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Minimization problems on the Hardy-Sobolev inequality

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Abstract We study minimization problems on Hardy-Sobolev type inequality. We consider the case where singularity is in interior of bounded domain $\Omega \subset \mathbb{R}^N$. The attainability of best constants for Hardy-Sobolev type inequalities with boundary singularities have been studied so far, for example [5] [6] [9] etc.... According to their results, the mean curvature of $\partial\Omega$ at singularity affects the attainability of the best constants. In contrast with the case of boundary singularity, it is well known that the best Hardy-Sobolev constant

$$\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 dx \middle| u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1 \right\}$$

is never achieved for all bounded domain Ω if $0 \in \Omega$. We see that the position of singularity on domain is related to the existence of minimizer. In this paper, we consider the attainability of the best constant for the embedding $H^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega)$ for bounded domain Ω with $0 \in \Omega$. In this problem, scaling invariance doesn't hold and we can not obtain information of singularity like mean curvature.

Keywords critical exponent · Hardy-Sobolev inequality · minimization problem · Neumann

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1 Introduction

We study the minimization problems for the Hardy-Sobolev type inequalities. Let $N \geq 3$, Ω is bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, $0 < s < 2$, and $2^*(s) := 2(N-s)/(N-2)$. The Hardy-Sobolev inequality asserts that there exists a positive constant C such that

$$C \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla u|^2 dx \tag{1}$$

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for all $u \in H_0^1(\Omega)$. For $s = 0$, the inequality (1) is called Sobolev inequality and for $s = 2$, the inequality (1) is called Hardy inequality.

In the non-singular case ($s = 0$), it is well known that the best Sobolev constant S is independent of domain Ω and S is never achieved for all bounded domains. But if $\Omega = \mathbb{R}^N$ and $H^1(\Omega)$ is replaced by the function space of $u \in L^{2N/(N-2)}(\Omega)$ with $\nabla u \in L^2(\Omega)$, then S is achieved by the function $u(x) = c(1+|x|^2)^{(2-N)/2}$ and hence the value $S = N(N-2)\pi[\Gamma(N/2)/\Gamma(N)]^{2/N}$ explicitly (see [1], [13] and [16]).

In the case of $s = 2$, the best constant for the Hardy inequality is $[(N-2)/2]^2$ and this constant is never achieved for all bounded domains and $\mathbb{R}^N$. This fact suggests that it is possible to improve this inequality. For example Brezis and Vazquez [2], many people research the optimal inequality of (1). In other words, the best remainder term for (1) is studied actively.

In the case of $0 < s < 2$, the best Hardy-Sobolev constant is defined by

$$\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 dx \,\middle|\, u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1 \right\}.$$

This constant has some similar properties to these of the best Sobolev constant. Indeed, due to scaling invariance, $\mu_s(\Omega)$ is independent of Ω, and thus $\mu_s := \mu_s(\Omega) = \mu_s(\mathbb{R}^N)$ is not attained for all bounded domains. If $\Omega = \mathbb{R}^N$, then μ_s is attained by

$$y_a(x) = [a(N-s)(N-2)]^{\frac{N-2}{2(2-s)}} \left(a + |x|^{2-s}\right)^{\frac{2-N}{2-s}}$$

for some $a > 0$ and hence

$$\mu_s = (N-2)(N-s) \left(\frac{\omega_{N-1}}{2-s} \frac{\Gamma^2\left(\frac{N-s}{2-s}\right)}{\Gamma\left(\frac{2(N-s)}{2-s}\right)} \right)^{\frac{2-s}{N-s}}$$

(see [9] and [13]) where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$.

On the other hand, for $0 \in \partial\Omega$, the result of the attainability for $\mu_s(\Omega)$ is quite different from that in the situation of $0 \in \Omega$. By Ghoussoub-Robert [6], it has proved that if Ω has smooth boundary and the mean curvature of $\partial\Omega$ at 0 is negative, then the extremal of $\mu_s(\Omega)$ exists for all $N \geq 3$. Recently, Lin and Wadade [14] have studied the following minimization problem;

$$\mu_{s,p}^\lambda(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 dx + \lambda \left(\int_\Omega |u|^p dx \right)^{\frac{2}{p}} \,\middle|\, u \in H_0^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}$$

where $\lambda \in \mathbb{R}$ and $2 \leq p \leq 2N/(N-2)$. Furthermore, as related results, Hsia, Lin and Wadade [10] studied the existence of the solution of double critical elliptic equations related with $\mu_{s,2^*}^\lambda(\Omega)$, that is, they have showed the existence of the solution for

$$\begin{cases} -\Delta u + \lambda u^{2^*-1} + \frac{u^{2^*(s)-1}}{|x|^s} = 0, \quad u > 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

under the appropriate conditions where $2^* = 2N/(N-2)$. To prove these results, we use the theorem of Egnell [4]. He showed that the existence of the extremal for $\mu_s(\Omega)$ if Ω is a half space $\mathbb{R}_+^N$ or an open cone. The open cone $\mathscr{C}$ is written of the form $\mathscr{C} := \{x \in \mathbb{R}^N | x =$

$r\theta,\ \theta \in \Sigma\}$ where Σ is connected domain on the unit sphere $\mathscr{S}^{N-1}$ in $\mathbb{R}^N$. By this result, $\mu_s(\mathscr{C}) > \mu_s(\mathbb{R}^N)$ and there is a positive solution for

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(s)-1}}{|x|^s} & \text{in } \mathscr{C}, \\ u = 0 \quad \text{on } \partial\mathscr{C}, & \text{and} \quad u(x) = o(|x|^{2-N}) \text{ as } x \to \infty. \end{cases}$$

The Neumann case also has been studied. Let Ω has C^2 boundary and the mean curvature of $\partial\Omega$ at 0 is positive. Ghoussoub and Kang [5] have showed that there is a least energy solution for

$$\begin{cases} -\Delta u + \lambda u = \frac{|u|^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for $N \geq 3$, $\lambda > 0$.

Like these results, if $0 \in \partial\Omega$, we can use the benefit of the mean curvature of $\partial\Omega$ at 0 to show the results. However if $0 \in \Omega$, we cannot obtain the information of singularity such the mean curvature, and the fact causes some technical difficulties.

In this paper, we consider the attainability for the following minimization problem

$$\mu_s^N(\Omega) := \inf\left\{ \int_\Omega (|\nabla u|^2 + u^2)dx \,\middle|\, u \in H^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\}.$$

The main theorem is as follows.

Theorem 1 *Let $\partial\Omega$ has a smoothness which the Sobolev embeddings hold, then the following statements hold true.*

(I) If Ω is sufficiently small, then $\mu_s^N(\Omega)$ is attained. Especially, if Ω satisfies the following;

$$|\Omega| \left(\int_\Omega |x|^{-s} dx \right)^{-\frac{2}{2^*(s)}} \leq \mu_s$$

then $\mu_s^N(\Omega)$ is attained, where $|\Omega|$ is the N-dimensional Lebesgue measure of domain Ω.

(II) There is a positive constant M which depends on only Ω such that $\mu_s^N(r\Omega)$ is never attained if $r > M$.

Eventually, the size of domain affects the attainability of $\mu_s^N(\Omega)$.

The rest of the paper is organized as follows. In Section 2 we introduce three lemmas to prove Theorem 1. Then in Section 3 we prove Theorem 1 using the lemmas in Section 2. In Section 4, as an application, we consider the case when singularity is in the boundary of domain. Then we introduce a new result concerning the attainability of $\mu_s^N(\Omega)$ with boundary singularity.

2 Preparation

In this section, we prepare some lemmas to prove Theorem 1.

Lemma 1 *For $r>0$, the value $\mu_{s,r}^N(\Omega)$ is defined by*

$$\mu_{s,r}^N(\Omega) := \inf\left\{ \int_\Omega (|\nabla u|^2 + ru^2)dx \,\middle|\, u \in H^1(\Omega),\ \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1 \right\}.$$

We have

$$\mu_{s,r}^N(\Omega) = \mu_s^N(r\Omega).$$

Proof For $r>0$ and $u \in H^1(\Omega)$, u_r is defined by the scaling of u, that is $u_r(x) := r^{\frac{2-N}{2}}u(x/r) \in H^1(r\Omega)$. Note that

$$\begin{aligned}
\int_{r\Omega} |\nabla u_r|^2 dx &= \int_\Omega |\nabla u|^2 dx, \\
\int_{r\Omega} u_r^2 dx &= r^2 \int_\Omega u^2 dx, \\
\int_{r\Omega} \frac{{u_r}^{2^*(s)}}{|x|^s} dx &= \int_\Omega \frac{u^{2^*(s)}}{|x|^s} dx.
\end{aligned}$$

With these facts in mind, taking $u \in H^1(\Omega)$ such that

$$\int_\Omega \frac{u^{2^*(s)}}{|x|^s}dx = 1, \quad \int_\Omega (|\nabla u|^2 + r^2u^2)dx \le \mu_{s,r}^N(\Omega) + \varepsilon$$

for $\varepsilon > 0$ sufficiently small, we have

$$\mu_s^N(r\Omega) \le \int_{r\Omega} (|\nabla u_r|^2 + u_r^2)dx = \int_\Omega (|\nabla u|^2 + r^2u^2)dx \le \mu_{s,r}^N(\Omega) + \varepsilon.$$

Hence we have $\mu_s^N(r\Omega) \le \mu_{s,r}^N(\Omega)$.

The inverse also holds by replacing Ω with $r\Omega$.

Lemma 2 *There exists a positive constant C which depends on only Ω such that*

$$\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u|^2 dx + C\int_\Omega u^2 dx \quad (u \in H^1(\Omega)). \tag{2}$$

Before beginning the proof, we make a remark. H. Jaber [12] has shown that the following theorem.

Theorem 2 *([12]) If (M,g) is a compact Riemannian manifold without boundary and $0 \in M$, there is a constant $C = C(M,g)$ such that*

$$\mu_s \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x,0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \le \int_M |\nabla u|^2 dv_g + C\int_\Omega u^2 dv_g \quad (u \in H^1(M))$$

where d_g is the Riemannian distance on M.

Different from Theorem 2, Ω is bounded domain of $\mathbb{R}^N$ and therefore Ω has a boundary, thus we can show the inequality (2) simply.

Proof Let $0 \in \Omega_1 \subset \Omega_2 \subset \Omega$ and these two subdomain are taken suitable again later. A cut-off function is defined by ϕ which satisfies

$$\phi \in C_c^\infty(\mathbb{R}^N), \quad 0 \le \phi \le 1 \text{ in } \Omega, \quad \phi = 1 \text{ on } \Omega_1, \quad \phi = 0 \text{ on } \Omega \setminus \Omega_2.$$

Here, we construct a partition of unity $\eta_1,\ \eta_2$ defined by

$$\eta_1 := \frac{\phi^2}{\phi^2 + (1-\phi)^2}, \quad \eta_2 := \frac{(1-\phi)^2}{\phi^2 + (1-\phi)^2}.$$

Note that $\eta_1^{\frac{1}{2}},\ \eta_2^{\frac{1}{2}} \in C^2(\Omega)$ by the definition. We may assume that $u \in C^\infty(\Omega) \cap H^1(\Omega)$ by density. We have

$$\begin{aligned}
\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &= \mu_s \left\| u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} = \mu_s \left\| \sum_{i=1}^2 \eta_i u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \\
&\le \mu_s \sum_{i=1}^2 \left\| \eta_i u^2 \right\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \\
&= \mu_s \sum_{i=1}^2 \left(\int_\Omega \frac{|\eta_i^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1, I_2 for each.

For I_1, since $\mathrm{supp}\eta_1 \subset \Omega$ we can use the Hardy-Sobolev inequality. We get that

$$\begin{aligned}
I_1 &= \mu_s \left(\int_\Omega \frac{|\eta_1^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla(\eta_1^{\frac{1}{2}} u)|^2 dx \\
&= \int_\Omega |\nabla u|^2 \eta_1 dx + \int_\Omega \nabla(\eta_1^{\frac{1}{2}}) \cdot \nabla(\eta_1^{\frac{1}{2}} u^2) dx.
\end{aligned}$$

Since $\eta_1^{\frac{1}{2}} \in C^2(\Omega)$ we may integrate by parts the second term and hence we obtain

$$I_1 \le \int_\Omega |\nabla u|^2 \eta_1 dx - \int_\Omega \Delta(\eta_1^{\frac{1}{2}}) \eta_1^{\frac{1}{2}} u^2 dx \tag{3}$$

For I_2, since $0 \notin \mathrm{supp}\eta_2$ and taking account to that $\eta = 0$ on Ω_1 we have

$$\begin{aligned}
I_2 &= \mu_s \left(\int_\Omega \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} = \mu_s \left(\int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\
&\le \mu_s \cdot a \left(\int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u|^{2^*(s)} dx \right)^{\frac{2}{2^*(s)}} \\
&\le \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} \left(\int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u|^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\le \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} S(\Omega, \Omega_1)^{-1} \int_{\Omega \setminus \Omega_1} |\nabla(\eta_2^{\frac{1}{2}} u)|^2 dx \\
&= \mu_s \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{2^*}} S(\Omega, \Omega_1)^{-1} \int_\Omega |\nabla(\eta_2^{\frac{1}{2}} u)|^2 dx
\end{aligned}$$

where $a := \mathrm{dist}(0, \partial\Omega_1)^{-2s/2^*(s)}$ and

$$S(\Omega, \Omega_1) := \inf\left\{ \int_{\Omega\setminus\Omega_1} |\nabla u|^2 dx \middle| u \in H^1(\Omega),\ u = 0 \text{ on } \partial\Omega_1,\ \int_{\Omega\setminus\Omega_1} |u|^{2^*} = 1 \right\}.$$

Here, let us take $\Omega_0 \subset \Omega_1$. It is clearly that $a \leq \mathrm{dist}(0, \partial\Omega_0)^{-2s/2^*(s)}$. On the other hand, for $u \in H^1(\Omega \setminus \Omega_1)$ such that $u = 0$ on $\partial\Omega_1$, we define $v \in H^1(\Omega \setminus \Omega_0)$ by

$$v := \begin{cases} u & \text{in } \Omega \setminus \Omega_1 \\ 0 & \text{in } \Omega_1 \setminus \Omega_0. \end{cases}$$

By identifying $u \in H^1(\Omega \setminus \Omega_1)$ with $v \in H^1(\Omega \setminus \Omega_0)$ concerning the calculation of the Sobolev quotient, we may see that

$$\{u \in H^1(\Omega \setminus \Omega_1) | u = 0 \text{ on } \partial\Omega_1\} \subset \{u \in H^1(\Omega \setminus \Omega_0) | u = 0 \text{ on } \partial\Omega_0\}.$$

Hence we obtain $S(\Omega, \Omega_1) \geq S(\Omega, \Omega_0)$. Consequently, if Ω_1 is sufficiently large, a and $S(\Omega, \Omega_1)^{-1}$ is bounded from above uniformly. By choosing Ω_1 and Ω_2 close to Ω we obtain

$$I_2 \leq \frac{1}{2} \int_\Omega |\nabla(\eta_2^{\frac{1}{2}} u)|^2 dx.$$

Therefore

$$I_2 \leq \int_\Omega |\nabla u|^2 \eta_2 dx + \int_\Omega |\nabla \eta_2^{\frac{1}{2}}|^2 u^2 dx. \tag{4}$$

Here, since $\eta_1^{\frac{1}{2}},\ \eta_2^{\frac{1}{2}} \in C^2(\Omega)$ there is a positive constant C such that

$$\max_{x\in\Omega} |\Delta(\eta_1^{\frac{1}{2}})| \leq \frac{C}{2}, \quad \max_{x\in\Omega} |\nabla \eta_2^{\frac{1}{2}}|^2 \leq \frac{C}{2}. \tag{5}$$

This constant depends on only Ω.

Consequently (3), (4) and (5) yield that

$$\mu_s \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq I_1 + I_2 \leq \int_\Omega |\nabla u|^2 dx + C \int_\Omega u^2 dx.$$

Lemma 3 *$\mu_s^N(\Omega) \leq \mu_s$ holds (see [9], Lemma 11.1). Furthermore, the following statements hold true;*

(I) If $\mu_s^N(\Omega) < \mu_s$, then $\mu_s^N(\Omega)$ is attained.
(II) If $\mu_s^N(\Omega) = \mu_s$, then $\mu_s^N(r\Omega)$ is not attained for all $r > 1$.

Firstly, we prove Lemma 3 (I).

Proof (Proof of Lemma 3 (I))
Assume $\{u_n\}_{n=1}^\infty \subset H^1(\Omega)$ is a minimizing sequence of $\mu_s^N(\Omega)$. Without loss of generality, we may assume

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = 1 \tag{6}$$

for all $n \in \mathbb{N}$ and which implies

$$\int_\Omega (|\nabla u_n|^2 + u_n^2) dx = \mu_s^N(\Omega) + o(1) \quad (n \to \infty). \tag{7}$$

Thus u_n is bounded in $H^1(\Omega)$. So we can suppose, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1(\Omega) \\ u_n &\to u \quad \text{in } L^p(\Omega) \quad (1 \le p < 2^*) \\ u_n &\to u \quad \text{in } L^q(\Omega, |x|^{-s}) \quad (1 \le q < 2^*(s)) \\ u_n &\to u \quad \text{a.e. in } \Omega \end{aligned}$$

as $n \to \infty$.

For this limit function u, we show that $u \not\equiv 0$ a.e. in Ω. Assume that $u \equiv 0$ a.e. in Ω. By the inequality (2) in Lemma 2,

$$\mu_s \left(\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u_n|^2 dx + C \int_\Omega u_n^2 dx \tag{8}$$

holds for all n. Thus (6), (7), (8) and $u_n \to u$ in $L^2(\Omega)$ yield

$$\mu_s \le \mu_s^N(\Omega) + o(1).$$

Letting n tend to infinity, we obtain $\mu_s \le \mu_s^N(\Omega)$ and which is contradiction in the assumption of $\mu_s^N(\Omega) < \mu_s$. Consequently $u \not\equiv 0$.

By the theorem of Brezis and Lieb (see [3]), we obtain

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx = \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx + o(1)$$

and it follows that

$$\begin{aligned} 1 &= \left(\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ &= \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx + \int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} + o(1) \\ &\le \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} + \left(\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} + o(1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} + \left(\int_\Omega \frac{|u_n - u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ &\le \frac{\int_\Omega (|\nabla u|^2 + u^2) dx}{\mu_s^N(\Omega)} + \frac{\int_\Omega (|\nabla (u_n - u)|^2 + (u_n - u)^2 dx}{\mu_s^N(\Omega)} \\ &= \frac{\int_\Omega (|\nabla u_n|^2 + u_n^2) dx}{\mu_s^N(\Omega)} + o(1) \\ &= 1 + o(1). \end{aligned}$$

Hence there exist a limit and we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx+\int_\Omega \frac{|u_n-u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}\\
&=\lim_{n\to\infty}\left[\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}+\left(\int_\Omega \frac{|u_n-u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}\right]\\
&=1.
\end{aligned}$$

By the equality condition of the above, we get either

$$u\equiv 0 \quad \text{a.e. in } \Omega \quad \text{or} \quad u_n\to u\not\equiv 0 \quad \text{in } L^{2^*(s)}(\Omega,|x|^{-s}).$$

Since $u\not\equiv 0$ we obtain $u_n\to u\not\equiv 0$ in $L^{2^*(s)}(\Omega,|x|^{-s})$ and hence this u is the minimizer of $\mu_s^N(\Omega)$.

Next, we prove Lemma 3 (II).

Proof (Proof of Lemma 3 (II)) We assume the existence of the minimizer of $\mu_s^N(r\Omega)$ and derive a contradiction. Let $u\in H^1(r\Omega)$ be a minimizer of $\mu_s^N(r\Omega)$, then we have

$$\mu_s^N(r\Omega)=\int_{r\Omega}(|\nabla u|^2+u^2)dx>\int_{r\Omega}(|\nabla u|^2+\frac{1}{r^2}u^2)dx\geq \mu_{s,1/r}^N(r\Omega).$$

By Lemma 1, the assumption $\mu_s^N(\Omega)=\mu_s$ and $\mu_s^N(r\Omega)\leq\mu_s$, we have

$$\mu_s\geq\mu_s^N(r\Omega)>\mu_{s,1/r}^N(r\Omega)=\mu_s^N(\Omega)=\mu_s.$$

This is a contradiction.

3 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof (Proof of Theorem 1 (I)) We recall that

$$\mu_s^N(\Omega):=\inf\left\{\int_\Omega(|\nabla u|^2+u^2)dx \,\middle|\, u\in H^1(\Omega),\ \int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx=1\right\}.$$

Taking a constant C such that $\int_\Omega \frac{C^{2^*(s)}}{|x|^s}=1$ and $u\equiv C$ as a test function, it follows that

$$\mu_s^N(\Omega)\leq|\Omega|\left(\int_\Omega|x|^{-s}\right)^{-\frac{2}{2^*(s)}}.$$

If this C is a minimizer of $\mu_s^N(\Omega)$, then by Lagrange multiplier theorem C is a classical solution of

$$\begin{cases}-\Delta u+u=\mu_s^N(\Omega)\frac{u^{2^*(s)}}{|x|^s} & \text{in } \Omega\\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega.\end{cases}$$

This contradicts and therefore

$$\mu_s^N(\Omega)<|\Omega|\left(\int_\Omega|x|^{-s}\right)^{-\frac{2}{2^*(s)}}.$$

Combining this estimate and Lemma 3 (I), Theorem 1 (I) holds true.

Proof (Proof of Theorem 1 (II)) Since Lemma 2, We can define a constant m by

$$m := \inf\{C > 0 |\ (2) \text{ holds.}\}.$$

M is defined by $M := \sqrt{m}$. In inequality (2), C is replaced by M^2 and hence we have

$$\mu_s \le \frac{\int_\Omega (|\nabla u|^2 + M^2u^2)dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}} \tag{9}$$

for all $u \in H^1(\Omega)$. Therefore by Lemma 1 we obtain

$$\begin{aligned}\mu_s &\le \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + M^2u^2)dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}}\\ &= \mu^N_{s,M}(\Omega)\\ &= \mu^N_s(M\Omega).\end{aligned}$$

Recall that $\mu^N_s(\Omega) \le \mu_s$ holds for all bounded domain Ω and thus $\mu^N_s(M\Omega) = \mu_s$. Consequently we obtain the result of Theorem 1 (II) by Lemma 3 (II).

4 Singularity on the boundary

Throughout this section, assume that $0 \in \partial\Omega$. If the mean curvature of $\partial\Omega$ at 0 is positive, we have obtained the results in Section 1. However, if the mean curvature of $\partial\Omega$ at 0 vanishes, we don't obtain results so far, even if the attainability of $\mu^N_s(\Omega)$. In this section, we show the following results by using the strategy in Section 2 and Section 3.

Theorem 3 *Let $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $0 \in \partial\Omega$ and $\partial\Omega$ is flat near the origin. Then the following statements hold;*

(I) If Ω is sufficiently small, then $\mu^N_s(\Omega)$ is attained. Especially, if Ω satisfies the following;

$$|\Omega|\left(\int_\Omega |x|^{-s}dx\right)^{-\frac{2}{2^*(s)}} \le \frac{\mu_s}{2^{\frac{2-s}{N-s}}}$$

then $\mu^N_s(\Omega)$ is attained.

(II) There is a positive constant M which depends on only Ω such that $\mu^N_s(r\Omega)$ is never attained if $r > M$.

This condition of the boundary in this theorem is a special case of vanishing of the mean curvature of $\partial\Omega$ at 0.

We prove the theorem in the same way as in Section 2 and Section 3. Different from the proof of Theorem 1, we need the following lemma instead of Lemma 2.

Lemma 4 *There is a positive constant C depends on only Ω such that*

$$\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u|^2dx + C\int_\Omega u^2dx \quad (u \in H^1(\Omega)). \tag{10}$$

Proof We introduce some notation. $B_R(0)$ is an open ball which center is origin and radius is R. $\mathbb{R}^N_+$ is a half space which is defined by $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N | x_n > 0\}$ where $x' := (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.

Since $\partial\Omega$ is flat near the origin, by rotating coordinate there is a constant $r > 0$ such that $B_r(0) \cap \Omega = B_r^+(0) := B_r(0) \cap \mathbb{R}^N_+$. For $u \in H^1(\Omega)$ we have

$$\begin{aligned}\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} &= \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx + \int_{\Omega\setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \\ &\le \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} + \left(\int_{\Omega\setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \\ &= J_1 + J_2.\end{aligned}$$

For $u \in H^1(B_r^+(0))$, $\tilde{u} \in H^1(B_r(0))$ is defined by the even reflection for the direction x_N, that is,

$$\tilde{u}(x', x_N) := \begin{cases} u(x', x_N) & \text{if } 0 \le x_N < 1 \\ u(x', x_N) & \text{if } -1 < x_N < 0. \end{cases}$$

Concerning J_1, by Lemma 2 we have

$$\begin{aligned} J_1 &= \left(\int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \\ &= \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}} \left(\int_{B_r(0)} \frac{|\tilde{u}|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \\ &\le \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \left(\int_{B_r(0)} |\nabla \tilde{u}|^2 dx + C_1 \int_{B_r(0)} \tilde{u}^2 dx\right) \\ &= \left(\frac{1}{2}\right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \cdot 2 \left(\int_{B_r^+(0)} |\nabla u|^2 dx + C_1 \int_{B_r^+(0)} u^2 dx\right) \\ &= \left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\right)^{-1} \left(\int_{B_r^+(0)} |\nabla u|^2 dx + C_1 \int_{B_r^+(0)} u^2 dx\right) \end{aligned}$$

for some positive constant C_1 depends on only $B_r(0)$.

Next, we estimate J_2. Let $\delta > 0$ for sufficiently small. We consider $\{\phi_i\}_{i=1}^m$ a partition of unity on $\overline{\Omega \setminus B_r^+(0)}$ such that $\phi_i^{\frac{1}{2}} \in C^1$ and $|\mathrm{supp}\phi_i| \le \delta$ for all i. Since $|x|^{-s} \le r^{-s}$ for $x \in \Omega \setminus B_r^+(0)$ we have

$$\begin{aligned} J_2 &= \left(\int_{\Omega\setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \le \sum_{i=1}^m \left(\int_{\Omega\setminus B_r^+(0)} \frac{|\phi_i^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}} \\ &\le r^{-\frac{2s}{2^*(s)}} \sum_{i=1}^m \left(\int_{\Omega\setminus B_r^+(0)} |\phi_i^{\frac{1}{2}} u|^{2^*(s)} dx\right)^{\frac{2}{2^*(s)}}. \end{aligned}$$

By Hölder inequalities it follows that

$$\begin{aligned}\left(\int_{\Omega\setminus B_r^+(0)}|\phi_i^{\frac{1}{2}}u|^{2^*(s)}dx\right)^{\frac{2}{2^*(s)}} &\leq |\mathrm{supp}\phi_i|^{\frac{2}{2^*(s)}-\frac{2}{2^*}}\|\phi_i^{\frac{1}{2}}u\|^2_{L^{2^*}(\Omega\setminus B_r^+(0))}\\ &\leq \delta^{\frac{2}{2^*(s)}-\frac{2}{2^*}}\|\phi_i^{\frac{1}{2}}u\|^2_{L^{2^*}(\Omega\setminus B_r^+(0))}\end{aligned}$$

for each $i\in\mathbb{N}$. Since δ is sufficiently small, by using the Sobolev inequalities (If necessary we use the Sobolev inequalities of mixed boundary condition version.) we have

$$J_2\leq\left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\right)^{-1}\cdot\frac{1}{2}\sum_{i=1}^m\int_{\Omega\setminus B_r^+(0))}|\nabla(\phi_i^{\frac{1}{2}}u)|^2dx.$$

Consequently we have

$$J_2\leq\left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\right)^{-1}\left(\int_{\Omega\setminus B_r^+(0)}|\nabla u|^2dx+C_2\int_{\Omega\setminus B_r^+(0)}u^2dx\right).$$

for some positive constant C_2 depends on only $\Omega\setminus B_r^+(\Omega)$. Combining the estimates of J_1 and J_2 we obtain

$$\left(\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}\right)^{\frac{2}{2^*(s)}}\leq J_1+J_2\leq\left(\frac{\mu_s}{2^{\frac{2-s}{N-s}}}\right)^{-1}\left(\int_\Omega|\nabla u|^2dx+C\int_\Omega u^2dx\right)$$

for some positive constant C depends on Ω.

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