = k$ as the **k-th lower diagonal**. (Equivalently, the k-th lower diagonal is the "usual" diagonal of the lower-left $(n-k) \times (n-k)$ submatrix.) The usual diagonal is the 0-th lower diagonal in this terminology. We now define the $\Delta_{i,j}$ as follows.

(1) First place the linear polynomial $x_i - it$ in the i-th entry along the 0-th lower (i.e. main) diagonal, so $\Delta_{i,i} := x_i - it$.
(2) Suppose that $\Delta_{i,j}$ for the $k-1$-st lower diagonal have already been defined. Let (i,j) be on the k-th lower diagonal, so $i - j = k$. Define

$$\Delta_{i,j} := \left(\sum_{\ell=1}^{j} \Delta_{i-j+\ell-1,\ell} \right) (x_j - x_i - t).$$

In words, this means the following. Suppose $k = i - j > 0$. Then $\Delta_{i,j}$ is the product of $(x_j - x_i - t)$ with the sum of the entries in the boxes which are in the "diagonal immediately above the (i,j) box" (i.e. the boxes which are in the $(k-1)$-st lower diagonal), but we omit any boxes to the right of the (i,j) box (i.e. in columns $j+1$ or higher). Finally, the polynomial $f_{i,j}$ is obtained by taking the sum of the entries in the (i,j)-th box and any boxes "to its left" in the same lower diagonal. More precisely,

$$f_{i,j} = \sum_{k=1}^{j} \Delta_{i-j+k,k}. \tag{2.7}$$

We are now ready to state our main result.

Theorem 2.1. *Let n be a positive integer and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function. Let $\mathrm{Hess}(h) \subset \mathcal{F}\ell ags(\mathbb{C}^n)$ denote the corresponding regular nilpotent Hessenberg variety equipped with the circle S-action described above. Then the restriction map*

$$H_T^*(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H_S^*(\mathrm{Hess}(h))$$

is surjective. Moreover, there is an isomorphism of $\mathbb{Q}[t]$-algebras

$$H_S^*(\mathrm{Hess}(h)) \cong \mathbb{Q}[x_1, \ldots, x_n, t]/I(h)$$

sending x_i to τ_i and t to t and we identify $H^*(BS) = \mathbb{Q}[t]$. Here the ideal $I(h)$ is defined by

$$I(h) := (f_{h(j),j} \mid 1 \leq j \leq n). \tag{2.8}$$

We can also describe the ideal $I(h)$ defined in (2.8) as follows. Any Hessenberg function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ determines a subspace of the vector space $M(n \times n, \mathbb{C})$ of matrices as follows: an (i, j)-th entry is required to be 0 if $i > h(j)$. If we represent a Hessenberg function h by listing its values $(h(1), h(2), \cdots, h(n))$, then the Hessenberg subspace can be described in words as follows: the first column (starting from the left) is allowed $h(1)$ non-zero entries (starting from the top), the second column is allowed $h(2)$ non-zero entries, et cetera. For example, if $h = (3, 3, 4, 5, 7, 7, 7)$ then the Hessenberg subspace is

$$\left\{ \begin{pmatrix} \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star \end{pmatrix} \right\} \subseteq M(7 \times 7, \mathbb{C}).$$

Then, using the association of the polynomials $f_{i,j}$ with the (i, j)-th entry of the matrix (2.6), the ideal $I(h)$ can be described as being "generated by the $f_{i,j}$ in the boxes at the bottom of each column in the Hessenberg space". For instance, in the $h = (3, 3, 4, 5, 7, 7, 7)$ example above, the generators are $\{f_{3,1}, f_{3,2}, f_{4,3}, f_{5,4}, f_{7,5}, f_{7,6}, f_{7,7}\}$.

Our main result generalizes previous known results.

Remark 1. Consider the special case $h = (2, 3, \ldots, n, n)$. In this case the corresponding regular nilpotent Hessenberg variety has been well-studied and it is called a **Peterson variety** Pet_n (of type A). Our result above is a generalization of the result in [11] which gives a presentation of $H^*_S(Pet_n)$. Indeed, for $1 \leq j \leq n-1$, we obtain from (2.5) and (2.3) that

$$\begin{aligned} f_{j+1,j} &= f_{j,j-1} + (x_j - x_{j+1} - t) f_{j,j} \\ &= f_{j,j-1} + (-p_{j-1} + 2p_j - p_{j+1} - 2t) p_j \end{aligned}$$

and since $f_{n,n} = p_n$ we have

$$\begin{aligned} H^*_S(Pet_n) &\cong \mathbb{Q}[x_1, \ldots, x_n, t] / \big(f_{j,j-1} + (-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j,\ p_n \mid 1 \leq j \leq n-1\big) \\ &= \mathbb{Q}[x_1, \ldots, x_n, t] / \big((-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j,\ p_n \mid 1 \leq j \leq n-1\big) \\ &\cong \mathbb{Q}[p_1, \ldots, p_{n-1}, t] / \big((-p_{j-1} + 2p_j - p_{j+1} - 2t)p_j \mid 1 \leq j \leq n-1\big) \end{aligned}$$

which agrees with [11]. (Note that we take by convention $p_0 = p_n = 0$.)

The main theorem above also immediately yields a computation of the ordinary cohomology ring. Indeed, since the odd degree cohomology groups of $\mathrm{Hess}(h)$ vanish [29], by setting $t = 0$ we obtain the ordinary cohomology. Let $\check{f}_{i,j} := f_{i,j}(x, t = 0)$ denote the polynomials in the variables x_i obtained by setting $t = 0$. A computation then shows that

$$\check{f}_{i,j} = \sum_{k=1}^{j} x_k \prod_{\ell=j+1}^{i} (x_k - x_\ell).$$

(For the case $i = j$ we take by convention $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$.) We have the following.

Corollary 2.2. *Let the notation be as above. There is a ring isomorphism*

$$H^*(\mathrm{Hess}(h)) \cong \mathbb{Q}[x_1, \ldots, x_n] / \check{I}(h)$$

where $\check{I}(h) := \big(\check{f}_{h(j),j} \mid 1 \leq j \leq n\big)$.

Remark 2. Consider the special case $h = (n, n, \ldots, n)$. In this case the condition in (1.1) is vacuous and the associated regular nilpotent Hessenberg variety is the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$. In this case we can explicitly relate the generators $\check{f}_{h(j)=n,j}$ of our ideal $\check{I}(h) = \check{I}(n, n, \ldots, n)$ with the power sums $\mathsf{p}_r(x) = \mathsf{p}_r(x_1, \ldots, x_n) := \sum_{k=1}^{n} x_k^r$, thus relating our presentation with the usual Borel presentation as in (2.1), see e.g. [13]. More explicitly, for r be an integer, $1 \leq r \leq n$, define

$$\mathsf{q}_r(x) = \mathsf{q}_r(x_1, \ldots, x_n) := \sum_{k=1}^{n+1-r} x_k \prod_{\ell=n+2-r}^{n} (x_k - x_\ell).$$

Note that by definition $\mathsf{q}_r(x) = \check{f}_{n,n+1-r}$ so these are the generators of $\check{I}(n,n,\dots,n)$. The polynomials $\mathsf{q}_r(x)$ and the power sums $\mathsf{p}_r(x)$ can then be shown to satisfy the relations

$$\mathsf{q}_r(x) = \sum_{i=0}^{r-1} (-1)^i e_i(x_{n+2-r},\dots,x_n)\mathsf{p}_{r-i}(x). \tag{2.9}$$

Remark 3. In the usual Borel presentation of $H^*(\mathcal{F}\ell ags(\mathbb{C}^n))$, the ideal I of relations is taken to be generated by the elementary symmetric polynomials. The power sums p_r generate this ideal I when we consider the cohomology with $\mathbb{Q}$ coefficients, but this is not true with $\mathbb{Z}$ coefficients. Thus our main Theorem 2.1 does not hold with $\mathbb{Z}$ coefficients in the case when $h = (n,n,\dots,n)$, suggesting that there is some subtlety in the relationship between the choice of coefficients and the choice of generators of the ideal $I(h)$.

3. Sketch of the proof of the main theorem

We now sketch the outline of the proof of the main result (Theorem 2.1) above. As a first step, we show that the elements τ_i satisfy the relations $f_{h(j),j} = f_{h(j),j}(\tau_1,\dots,\tau_n,t) = 0$. The main technique of this part of the proof is (equivariant) localization, i.e. the injection

$$H^*_S(\mathrm{Hess}(h)) \to H^*_S(\mathrm{Hess}(h)^S). \tag{3.1}$$

Specifically, we show that the restriction $f_{h(j),j}(w)$ of each $f_{h(j),j}$ to an S-fixed point $w \in \mathrm{Hess}(h)^S$ is equal to 0; by the injectivity of (3.1) this then implies that $f_{h(j),j} = 0$ as desired. This part of the argument is rather long and requires a technical inductive argument based on a particular choice of total ordering on $\mathrm{Hess}(h)^S$ which refines a certain natural partial order on Hessenberg functions. Once we show $f_{h(j),j} = 0$ for all j, we obtain a well-defined ring homomorphism which sends x_i to τ_i and t to t:

$$\varphi_h : \mathbb{Q}[x_1,\dots,x_n,t]/(f_{h(j),j} \mid 1 \le j \le n) \to H^*_S(\mathrm{Hess}(h)). \tag{3.2}$$

We then show that the two sides of (3.2) have identical Hilbert series. This part of the argument is rather straightforward, following the techniques used in e.g. [11].

The next key step in our proof of Theorem 2.1 relies on the following two key ideas: firstly, we use our knowledge of the special case where the Hessenberg function h is $h = (n,n,\dots,n)$, for which the associated regular nilpotent Hessenberg variety is the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$, and secondly, we consider localizations of the rings in question with respect to $R := \mathbb{Q}[t]\backslash\{0\}$. For the following, for $h = (n,n,\dots,n)$ we let $\mathcal{H} := \mathrm{Hess}(h = (n,n,\dots,n)) = \mathcal{F}\ell ags(\mathbb{C}^n)$ denote the full flag variety and let I denote the associated ideal $I(n,n,\dots,n)$. In this case we know that the map $\varphi := \varphi_{(n,n,\dots,n)}$ is surjective since the Chern classes τ_i are known to generate the cohomology ring of $\mathcal{F}\ell ags(\mathbb{C}^n)$. Since the Hilbert series of both sides are identical, we then know that φ is an isomorphism.

The following commutative diagram is crucial for the remainder of the argument.

$$\begin{tikzcd}
R^{-1}\big(\mathbb{Q}[x_1,\dots,x_n,t]/I\big) \arrow[r, "R^{-1}\varphi", "\cong"'] \arrow[d, "\text{surj}"] & R^{-1}H^*_S(\mathcal{H}) \arrow[r, "\cong"'] \arrow[d] & R^{-1}H^*_S(\mathcal{H}^S) \arrow[d, "\text{surj}"] \\
R^{-1}\big(\mathbb{Q}[x_1,\dots,x_n,t]/I(h)\big) \arrow[r, "R^{-1}\varphi_h"] & R^{-1}H^*_S(\mathrm{Hess}(h)) \arrow[r, "\cong"'] & R^{-1}H^*_S(\mathrm{Hess}(h)^S)
\end{tikzcd}$$

The horizontal arrows in the right-hand square are isomorphisms by the localization theorem. Since φ is an isomorphism, so is $R^{-1}\varphi$. The rightmost and leftmost vertical arrows are easily seen to be surjective, implying that $R^{-1}\varphi_h$ is also surjective. A comparison of Hilbert series shows that $R^{-1}\varphi_h$ is an isomorphism. Finally, to complete the proof we consider the commutative diagram

$$\begin{tikzcd}
\mathbb{Q}[x_1,\dots,x_n,t]/I(h) \arrow[r, "\varphi_h"] \arrow[d, "\text{inj}"] & H^*_S(\mathrm{Hess}(h)) \arrow[d, "\text{inj}"] \\
R^{-1}\mathbb{Q}[x_1,\dots,x_n,t]/I(h) \arrow[r, "R^{-1}\varphi_h", "\cong"'] & R^{-1}H^*_S(\mathrm{Hess}(h))
\end{tikzcd}$$

for which it is straightforward to see that the vertical arrows are injections. From this it follows that φ_h is an injection, and once again a comparison of Hilbert series shows that φ_h is in fact an isomorphism.

4. Open questions

We outline a sample of possible directions for future work.

- In [24], Mbirika and Tymoczko suggest a possible presentation of the cohomology rings of regular nilpotent Hessenberg varieties. Using our presentation, we can show that the Mbirika-Tymoczko ring is not isomorphic to $H^*(\mathrm{Hess}(h))$ in the special case of Peterson varieties for $n-1 \geq 2$, i.e. when $h(i) = i+1, 1 \leq i < n$ and $n \geq 3$. (However, they do have the same Betti numbers.) In the case $n = 4$, we have also checked explicitly for the Hessenberg functions $h = (2,4,4,4), h = (3,3,4,4)$, and $h = (3,4,4,4)$ that the relevant rings are not isomorphic. It would be of interest to understand the relationship between the two rings in some generality.
- In [15], the last three authors give a presentation of the (equivariant) cohomology rings of Peterson varieties for general Lie type in a pleasant uniform way, using entries in the Cartan matrix. It would be interesting to give a similar uniform description of the cohomology rings of regular nilpotent Hessenberg varieties for all Lie types.
- In the case of the Peterson variety (in type A), a basis for the S-equivariant cohomology ring was found by the second author and Tymoczko in [16]. In the general regular nilpotent case, and following ideas of the second author and Tymoczko [17], it would be of interest to construct similar additive bases for $H^*_S(\mathrm{Hess}(h))$. Additive bases with suitable geometric or combinatorial properties could lead to an interesting 'Schubert calculus' on regular nilpotent Hessenberg varieties.
- Fix a Hessenberg function h and let $S : \mathbb{C}^n \to \mathbb{C}^n$ be a *regular semisimple* linear operator, i.e. a diagonalizable operator with distinct eigenvalues. There is a natural Weyl group action on the cohomology ring $H^*(\mathrm{Hess}(S,h))$ of the regular semisimple Hessenberg variety corresponding to h (cf. for instance [30, p. 381] and also [28]). Let $H^*(\mathrm{Hess}(S,h))^W$ denote the ring of W-invariants where W denotes the Weyl group. It turns out that there exists a surjective ring homomorphism $H^*(\mathrm{Hess}(N,h)) \to H^*(\mathrm{Hess}(S,h))^W$ which is an isomorphism in the special case of the Peterson variety. (Historically this line of thought goes back to Klyachko's 1985 paper [21].) In an ongoing project, we are investigating properties of this ring homomorphism for general Hessenberg functions h.

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