IMPROVED HARDY INEQUALITIES IN A LIMITING CASE AND THEIR APPLICATIONS

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ABSTRACT. We show that the Hardy inequality in a limiting case can be improved by adding remainder terms with singular weights. We discuss the optimality of remainder terms from the view point of the weights. Also we consider the existence of weak solutions of a weighted eigenvalue problem and study the asymptotic behavior of the first eigenvalue as parameter involved varies.

1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $0 \in \Omega$. The classical Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \tag{1.1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, where $1 \leq p < N$. It is known that, for $p > 1$, the constant $(\frac{N-p}{p})^p$ is optimal and is never attained in $W_0^{1,p}(\Omega)$. Therefore, one can expect the existence of remainder terms on the right-hand side of the inequality (1.1). Indeed, there are many papers that deal with the improvements of the inequality (1.1) (see [1], [2], [7], [8], [9], [10], [11], [12], [15] and the references therein). For the case $p = 2$, Chaudhuri and Ramaswamy [8] have proved that, for $0 \leq \beta < 2$ and $1 < q < 2^*_\beta := \frac{2(N-\beta)}{N-2}$, there exists a constant $C > 0$ depending on N, β, q and Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx + C \left(\int_\Omega \frac{|u|^q}{|x|^\beta}\, dx\right)^{\frac{2}{q}} \tag{1.2}$$

holds for all $u \in W_0^{1,2}(\Omega)$. Vázquez and Zuazua [18] applied the remainder term in (1.1) to study the large-time behavior of solutions to the linear heat equation with a singular potential.

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In this paper, we focus on the critical case $p = N$. In this case, (1.1) loses its meaning as it is and instead,

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx \tag{1.3}$$

holds for all $u \in W_0^{1,N}(\Omega)$, here and henceforth, $\tilde{R} := \sup_{x\in\Omega} |x|$. We call (1.3) *the Hardy inequality in a limiting case*. It is known that the constant $(\frac{N-1}{N})^N$ is optimal for any bounded domain $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$, see Adimurthi and Sandeep [3]. We show a simple proof of this fact in Appendix. A main aim of this paper is to obtain remainder terms for the Hardy inequality in a limiting case (1.3). Next, we discuss the optimality of remainder terms from the view point of the weights. Finally, we consider the existence of weak solutions of a weighted eigenvalue problem with singular weights.

Adimurthi-Chaudhuri-Ramaswamy [1] have proved that for given $T > 0$ and for $R = e^{e^{\cdot^{e(\text{k-times})}}} T$, the inequality

$$\int_{B_T^2(0)} |\nabla h|^2 dx \geq \frac{1}{4} \sum_{j=1}^{k} \int_{B_T^2(0)} \frac{|h|^2}{\left(|x| \prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|}\right)^2} dx \tag{1.4}$$

holds for all $h \in W_0^{1,2}(B_T^2(0))$, where $B_T^2(0) \subset \mathbb{R}^2$ is a ball of radius T with center 0 and k is the first integer for which $0 < \log^{(k)} \frac{R}{T} \leq 1$. Here $\log^{(k)}$ is defined inductively by $\log^{(1)}(\cdot) = \log(\cdot)$, $\log^{(k)}(\cdot) = \log\left(\log^{(k-1)}(\cdot)\right)$ for $k \geq 2$. Note that, the more terms we have on the right-hand side of (1.4), the larger R must be in the weights, and if we choose $R = eT$, then k must be 1. By this reason, it seems difficult to claim that we have obtained the remainder terms for the inequality (1.3). In [3], the authors claim that there exists $C > 0$ such that the inequality

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + C \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N (\log^{(2)} \frac{R_1}{|x|})^N} dx$$

holds true for any $u \in W_0^{1,N}(\Omega)$, where $R_1 \geq (e^e)^{2/N} \tilde{R}$. However, the proof of it is omitted in [3]. The motivation of the present paper is to extend the studies done for the *subcritical* Hardy inequality (1.1) by Adimurthi et.al.[1] to the critical Hardy inequality (1.3). Our main tools are, basically, the symmetrization of functions and a transformation invented by Brezis and Vázquez [7], and the way of arguments is now well known. However, we need a new type of transformation of functions, which is relevant to our study. This new transformation, which is a combination of the usual Brezis-Vázquez's transformation and a "nonlinear" scaling, is the clue to obtain the results in this paper; see (2.2) below.

Our main result on the improvement of the inequality (1.3) is stated in the following theorem.

Theorem 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $0 \in \Omega$, and put $\tilde{R} = \sup_{x\in\Omega} |x|$. For any $-1 < L < N-2$, let $q > 0$ be such that*

$$\alpha = \alpha(q, L) = \frac{N-1}{N}q + L + 2 \leq N. \tag{1.5}$$

Then the inequality

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + \omega_N^{1-\frac{N}{q}} C(L, N, q)^{\frac{N}{q}} \left(\int_\Omega \frac{|u|^q}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\alpha} dx\right)^{\frac{N}{q}} \tag{1.6}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where ω_N is the area of the unit sphere in $\mathbb{R}^N$ and

$$C(L, N, q)^{-1} := \int_0^1 s^L \left(\log \frac{1}{s}\right)^{\frac{N-1}{N}q} ds = (L+1)^{-(\frac{N-1}{N}q+1)} \Gamma\left(\frac{N-1}{N}q + 1\right),$$

here $\Gamma(\cdot)$ is the Gamma function.

Remark 2. Let $L > -1$ be given. Note that the map

$$r \mapsto \frac{1}{r^N (\log \frac{Re}{r})^\alpha}$$

is decreasing in $r \in [0, R]$ if and only if $\alpha \leq N$. Thus if $q > 0$ be such that $\alpha = \alpha(q, L) > N$, then we cannot apply the rearrangement argument (see Step 2 in the proof of Theorem 1). This is the case, for example, if $q \geq N$. However even in this case, the inequality

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + \omega_N^{1-\frac{N}{q}} C(L, N, q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u^\#|^q}{|x|^N \left(\log \frac{Re}{|x|}\right)^\alpha} dx\right)^{\frac{N}{q}} \tag{1.7}$$

holds for all $u \in W_0^{1,N}(\Omega)$. Here and henceforth, $B_R(0) \subset \mathbb{R}^N$ be a ball such that $|\Omega| = |B_R(0)|$, $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$, and

$$u^\#(x) = u^\#(|x|) = \inf\{\lambda > 0 \mid |\{x \in \Omega \mid |u(x)| > \lambda\}| \leq |B_{|x|}(0)|\}$$

is the symmetric decreasing rearrangement (the Schwarz symmetrization) of u.

We note that $\tilde{R} \ge R$ for any bounded domain Ω. Moreover, the map

$$r \mapsto \frac{1}{r^N (\log \frac{\tilde{R}e}{r})^\gamma}$$

is monotonically decreasing in $r \in [0, R]$ if and only if $\gamma \le N \log \frac{\tilde{R}e}{R}$. Therefore, if the domain Ω satisfies a geometric condition $\alpha \le N \log \frac{\tilde{R}e}{R}$ (this is valid, for example, if Ω has a "long thin nose"), the last integral of (1.7) is estimated from below as

$$\int_{B_R(0)} \frac{|u^\#|^q}{|x|^N \left(\log \frac{Re}{|x|}\right)^\alpha} dx \ge \int_{B_R(0)} \frac{|u^\#|^q}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\alpha} dx \ge \int_\Omega \frac{|u|^q}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\alpha} dx$$

by the symmetrization argument. Therefore, for some domains Ω satisfying a geometric condition, we have (1.6) even when $\alpha > N$.

Remark 3. (1.6) does not hold when $L \le -1$ (see Theorem 9). Therefore we see that the weight function in the remainder term of (1.6) is optimal.

Direct application of Theorem 1 yields the following Corollary 4, which was first proved in [3]. We believe that the proof here is much simpler than that in [3].

Corollary 4. *(Adimurthi-Sandeep* [3]*: Theorem 1.3) Let $N \ge 2$. The best constant $(\frac{N-1}{N})^N$ in the inequality (1.3) is never attained in $W_0^{1,N}(\Omega)$.*

We also obtain different types of remainder terms involving the symmetrization of functions.

Theorem 5. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \ge 2$, with $0 \in \Omega$, and put $\tilde{R} = \sup_{x\in\Omega} |x|$. Let $B_R(0) \subset \mathbb{R}^N$ be a ball such that $|\Omega| = |B_R(0)|$. Then we have the following statements:*

(I) *The inequality*

$$\int_\Omega |\nabla u|^N dx \ge \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + \lambda_1(B_1^N(0)) \int_{B_R(0)} \frac{|u^\#|^N}{|x|^N (\log \frac{Re}{|x|})^{2N}} dx \tag{1.8}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $\lambda_1(B_1^N(0))$ is the first eigenvalue of $-\Delta_N$ acting on $W_0^{1,N}(B_1^N(0))$.

(II) *The inequality*

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log \frac{\tilde{R}e}{|x|})^N} dx + \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u^\#|^N}{|x|^N \left(\log \frac{Re}{|x|}\right)^N \left(\log\log \frac{Re}{|x|}\right)^N} dx \tag{1.9}$$

holds for all $u \in W_0^{1,N}(\Omega)$.

(III) *For* $0 < q < N$ *and* $L > -1$, *put* $\alpha = \alpha(q, L) = \frac{N-1}{N}q + L + 2$. *The inequalities:*

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log \frac{\tilde{R}e}{|x|})^N} dx + B\omega_N^{1-\frac{N}{q}} C(L, N, q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u^\#|^q}{|x|^N \left(\log \frac{Re}{|x|}\right)^\alpha} dx\right)^{\frac{N}{q}} + \frac{C}{N^2}\left(\frac{N-1}{N}\right)^{N-2} \int_{B_R(0)} \frac{|u^\#|^N}{\left(|x|^N \log \frac{Re}{|x|}\right)^N \left(\log\log \frac{Re}{|x|}\right)^2} dx \tag{1.10}$$

and

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log \frac{\tilde{R}e}{|x|})^N} dx + B\omega_N^{1-\frac{N}{q}} C(L, N, q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u^\#|^q}{|x|^N \left(\log \frac{Re}{|x|}\right)^\alpha} dx\right)^{\frac{N}{q}} + \lambda_1(B_1^2(0)) \int_{B_R(0)} \frac{|u^\#|^N}{|x|^N \left(\log \frac{Re}{|x|}\right)^{N+2}} dx \tag{1.11}$$

hold for all $u \in W_0^{1,N}(\Omega)$, *where* B, C *are positive constants in (2.7) below, and* $\lambda_1(B_1^2(0))$ *is the first eigenvalue of* $-\Delta$ *acting on* $H_0^1(B_1^2(0))$, *here* $B_1^2(0)$ *is the unit ball in* $\mathbb{R}^2$.

Remark 6. Note that the functions

$$r \mapsto \frac{1}{r^N(\log \frac{\tilde{R}e}{r})^\gamma}, \quad \left(\gamma \leq N \log \frac{\tilde{R}e}{R}\right)$$

$$r \mapsto \frac{1}{r^N(\log \frac{\tilde{R}e}{r})^N(\log\log \frac{\tilde{R}e}{r})^\gamma}, \quad \left(\gamma \leq N \log \frac{\tilde{R}}{R}\left(\log\log \frac{\tilde{R}e}{R}\right)\right)$$

are monotonically decreasing on $[0, R]$. Thus for some domains Ω, we can replace the integrals of $u^\#$ on $B_R(0)$ in the right-hand sides of the inequalities (1.8)–(1.11) by those of u on Ω, by the argument as in Remark 2. For example, if Ω satisfies a geometrical condition $1 \le \log \frac{\tilde{R}}{R} \left(\log \log \frac{\tilde{R}e}{R} \right)$, then, the inequality

$$\int_\Omega |\nabla u|^N dx \ge \left(\frac{N-1}{N} \right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + \left(\frac{N-1}{N} \right)^N \int_\Omega \frac{|u|^N}{|x|^N \left(\log \frac{\tilde{R}e}{|x|} \right)^N \left(\log \log \frac{\tilde{R}e}{|x|} \right)^N} dx$$

holds true for all $u \in W_0^{1,N}(\Omega)$ instead of (1.9).

Let $1 < p < \infty$ and $\beta \neq -1$. A. Kufner [14] obtained the following one-dimensional Hardy inequality:

$$\int_0^R |v'(r)|^p r^{p-1} \left(\log \frac{R}{r} \right)^{\beta+p} dr \ge \left(\frac{|\beta+1|}{p} \right)^p \int_0^R |v(r)|^p \frac{1}{r} \left(\log \frac{R}{r} \right)^\beta dr$$

for functions $v \in W^{1,2}(0, R)$ satisfying $v(0) = 0$ when $\beta < -1$, or $v(R) = 0$ when $\beta > -1$ (see [14] Example 5.13). When $\beta = -1$, the above inequality is meaningless and it may not be known whether the similar inequality holds true or not. Here, we show the following one-dimensional Sobolev inequality including the weight $r \log \frac{Re}{r}$, by using Theorem 1.

Corollary 7. *For any $-1 < L < 0$, let $0 < q \le -2L$. Then the inequality*

$$\int_0^R |v'(r)|^2 r \log \frac{Re}{r} dr \ge \left(C \int_0^R \frac{|v(r)|^q}{r \left(\log \frac{Re}{r} \right)^{2+L}} dr \right)^{\frac{2}{q}}$$

holds for all $v \in W_0^{1,2}(0, R)$, where $C = C(L, 2, q)$ is the constant in Theorem 1.

Remark 8. The sharper Hardy inequality

$$\int_{B_R(0)} |\nabla u|^N dx \ge \left(\frac{N-1}{N} \right)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N (\log \frac{R}{|x|})^N} dx \tag{1.12}$$

holds for all $u \in W_0^{1,N}(B_R(0))$, where $N \ge 2$. Recently, Ioku and Ishiwata [13] showed that the constant $(\frac{N-1}{N})^N$ in the inequality (1.12) is optimal and never attained in $W_0^{1,N}(B_R(0))$. Furthermore, the author's [16] provided a remainder term for the inequality (1.12). When $\Omega \subset \mathbb{R}^N$ is a general bounded domain, the inequality

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}}{|x|})^N} dx$$

holds true for all $u \in W_0^{1,N}(\Omega)$. A simple proof of this fact can be seen in [17].

Now, we discuss the optimality of the remainder term (1.6) in Theorem 1.

Theorem 9. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 \in \Omega$, with $\tilde{R} = \sup_{x\in\Omega} |x|$. For $0 < q < N$, put $\alpha^* := \frac{N-1}{N} q + 1$. Define*

$$F_N := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{loc}(\Omega\backslash\{0\}) \text{ and } {}^\exists\alpha \in (\alpha^*, N] \text{ s.t. } \limsup_{|x|\to 0} f(x)|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\alpha < \infty \right\}, \text{ and}$$

$$G_N := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{loc}(\Omega\backslash\{0\}) \text{ and } \liminf_{|x|\to 0} f(x)|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^{\alpha^*} > 0 \right\}.$$

If $f \in F_N$, then there exists $\lambda(f) > 0$ such that the inequality

(1.13)
$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{\tilde{R}e}{|x|})^N} dx + \lambda(f) \left(\int_\Omega f(x)|u|^q \, dx \right)^{\frac{N}{q}}$$

holds true for all $u \in W_0^{1,N}(\Omega)$.

If $f \in G_N$, then no inequality of type (1.13) can hold. Especially, we cannot replace α in the remainder term of (1.6) by α^*.

We remark that there exist functions f with $f \notin F_N$ and $f \notin G_N$, for example, $f(x) = |x|^{-N} \left(\log \frac{\tilde{R}e}{|x|}\right)^{-\alpha^*} \left(\log |\log \frac{\tilde{R}e}{|x|}|\right)^{-\gamma}$ for any $\gamma > 0$.

Next, let us consider the following quasilinear eigenvalue problem with singular weights:

$$(P)^\lambda_\mu \begin{cases} -\Delta_N u = \mu \dfrac{|u|^{N-2}u}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} + \lambda f(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Delta_N u = \mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian, $0 < q < N$, $0 \leq \mu < (\frac{N-1}{N})^N$, $\lambda \in \mathbb{R}$ and the weight function f satisfies $f \in F_N$.

We call a function $u \in W_0^{1,N}(\Omega)$ a weak solution of the problem $(P)_\mu^\lambda$ if

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla \phi dx = \mu \int_\Omega \frac{|u|^{N-2}u\phi}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} dx + \lambda \int_\Omega |u|^{q-2}u\phi f dx$$

holds whenever $\phi \in W_0^{1,N}(\Omega)$. We look for a weak solution $u \in W_0^{1,N}(\Omega)$ of $(P)_\mu^\lambda$ by a constrained minimization argument. The solution obtained here corresponds to the first eigenvalue of $\lambda_\mu(f)$ of the operator $-\Delta_N u - \mu \frac{|u|^{N-2}u}{|x|^N\left(\log \frac{\tilde{R}e}{|x|}\right)^N}$ acting on $W_0^{1,N}(\Omega)$. Furthermore we study the asymptotic behavior of $\lambda_\mu(f)$ as $\mu \nearrow (\frac{N-1}{N})^N$. Recall $\alpha^* = \frac{N-1}{N}q + 1$ for $q \in (0, N)$.

Theorem 10. *For all $f \in F_N$, the problem $(P)_\mu^\lambda$ admits a positive weak solution $u \in W_0^{1,N}(\Omega)$ corresponding to $\lambda = \lambda_\mu(f) > 0$, which satisfies $\lambda_\mu(f) \to \lambda(f)$ for a limit $\lambda(f) > 0$ as $\mu \to (\frac{N-1}{N})^N$.*

The organization of this paper is as follows: In §1, Theorem 1, Theorem 5 and Corollary 7 are proved. In §2, Theorem 9 and Theorem 10 are studied. In Appendix, we show Proposition 16 to make this paper self-contained.

2. Improved Hardy inequalities in a limiting case

In this section, we prove Theorem 1, Theorem 5 and Corollary 7. As mentioned in §1, main tool of the proof is a new transformation of functions, which is inspired by the idea of Brezis and Vázquez [7].

First, we prepare a simple lemma.

Lemma 11 ([11] Lemma 1.1)**.** *Let $N \geq 2$, and ξ, η be real numbers such that $\xi \geq 0$ and $\xi - \eta \geq 0$. Then*

$$(\xi - \eta)^N + N\xi^{N-1}\eta - \xi^N \geq |\eta|^N. \tag{2.1}$$

Proof. Taylor's formula implies

$$(\xi - \eta)^N + N\xi^{N-1}\eta - \xi^N = N(N-1)\eta^2 \int_0^1 (1-t)(\xi - t\eta)^{N-2} dt.$$

Thus if $\eta \leq 0$, the estimate $\xi - t\eta \geq t|\eta|$ yields (2.1), and if $\eta \geq 0$, the estimate $\xi - t\eta \geq (1-t)|\eta|$ yields (2.1). □

In the proof of Theorem 1, we utilize the well-known transformation of Brezis and Vázquez [7] combined with the new change of variables; see (2.2). This new transformation is the key to the proof.

Proof of Theorem 1 . (Step 1): First we prove the inequality (1.6) when Ω is a ball $B_R(0)$ (i.e. $\tilde{R} = R$) and for smooth positive radially nonincreasing functions $u \in C_0^\infty(B_R(0))$. We define the new transformation

$$(2.2)\qquad v(s) = \left(\log \frac{Re}{r}\right)^{-\frac{N-1}{N}} u(r),\ \text{ where } r = |x|, s = s(r) = \left(\log \frac{Re}{r}\right)^{-1},$$
$$s'(r) = \frac{s(r)}{r \log \frac{Re}{r}} \geq 0.$$

Note that $v(0) = v(1) = 0$ since $u(R) = 0$, and

$$(2.3)\quad u'(r) = -\left(\frac{N-1}{N}\right)\left(\log \frac{Re}{r}\right)^{-\frac{1}{N}} \frac{v(s(r))}{r} + \left(\log \frac{Re}{r}\right)^{\frac{N-1}{N}} v'(s(r))s'(r) \leq 0.$$

Now we observe that

(2.4)

$$I := \int_{B_R(0)} |\nabla u|^N dx - \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx$$
$$= \omega_N \int_0^R |u'(r)|^N r^{N-1} dr - \left(\frac{N-1}{N}\right)^N \omega_N \int_0^R \frac{|u(r)|^N}{r(\log \frac{Re}{r})^N} dr$$
$$= \omega_N \int_0^R \left(\frac{N-1}{N}\left(\log \frac{Re}{r}\right)^{-\frac{1}{N}} \frac{v(s(r))}{r} - \left(\log \frac{Re}{r}\right)^{\frac{N-1}{N}} v'(s(r))s'(r)\right)^N r^{N-1} dr$$
$$- \left(\frac{N-1}{N}\right)^N \omega_N \int_0^R \frac{|u(r)|^N}{r(\log \frac{Re}{r})^N} dr.$$

Here, we can apply Lemma 11 with the choice

$$\xi = \frac{N-1}{N}\left(\log \frac{Re}{r}\right)^{-\frac{1}{N}} \frac{v(s(r))}{r} \quad \text{and} \quad \eta = \left(\log \frac{Re}{r}\right)^{\frac{N-1}{N}} v'(s(r))s'(r).$$

Dropping $\xi^N \geq 0$ in (2.1) and using the boundary conditions $v(0) = v(1) = 0$, we obtain

$$(2.5)\qquad I \geq -\omega_N N \left(\frac{N-1}{N}\right)^{N-1} \int_0^R v(s(r))^{N-1} v'(s(r))s'(r) dr$$
$$+ \omega_N \int_0^R |v'(s(r))|^N (s'(r))^N \left(r \log \frac{Re}{r}\right)^{N-1} dr$$
$$= -\omega_N N \left(\frac{N-1}{N}\right)^{N-1} \int_0^1 v(s)^{N-1} v'(s) ds + \omega_N \int_0^1 |v'(s)|^N s^{N-1} ds$$
$$= \omega_N \int_0^1 |v'(s)|^N s^{N-1} ds.$$

On the other hand, by using the estimate

$$|v(s)| = \left|\int_s^1 v'(t)dt\right| = \left|\int_s^1 v'(t)t^{\frac{N-1}{N}-\frac{N-1}{N}}dt\right| \le \left(\int_0^1 |v'(t)|^N t^{N-1}\,dt\right)^{\frac{1}{N}} \left(\log\frac{1}{s}\right)^{\frac{N-1}{N}},$$

we get

$$\int_0^1 |v(s)|^q s^L ds \le \left(\int_0^1 |v'(s)|^N s^{N-1} ds\right)^{\frac{q}{N}} \int_0^1 s^L \left(\log\frac{1}{s}\right)^{\frac{N-1}{N}q} ds.$$

Therefore, we have

$$\int_0^1 |v'(s)|^N s^{N-1} ds \ge C(L,N,q)^{\frac{N}{q}} \left(\int_0^1 |v(s)|^q s^L\,ds\right)^{\frac{N}{q}}. \tag{2.6}$$

Consequently, by (2.5) and (2.6), we obtain

$$I \ge \omega_N C(L,N,q)^{\frac{N}{q}} \left(\int_0^1 |v(s)|^q s^L\,ds\right)^{\frac{N}{q}} = \omega_N C(L,N,q)^{\frac{N}{q}} \left(\int_0^R \frac{|u(r)|^q}{r\left(\log\frac{Re}{r}\right)^{\alpha}} dr\right)^{\frac{N}{q}}$$
$$= \omega_N^{1-\frac{N}{q}} C(L,N,q)^{\frac{N}{q}} \left(\int_{B_R} \frac{|u|^q}{|x|^N\left(\log\frac{Re}{|x|}\right)^{\alpha}} dx\right)^{\frac{N}{q}}.$$

where $\alpha = \alpha(q,L) = \frac{N-1}{N}q + L + 2$.

(Step 2): Let $u^{\#}$ denote the symmetric decreasing rearrangement of u. Assume $|\Omega| = |B_R(0)|$. Note that the function $r \mapsto \dfrac{1}{r^N(\log\frac{\tilde{R}e}{r})^{\alpha}}$ is monotonically decreasing on $[0,R]$ since $\alpha \le N$. Thus by using the symmetrization argument, we obtain

$$\int_\Omega |\nabla u|^N dx \ge \int_{B_R(0)} |\nabla u^{\#}|^N dx$$
$$\ge \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u^{\#}|^N}{|x|^N(\log\frac{Re}{|x|})^N} dx + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u^{\#}|^q}{|x|^N\left(\log\frac{Re}{|x|}\right)^{\alpha}} dx\right)^{\frac{N}{q}}$$
$$\ge \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u^{\#}|^N}{|x|^N(\log\frac{\tilde{R}e}{|x|})^N} dx + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u^{\#}|^q}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^{\alpha}} dx\right)^{\frac{N}{q}}$$
$$\ge \left(\frac{N-1}{N}\right)^N \int_{\Omega} \frac{|u|^N}{|x|^N(\log\frac{\tilde{R}e}{|x|})^N} dx + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{\frac{N}{q}} \left(\int_{\Omega} \frac{|u|^q}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^{\alpha}} dx\right)^{\frac{N}{q}}$$

where the first inequality comes from the Pólya-Szegö inequality, the second one comes from Step 1, the third one comes from the fact that $\tilde{R} \geq R$, and the last one comes from the Hardy-Littlewood inequality: $\int_{B_R(0)} f^\# g^\# \geq \int_\Omega fg$ for nonnegative measurable functions f and g. Finally, a density argument assures (1.6) holds true for all $u \in W_0^{1,N}(\Omega)$.

The proof of Theorem 1 is now complete. □

Proof of Theorem 5 . As in the proof of Theorem 1, it is enough to check that all inequalities hold true when $\Omega = B_R(0)$ and for all smooth positive radially decreasing functions $u \in C_0^\infty(B_R(0))$, since the rest of the argument is the same as Step 2 in the proof of Theorem 1. Thus as before, we put

$$I = \int_{B_R(0)} |\nabla u|^N dx - \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx$$

for smooth positive radially decreasing functions $u \in C_0^\infty(B_R(0))$.

We use the same transformation (2.2) in the proof of Theorem 1.

Proof of (I). From (2.5) and the Poincaré inequality, we obtain

$$\begin{aligned} I &\geq \int_0^1 |v'(s)|^N s^{N-1} ds = \int_{B_1^N(0)} |\nabla v|^N dx \geq \lambda_1(B_1^N) \int_{B_1^N(0)} |v|^N dx \\ &= \lambda_1(B_1^N)\, \omega_N \int_0^1 |v(s)|^N s^{N-1}\, ds = \lambda_1(B_1^N)\, \omega_N \int_0^R \frac{|u(r)|^N}{r \left(\log \frac{Re}{r}\right)^{2N}}\, dr \\ &= \lambda_1(B_1^N) \int_{B_R(0)} \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^{2N}} dx. \end{aligned}$$

This proves (I).

Proof of (II). From (2.5) and the sharper Hardy inequality in a limiting case (1.12), we obtain

$$\begin{aligned} I &\geq \int_{B_1^N(0)} |\nabla v|^N\, dx \geq \left(\frac{N-1}{N}\right)^N \int_{B_1^N(0)} \frac{|v|^N}{|x|^N (\log \frac{1}{|x|})^N}\, dx \\ &= \left(\frac{N-1}{N}\right)^N \omega_N \int_0^1 \frac{|v(s)|^N}{s (\log \frac{1}{s})^N}\, ds \\ &= \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N \left(\log \frac{Re}{|x|}\right)^N \left(\log \log \frac{Re}{|x|}\right)^N} dx \end{aligned}$$

This proves (II).

Proof of (III). We follow the argument by Adimurthi, Chaudhuri, and Ramaswamy [1]. By (2.4), we observe that

$$I = \omega_N \int_0^R \left(\frac{N-1}{N} \left(\log \frac{Re}{r} \right)^{-\frac{1}{N}} \frac{v(s(r))}{r} - \left(\log \frac{Re}{r} \right)^{\frac{N-1}{N}} v'(s(r)) s'(r) \right)^N r^{N-1} dr$$

$$- \left(\frac{N-1}{N} \right)^N \omega_N \int_0^R \frac{|u(r)|^N}{r (\log \frac{Re}{r})^N} dr$$

$$= \left(\frac{N-1}{N} \right)^N \omega_N \int_0^R \frac{|v(s(r))|^N}{r \log \frac{Re}{r}} \left\{ \left(1 - \frac{N}{N-1} \frac{v'(s(r))}{v(s(r))} r \log \frac{Re}{r} s'(r) \right)^N - 1 \right\} dr.$$

Put $x(r) = -\frac{N}{N-1} \frac{v'(s(r))}{v(s(r))} r \log \frac{Re}{r} s'(r)$. Since u is radially decreasing, we have $\left(\log \frac{Re}{r} \right)^{\frac{1}{N}} u'(r) = -\left(\frac{N-1}{N} \right) \frac{v(s(r))}{r} + \left(\log \frac{Re}{r} \right) v'(s(r)) s'(r) \leq 0$ which implies $x(r) \geq -1$. Thus by the inequality ([1] Lemma 2.1.):

$$(1+x)^N \geq 1 + Nx + Cx^2 + B|x|^N \quad ({}^\forall x \geq -1), \tag{2.7}$$

where C and B are positive constants, we obtain

$$I \geq \left(\frac{N-1}{N} \right)^{N-2} C \int_0^1 |v(s)|^{N-2} (v'(s))^2 s ds + B \int_0^1 |v'(s)|^N s^{N-1} ds$$

$$- \left(\frac{N^2}{N-1} \right) \int_0^1 v(s)^{N-1} v'(s) ds$$

$$= \frac{4C}{N^2} \left(\frac{N-1}{N} \right)^{N-2} \int_0^1 \left| \left(v^{\frac{N}{2}}(s) \right)' \right|^2 s ds + B \int_0^1 |v'(s)|^N s^{N-1} ds$$

$$\geq \max \left\{ \frac{C}{N^2} \left(\frac{N-1}{N} \right)^{N-2} \int_0^1 \left(\frac{v^{\frac{N}{2}}}{s \log \frac{1}{s}} \right)^2 s\, ds, \lambda_1(B_1^2(0)) \omega_N \int_0^1 \left| \left(v^{\frac{N}{2}}(s) \right) \right|^2 s\, ds \right\}$$

$$+ B \omega_N^{1-\frac{N}{q}} C(L,N,q)^{\frac{N}{q}} \left(\int_{B_R(0)} \frac{|u|^q}{|x|^N \left(\log \frac{Re}{|x|} \right)^{\alpha}} dx \right)^{\frac{N}{q}}$$

where the last inequality comes from the two-dimensional sharper Hardy inequality (1.12) and the Poincaré inequality on a ball $B_1^2(0) \subset \mathbb{R}^2$, both applied to $v^{\frac{N}{2}}$, and Theorem 1. Since

$$\int_0^1 \left(\frac{v^{\frac{N}{2}}(s)}{s \log \frac{1}{s}} \right)^2 s ds = \int_0^R \frac{|u(r)|^N}{\left(r \log \frac{Re}{r} \right)^N \left(\log \log \frac{Re}{r} \right)^2} dr,$$

$$\int_0^1 |\left(v^{\frac{N}{2}}(s) \right)|^2 s ds = \omega_N \int_0^R \frac{|u(r)|^N}{r \left(\log \frac{Re}{r} \right)^{N+2}} dr,$$

we have Theorem 5 (III).

The proof of Theorem 5 is now complete. □

Proof of Corollary 7. Let $v \in W_0^{1,2}(0,R)$ and we consider the following transformation

$$u(r) = \left(\log \frac{Re}{r}\right)^{\frac{1}{2}} v(r), \quad \text{for} \quad r = |x| \in [0,R],\ x \in B_R^2(0), \tag{2.8}$$

where $B_R^2(0) \subset \mathbb{R}^2$ is a 2-dimensional ball. Note that if $v \in W_0^{1,2}(0,R)$, then it holds $u \in W_0^{1,2}(B_R^2(0))$. Indeed, by (2.8), we see $u(R) = v(R) = 0$. Moreover,

$$\int_{B_R^2(0)} |u|^2 dx = \omega_2 \int_0^R |v(r)|^2 r \log \frac{Re}{r} dr \le \omega_2 R \int_0^R |v(r)|^2 dr < \infty,$$

(2.9)

$$\begin{aligned}
\int_{B_R^2(0)} |\nabla u|^2\, dx &= \omega_2 \int_0^R \left(\left(\log \frac{Re}{r}\right)^{\frac{1}{2}} v'(r) - \frac{1}{2}\left(\log \frac{Re}{r}\right)^{-\frac{1}{2}} \frac{v(r)}{r} \right)^2 r dr \\
&= \omega_2 \int_0^R |v'(r)|^2 r \log \frac{Re}{r}\, dr - \omega_2 \int_0^R v'(r) v(r)\, dr + \frac{\omega_2}{4} \int_0^R \frac{|v(r)|^2}{r \log \frac{Re}{r}} dr \\
&\le \omega_2 R \int_0^R |v'(r)|^2 dr + \frac{\omega_2}{4} \left(\int_0^R \frac{|v(r)|^2}{r^2} dr \right)^{\frac{1}{2}} \left(\int_0^R |v(r)|^2\, dr \right)^{\frac{1}{2}} \\
&\le \omega_2 R \int_0^R |v'(r)|^2 dr + \frac{\omega_2}{4} \left(4 \int_0^R |v'(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R |v(r)|^2\, dr \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Here, we have used the facts $r \log \frac{Re}{r} \le R$ on $[0,R]$, $\log \frac{Re}{r} \ge 1$ on $[0,R]$, and the one-dimensional Hardy inequality

$$\frac{1}{4} \int_0^R \frac{|v(r)|^2}{r^2} dr \le \int_0^R |v'(r)|^2 dr, \quad \text{for } v \in W^{1,2}(0,R), v(0) = 0,$$

see [6].

Therefore, we can apply Theorem 1 to $u \in W_0^{1,2}(B_R^2(0))$ to get

$$\begin{aligned}
I &:= \int_{B_R^2(0)} |\nabla u|^2 dx - \frac{1}{4} \int_{B_R^2(0)} \frac{|u|^2}{|x|^2 (\log \frac{Re}{|x|})^2} dx \\
&\ge \omega_2^{1-\frac{2}{q}} C(L,2,q)^{\frac{2}{q}} \left(\int_{B_R^2(0)} \frac{|u|^q}{|x|^2 \left(\log \frac{Re}{|x|}\right)^{\frac{q}{2}+2+L}} dx \right)^{\frac{2}{q}}.
\end{aligned} \tag{2.10}$$

Note that $\frac{q}{2} + 2 + L \leq 2$ by the choice of q, thus the assumption (1.5) in Theorem 1 is satisfied. On the other hand, by (2.9), we have

(2.11)
$$
\begin{aligned}
I &= \omega_2 \int_0^R |v'(r)|^2 r \log \frac{Re}{r} dr + \frac{\omega_2}{4} \int_0^R \frac{|v(r)|^2}{r \log \frac{Re}{r}} dr - \frac{\omega_2}{4} \int_0^R \frac{|u(r)|^2}{r (\log \frac{Re}{r})^2} dr \\
&= \omega_2 \int_0^R |v'(r)|^2 r \log \frac{Re}{r} dr.
\end{aligned}
$$

From (2.10) and (2.11), we get

$$
\begin{aligned}
\omega_2 \int_0^R |v'(r)|^2 r \log \frac{Re}{r} dr &\geq \omega_2^{1-\frac{2}{q}} C(L,2,q)^{\frac{2}{q}} \left(\int_{B_R^2(0)} \frac{|u|^q}{|x|^2 \left(\log \frac{Re}{|x|} \right)^{\frac{q}{2}+2+L}} dx \right)^{\frac{2}{q}} \\
&= \omega_2 C(L,2,q)^{\frac{2}{q}} \left(\int_0^R \frac{|v(r)|^q}{r \left(\log \frac{Re}{|x|} \right)^{2+L}} dr \right)^{\frac{2}{q}}.
\end{aligned}
$$

The proof of Corollary 7 is now complete. □

3. Optimality of the weight and application to the weighted eigenvalue problem

In order to prove Theorem 9, we follow the argument in the proof of Corollary 1.2. in [1].

Proof of Theorem 9. If $f \in F_N$, then there exists $\alpha \in (\alpha^*, N]$ such that

$$
\lim_{\varepsilon \to 0} \sup_{x \in B_\varepsilon} f(x) |x|^N \left(\log \frac{\tilde{R}e}{|x|} \right)^\alpha < \infty
$$

holds. Hence, for sufficiently small $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$
f(x) < \frac{C}{|x|^N \left(\log \frac{\tilde{R}e}{|x|} \right)^\alpha} \quad \text{in} \quad B_\varepsilon(0).
$$

Outside B_ε, both are bounded functions and hence C can be chosen so that this inequality holds in the whole of Ω. Then, it is easy to check that (1.13) follows from the improved Hardy inequality in a limiting case (1.6).

For the proof of the latter half part of Theorem, let $f \in G_N$. Then we can find $C > 0$, $b > 0$ such that $f(x) \geq \frac{C}{|x|^N \left(\log \frac{\tilde{R}e}{|x|} \right)^{\alpha^*}}$ in $0 \leq |x| \leq \frac{b\tilde{R}e}{2}$. We may assume that $B_{b\tilde{R}e}(0) \subset \Omega$ $(\subset B_{\tilde{R}}(0))$. Let $s < \frac{N-1}{N}$ be a positive parameter and we take u_s just as in (4.1) in the proof of Proposition 16. Direct calculations, as in the proof of Proposition 16, show that

$$\left(\int_\Omega \frac{|u_s|^q}{|x|^N\left(\log \frac{\tilde{R}e}{|x|}\right)^{\alpha^*}}dx\right)^{\frac{N}{q}} = \left(\omega_N \frac{1}{(\frac{N-1}{N}-s)q}\left(\log\frac{2}{b}\right)^{(s-\frac{N-1}{N})q}\right)^{\frac{N}{q}} + O(1), \tag{3.1}$$

$$\int_\Omega |\nabla u_s|^N dx = \omega_N \frac{-s^N}{(s-1)N+1}\left(\log\frac{2}{b}\right)^{(s-1)N+1} + O(1), \tag{3.2}$$

$$\int_\Omega \frac{|u_s|^N}{|x|^N\left(\log \frac{\tilde{R}e}{|x|}\right)^{N}}dx = \omega_N \frac{-1}{(s-1)N+1}\left(\log\frac{2}{b}\right)^{(s-1)N+1} + O(1) \tag{3.3}$$

as $s \to \frac{N-1}{N}$. By (3.1), (3.2), (3.3) and $\frac{N}{q} > 1$, we have

$$\frac{\int_\Omega |\nabla u_s|^N dx - \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u_s|^N}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^N}dx}{\left(\int_\Omega f(x)|u_s|^q dx\right)^{\frac{N}{q}}} \le \frac{\int_\Omega |\nabla u_s|^N dx - \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u_s|^N}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^N}dx}{C\left(\int_\Omega \frac{|u_s|^q}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^{\alpha^*}}dx\right)^{\frac{N}{q}}}$$

$$= C\left(\frac{N-1}{N} - s\right)^{\frac{N}{q}-1} \to 0$$

as $s \to \frac{N-1}{N}$. Thus the inequality (1.6) does not hold for f as above. □

Lastly, we prove Theorem 10. In order to prove the Theorem 10, we need the following lemmas.

Lemma 12. *([4] Theorem 2.1.) Let $(u_m)_{m=1}^\infty \subset W_0^{1,p}(\Omega)$ be such that, as $m \to \infty$, $u_m \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and satisfies*

$$-\Delta_p u_m = f_m + g_m \quad in \ \mathcal{D}'(\Omega),$$

where $f_m \to 0$ in $W_0^{-1,p'}(\Omega)$ and g_m is bounded in $\mathcal{M}(\Omega)$, the space of Radon measures on Ω, i.e.

$$| < g_m, \phi > | \le C_K \|\phi\|_\infty$$

for all $\phi \in \mathcal{D}(\Omega)$ with supp $\phi \subset K$. Then there exists a subsequence, say u_{m_k}, such that

$$u_{m_k} \to u \quad in \ W_0^{1,\gamma}(\Omega) \quad ({}^\forall\gamma < p).$$

Lemma 13. *([5]) For $p \in (0, +\infty)$, let $(g_m)_{m=1}^\infty \subset L^p(\Omega, \mu)$ be a sequence of functions on a measurable space (Ω, μ) such that*

(i) $\|g_m\|_{L^p(\Omega,\mu)} \le {}^\exists C < \infty$ *for all $m \in \mathbb{N}$, and*
(ii) $g_m(x) \to g(x)$ *μ-a.e. $x \in \Omega$ as $m \to \infty$.*

Then

$$\lim_{m\to\infty}\left(\|g_m\|^p_{L^p(\Omega,\mu)} - \|g_m - g\|^p_{L^p(\Omega,\mu)}\right) = \|g\|^p_{L^p(\Omega,\mu)}.$$

Note that we can apply Lemma 13 to $\mu(dx) = f(x)dx$, where f is any nonnegative $L^1(\Omega)$ function.

Lemma 14. *For any* $0 < q < N$ *and any* $\alpha > \alpha^*$, *there exists* $C > 0$ *such that the inequality*

$$\int_\Omega |\nabla u|^N dx \ge C\left(\int_\Omega \frac{|u|^q}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^\alpha}dx\right)^{\frac{N}{q}} \tag{3.4}$$

holds true for all $u \in W_0^{1,N}(\Omega)$. *Moreover, set* $f_\alpha(x) = \dfrac{1}{|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^\alpha}$. *Then the embedding* $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega, f_\alpha)$ *is compact.*

Proof. By Hölder inequality and the Hardy inequality (1.3), we have

$$\begin{aligned}\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{\tilde{R}e}{|x|})^\beta}dx &\le \left(\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{\tilde{R}e}{|x|})^N}dx\right)^{\frac{q}{N}}\left(\int_\Omega \frac{1}{|x|^N(\log\frac{\tilde{R}e}{|x|})^{\frac{N}{N-q}(\beta-q)}}dx\right)^{1-\frac{q}{N}}\\ &\le \left(\left(\frac{N-1}{N}\right)^{-N}\int_\Omega |\nabla u|^N dx\right)^{\frac{q}{N}}\left(\int_\Omega \frac{1}{|x|^N(\log\frac{\tilde{R}e}{|x|})^{\frac{N}{N-q}(\beta-q)}}dx\right)^{1-\frac{q}{N}}.\end{aligned}$$

Since $\beta > \alpha^* = \frac{N-1}{N}q + 1$, the exponent $\frac{N}{N-q}(\beta - q) > 1$, so the last integral is finite. Thus we have (3.4).

The continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega, f_\alpha)$ comes from the inequality (3.4). To prove that this embedding is compact, let $\{u_m\}$ be a bounded sequence in $W_0^{1,N}(\Omega)$. Then we have a subsequence $\{u_{m_k}\}$ such that

$$\begin{aligned}&u_{m_k} \rightharpoonup u \quad \text{weakly in } W_0^{1,N}(\Omega) \quad \text{as} \quad k\to\infty,\\ &u_{m_k} \to u \quad \text{strongly in } L^\gamma(\Omega) \quad \text{as} \quad k\to\infty \ \left(1 \le {}^\forall\gamma < \infty\right).\end{aligned}$$

Take β such that $\alpha > \beta > \alpha^*$ and note that $\lim_{|x|\to 0}|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^\beta f_\alpha(x) = 0$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sup_{B_\delta(0)}|x|^N\left(\log\frac{\tilde{R}e}{|x|}\right)^\beta f_\alpha(x) \le \varepsilon \ \text{ and } \ \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))} < \infty.$$

Thus

$$\begin{aligned}
&\|u_{m_k} - u\|^q_{L^q(\Omega, f_\alpha)} = \int_{\Omega\setminus B_\delta(0)} |u_{m_k} - u|^q f_\alpha(x)dx + \int_{B_\delta(0)} |u_{m_k} - u|^q f_\alpha(x)dx \\
&\le \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))} \|u_{m_k} - u\|^q_{L^q(\Omega)} + \varepsilon \int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\beta} dx \\
&\le \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))} \|u_{m_k} - u\|^q_{L^q(\Omega)} + \varepsilon C \|\nabla(u_{m_k} - u)\|^q_{L^N(\Omega)} \\
&= o(1) + \varepsilon O(1) \quad \text{as } k \to \infty,
\end{aligned}$$

here the second inequality comes from (3.4). Finally, letting $\varepsilon \to 0$, we obtain $\|u_{m_k} - u\|^q_{L^q(\Omega, f_\alpha)} \to 0$ and the proof is completed. □

Remark 15. By using the test function u_s defined by (4.1) in Proposition 16, we check that

$$\inf_{u \in W_0^{1,N}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{\tilde{R}e}{|x|})^{\alpha^*}} dx\right)^{\frac{N}{q}}} = 0.$$

Thus we cannot replace β in the inequality (3.4) by α^*. By this reason, if we define the class of weight functions

$$\mathcal{F}_N = \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{\text{loc}}(\Omega\setminus\{0\}) \text{ and } \limsup_{|x|\to 0} f(x)|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^{\alpha^*} < \infty \right\},$$

then we do not know the solvability of the problem $(P)^\lambda_\mu$ for $f \in \mathcal{F}_N$.

Proof of Theorem 10. We will use the methods similar to the proof of Theorem 1.2. in [1]. Let p' be Hölder conjugate exponent of p and $W^{-1,p'}(\Omega) := \left(W_0^{1,p}(\Omega)\right)^*$.

We look for a minimizer of the functional

$$J_\mu(u) := \int_\Omega |\nabla u|^N \, dx - \mu \int_\Omega \frac{|u|^N}{\left(|x| \log \frac{\tilde{R}e}{|x|}\right)^N} \, dx \quad ({}^\forall u \in W_0^{1,N}(\Omega))$$

over the manifold $M := \{u \in W_0^{1,N}(\Omega) \mid \int_\Omega |u|^q f(x)dx = 1\}$. Note that J_μ is continuous, Gâteaux differentiable and coercive on $W_0^{1,N}(\Omega)$ for any $\mu \in [0, \left(\frac{N-1}{N}\right)^N)$ thanks to the Hardy inequality (1.3). Thus it is clear that $\lambda_\mu(f) := \inf_{u\in M} J_\mu(u)$ is positive. Let $(u_m)_{m=1}^\infty \subset M$ be minimizing sequence of $\lambda_\mu(f)$. By Ekeland's Variational Principle, we may assume $J'_\mu(u_m) \to 0$ in $W_0^{-1,N'}(\Omega)$ as $m \to \infty$ without loss of generality. The coercivity of J_μ implies that $(u_m)_{m=1}^\infty$ is a bounded sequence in $W_0^{1,N}(\Omega)$, hence we have a subsequence $(u_{m_k})_{k=1}^{\infty}$ and $u \in W_0^{1,N}(\Omega)$ such that

$$u_{m_k} \rightharpoonup u \quad \text{weakly in } W_0^{1,N}(\Omega) \quad \text{as} \quad k \to \infty, \tag{3.5}$$

$$u_{m_k} \rightharpoonup u \quad \text{weakly in } L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right) \quad \text{as} \quad k \to \infty, \tag{3.6}$$

$$u_{m_k} \to u \quad \text{strongly in } L^\gamma(\Omega) \quad \text{as} \quad k \to \infty \ \left(1 \le {}^\forall\gamma < \infty\right), \tag{3.7}$$

$$u_{m_k} \to u \quad \text{a.e. in } \Omega \quad \text{as} \quad k \to \infty \tag{3.8}$$

for some $u \in W_0^{1,N}(\Omega)$. Note that the second convergence (3.6) comes from the fact $\left(L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)\right)^* \subset W^{-1,N'}(\Omega)$, which is a consequence of the Hardy inequality (1.3), and (3.5). Recall that for $f \in F_N$, there exist $C > 0$ and $\alpha \in (\alpha^*, N]$ such that

$$f(x) \le \frac{C}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^\alpha} \quad \text{in } \Omega.$$

Thus $W_0^{1,N}(\Omega)$ is compactly embedded in $L^q(\Omega, f)$ by Lemma 14. Hence M is weakly closed in $W_0^{1,N}(\Omega)$ and $u \in M$.

Furthermore since $\|J'_\mu(u_m)\|_{W^{-1,N'}(\Omega)} \to 0$, u_m satisfies

$$-\Delta_N u_m = \mu \frac{|u_m|^{N-2}u_m}{\left(|x| \log \frac{\tilde{R}e}{|x|}\right)^N} + \lambda_m |u_m|^{q-2} u_m f + f_m$$

in $\mathcal{D}'(\Omega)$, where $f_m \to 0$ in $W^{-1,N'}(\Omega)$ and $\lambda_m \to \lambda$ as $m \to \infty$. Putting $g_m = \mu \frac{|u_m|^{N-2}u_m}{\left(|x| \log \frac{\tilde{R}e}{|x|}\right)^N} + \lambda_m |u_m|^{q-2} u_m f$, one can check that g_m is bounded in $\mathcal{M}(\Omega)$. Thus we have

$$\nabla u_{m_k} \to \nabla u \text{ a.e. in } \Omega \tag{3.9}$$

from Lemma 12. By using Lemma 13, (3.5), (3.6), (3.8), (3.9), and the Hardy inequality (1.3), we obtain

$$\begin{aligned}
\lambda_\mu(f) &= \|\nabla u_{m_k}\|_N^N - \mu \|u_{m_k}\|^N_{L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)} + o(1) \\
&= \|\nabla (u_{m_k} - u)\|_N^N - \mu \|u_{m_k} - u\|^N_{L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)} + \|\nabla u\|_N^N - \mu \|u\|^N_{L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)} + o(1) \\
&\ge \left(\left(\frac{N-1}{N}\right)^N - \mu\right) \|u_{m_k} - u\|^N_{L^N\left(\Omega, \left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)} + \lambda_\mu(f) + o(1)
\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. As $\mu < \left(\frac{N-1}{N}\right)^N$, we conclude that

$$||u_{m_k} - u||^N_{L^N\left(\Omega,\left(|x| \log \frac{\tilde{R}e}{|x|}\right)^{-N}\right)} \to 0 \quad \text{as} \quad k \to \infty,$$

$$||\nabla(u_{m_k} - u)||^N_N \to 0 \quad \text{as} \quad k \to \infty. \tag{3.10}$$

Hence we have the strong convergence of $\{u_{m_k}\}$ which implies $J_\mu(u) = \lambda_\mu(f)$ and $\lambda = \lambda_\mu(f)$. Since $J_\mu(|u|) = J_\mu(u)$ and the strong maximum principle of Δ_N, we can take $u > 0$ in Ω. Then using Lemma 12 and (3.10), we assure that u is a distributional solution of $(P)^\lambda_\mu$ corresponding to $\lambda = \lambda_\mu(f)$. Moreover u is a weak solution of $(P)^\lambda_\mu$ from density argument.

Finally, if $f \in F_N$, Theorem 9 implies

$$\lambda_\mu(f) \to \lambda(f) = \inf_{u \in W_0^{1,N}(\Omega\setminus\{0\})} \frac{\int_\Omega |\nabla u|^N dx - \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log \frac{\tilde{R}e}{|x|})^N} dx}{\left(\int_\Omega |u|^q f(x) dx\right)^{\frac{N}{q}}}$$

$$> 0 \text{ as } \mu \to \left(\frac{N-1}{N}\right)^N.$$

This completes the proof. □

4. Appendix

Proposition 16. *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain with* $0 \in \Omega$ *and* $\tilde{R} = \sup_{x\in\Omega} |x|$. *Set*

$$C_H(\Omega) = \inf_{0 \neq u \in W_0^{1,N}(\Omega)} \frac{\int_\Omega |\nabla u|^N \, dx}{\int_\Omega \frac{|u|^N}{|x|^N\left(\log \frac{\tilde{R}e}{|x|}\right)^N} \, dx}.$$

Then $C_H(\Omega) = (\frac{N-1}{N})^N$.

Proof. Let $s < \frac{N-1}{N}$ be a positive parameter and take $0 < b < 1$ small satisfying $B_{b\tilde{R}e}(0) \subset \Omega$ $(\subset B_{\tilde{R}}(0))$. We set

$$u_s(x) := \begin{cases} \left(\log \frac{\tilde{R}e}{|x|}\right)^s & \text{if } 0 \leq |x| \leq \frac{b\tilde{R}e}{2} \\ \text{smooth} & \text{if } \frac{b\tilde{R}e}{2} \leq |x| \leq b\tilde{R}e \\ 0 & \text{if } b\tilde{R}e \leq |x|. \end{cases} \tag{4.1}$$

Then

$$|\nabla u_s(x)| = \begin{cases} s\left(\log \frac{\tilde{R}e}{|x|}\right)^{s-1} \frac{1}{|x|} & \text{if } 0 \leq |x| \leq \frac{b\tilde{R}e}{2} \\ \text{smooth} & \text{if } \frac{b\tilde{R}e}{2} \leq |x| \leq b\tilde{R}e \\ 0 & \text{if } b\tilde{R}e \leq |x|. \end{cases}$$

Note that $u_s \in W_0^{1,N}(\Omega)$ for $s < \frac{N-1}{N}$. Direct calculations show that

$$\int_\Omega |\nabla u_s|^N \,dx = \int_{B_{\frac{b\tilde{R}e}{2}}(0)} |\nabla u_s|^N \,dx + \int_{\Omega \setminus B_{\frac{b\tilde{R}e}{2}}(0)} |\nabla u_s|^N \,dx$$

$$= \omega_N s^N \int_0^{\frac{b\tilde{R}e}{2}} \left(\log \frac{\tilde{R}e}{r}\right)^{(s-1)N} \frac{dr}{r} + O(1)$$

$$= \omega_N s^N \frac{-1}{(s-1)N+1} \left(\log \frac{2}{b}\right)^{(s-1)N+1} + O(1) \quad \text{as } s \to \frac{N-1}{N}.$$

On the other hand, we have

$$\int_\Omega \frac{|u_s|^N}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} dx = \int_{B_{\frac{b\tilde{R}e}{2}}(0)} \frac{|u_s|^N}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} dx + \int_{\Omega \setminus B_{\frac{b\tilde{R}e}{2}}(0)} \frac{|u_s|^N}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} dx$$

$$= \omega_N \int_0^{\frac{b\tilde{R}e}{2}} \left(\log \frac{\tilde{R}e}{r}\right)^{(s-1)N} \frac{dr}{r} + O(1)$$

$$= \omega_N \frac{-1}{(s-1)N+1} \left(\log \frac{2}{b}\right)^{(s-1)N+1} + O(1) \quad \text{as } s \to \frac{N-1}{N}.$$

Therefore

$$\frac{\int_\Omega |\nabla u_s|^N \,dx}{\int_\Omega \frac{|u_s|^N}{|x|^N \left(\log \frac{\tilde{R}e}{|x|}\right)^N} \,dx} \to \left(\frac{N-1}{N}\right)^N \quad \text{as } s \to \frac{N-1}{N}$$

and we conclude that $C_H(\Omega) = (\frac{N-1}{N})^N$. □

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