A note on a nonlinear elliptic problem with the nonlocal coefficient

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Abstract

In this paper, we investigate the nonlocal and nonlinear elliptic problem,

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + u^p \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $N \leq 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, a is a nondegenerate continuous function, $p > 1$ and $\lambda \in \mathbb{R}$. We show several effects of the nonlocal coefficient a on the structure of the solution set of (P). We first introduce a scaling observation and describe the solution set by using that of an associated semilinear problem. This allows us to get unbounded continua of solutions (λ, u) of (P). A rich variety of new bifurcation and multiplicity results are observed. We also prove that the nonlocal coefficient can induce up to uncountably many solutions by a convenient way. Lastly, we give some remarks from the variational point of view.

Keywords: nonlocal, elliptic, bifurcation, variational method, Kirchhoff type

1. Introduction

In this paper, we consider a nonlinear elliptic problem involving the Dirichlet energy,

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + u^p \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

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where $N \leq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. In addition, assume a is a continuous function such that $a \geq a_0$ for some constant $a_0 > 0$, $1 < p < \infty$ if $N = 1, 2$, $1 < p \leq 5$ if $N = 3$ and $\lambda \in \mathbb{R}$ is a parameter. Our aim is to give one perspective how the nonlocal coefficient a works on the solvability of (P). To do this, we study the structure of the solution set, the bifurcation phenomena and the multiplicity of solutions of (P) by a convenient way.

The nonlocal problems involving the Dirichlet energy are introduced by the suitable ways in several stages of natural sciences. In the theory of the nonlinear vibrations, it appears as a wave equation [19]. For the mathematical development, see [1]. On the other hand, a parabolic problem is introduced as a model equation for the dynamics of the population density of bacterias and also the heat conduction, see [12] and [13]. In particular, in [13], they indicate that it can admit several equilibria and has the energy structure. This motivates them to investigate the asymptotic behavior of the solution. More recently, the stationary and thus, elliptic problems with nonlinear reaction terms, such as (P), attract much attentions [7][10][15][21]-[28][30][34]. Using the variational or topological techniques, the authors investigate the existence of solutions. For example, the 3-superlinear at infinity case is considered in [34] and some references therein. That is, they consider (P) with $a(t) = a_0 + \alpha t$ where $a_0, \alpha > 0$ and the nonlinearity $f \in C(\mathbb{R})$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^3} = \infty.$$

This is a natural assumption in view of the mountain pass type geometry [4]. With the asymptotically linear condition at zero and some additional ones, they get the existence of solutions. On the other hand, Perera-Zhang [30] and Liang-Li-Shi [22] investigate a delicate problem including the asymptotically linearity at zero and the asymptotically 3-linearity at infinity, say,

$$\lim_{u\to\infty} \frac{f(u)}{u^3} = Const.$$

It is worth remarking that this case has a close relation to the nonlinear eigenvalue problem,

$$\begin{cases} -\int_\Omega |\nabla u|^2 dx\ \Delta u = \mu u^3 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Some interesting studies on its eigenvalues and functions are observed in their works. In addition, in [22], they indicate the difficulty caused by the lack of the Ambrosetti-Rabinowitz type condition [4]. Applying the tool in [18] based on the monotonicity trick [33], they get the solvability including a bifurcation result. Notice that our nonlinearities $\lambda u + u^p$ with $p > 3$ and $p = 3$ are typical examples of these two assumptions above respectively. In addition, we deal with the 3-sublinearity at infinity, $1 < p < 3$ which has not been considered yet to our best knowledge and the critical case, $N = 3$ and $p = 5$. Recently the critical case has been attacked by the author [25]. He solves the problem by utilizing the pioneering argument by Brezis-Nirenberg [8] with the concentration compactness result by Lions [23]. An interesting thing is that, he gets a solution which attains the local minimum of the energy in addition to a mountain pass type solution. See Theorem 5.1 there. Since if Ω is a ball, (P) with $a(t) = 1$ admits at most one positive solution, see [5], we may conclude that this multiplicity is, in fact, induced by the nonlocal coefficient. Now, this reminds us of the earlier works by Chipot et al. in [12] and [13] stated in the beginning of this paragraph. As in them, the author's result implies that the nonlocal coefficient can induce the multiplicity of stationary solutions even for the problem with the nonlinear reaction term. Indeed, we can find a related result in [10], which says that the concave-convex problem [3] may have the third positive solution. Readers can refer to its introduction or Theorem 2.4. Our work is inspired by these results. As noted in the first paragraph, one of the aims of this paper is to show how the nonlocal coefficient can affect the multiplicity of solutions of (P). Actually, we will see that (P) admits a rich variety of multiplicity results by the combined effect of the nonlocal coefficient and the nonlinear reaction term. Before beginning our main argument, we introduce a convenient observation below.

1.1. A scaling observation

Here, we introduce our basic idea throughout this paper. Let us reduce our problem (P) to a semilinear problem. Suppose that a, p and λ are as in the introduction and $\|\cdot\|$ is the usual $H_0^1(\Omega)$ norm. First, observe that for any $\lambda \in \mathbb{R}$, if u is a solution of (P), $v := a(\|u\|^2)^{1/(1-p)}u$ is a solution of the semilinear elliptic problem,

$$\begin{cases} -\Delta v = \mu v + v^p \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \tag{P$_0$}$$

with $\mu = \lambda/a(\|u\|^2)$. On the other hand, let $\lambda \in \mathbb{R}$, $t > 0$ and v be a solution of (P$_0$) with $\mu = \lambda/a(t)$. Then $u := a(t)^{1/(p-1)}v$ is a solution of

$$\begin{cases} -a(t)\Delta u = \lambda u + u^p \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Thus, if $t = \|u\|^2 = a(t)^{2/(p-1)}\|v\|^2$, u is nothing but a solution of (P). As a consequence, we conclude the following.

Proposition 1.1. *For $\lambda \in \mathbb{R}$, u is a solution of (P) if and only if there exists a pair $(t, v) \in \mathbb{R}^+ \times C^{2,\gamma}(\overline{\Omega})$ such that $u = a(t)^{1/(p-1)}v$ and*

$$t = a(t)^{\frac{2}{p-1}}\|v\|^2$$

where v is a solution of (P$_0$) with $\mu = \lambda/a(t)$.

Using Proposition 1.1, we can construct up to uncountably many solutions of (P) readily. See Section 4. Moreover, this observation allows us to obtain certain informations on the structure of the solution set and the bifurcation phenomena. We may expect that the nonlocal coefficient drastically affects the bifurcation diagram on (P) and so the multiplicity. See Sections 2 and 3. We note that, up to now, very many authors have already investigated the existence of solutions of (P) with several nonlinearities. But very few ones have concerned with its bifurcation phenomena or their diagrams. Recently some interests have begun to occur. See Theorem 1.2 and Remark 1.3 in [22], and, for another problem, [10] where the result by mathematical computations are shown. Furthermore, (after this work was finished) the author found [14] and [16] where the bifurcation techniques are applied. In particular, by Theorem 1.1 in [14], one gets an alternative on the global bifurcation for the general setting. It implies that $(\lambda, u) = (\lambda_*, 0)$ is the bifurcation point of (P) and there may exist an unbounded continuum of solutions of (P) which meets $(\lambda_*, 0)$. In this paper, we actually construct unbounded continua of solutions by a totally different idea and give additional informations around the trivial solution and infinity. Lastly, we remark that, although the variational method is very powerful on (P), it is necessary and interesting to develop non-variational techniques to proceed the analysis on the nonlocal problem. Actually, several authors now are developing them. See [6], [20], [22] and for another functional elliptic problem, [11]. The main argument in this paper is also non-variational. It is so helpful enough to answer the question, by the convenient way, "What can happen on the typical model problem (P) by the nonlocal coefficient?". In fact, we observe several new phenomena induced by the nonlocal coefficient here. In addition, our results can add certain interpretations and informations on the previous results [34], [22] and [14] etc.. We believe that it gives us good perspectives and motivations for the problem in the future.

1.2. Organization and notations of this note

This paper is organized as the following. In Section 2, we show the structure of the solution set of (P) using that of the semilinear problem (P_0). Next, in Section 3, we demonstrate the global bifurcation diagram of (P) for the simple case, Ω is a ball. Lastly, in Section 4, we put several remarks on the multiplicity results on (P). In particular, the former part, Subsection 4.1 is devoted to the construction of up to uncountably many solutions. The latter one, Subsection 4.2, is to the remarks from the variational point of view.

We put $\|u\| := (\int_\Omega |\nabla u|^2 dx)^{1/2}$ as the usual $H_0^1(\Omega)$ norm and $\|u\|_\infty := \sup_{x\in\Omega} |u(x)|$ as the $L^\infty(\Omega)$ norm. Furthermore, we define the norm in $C^{2,\gamma}(\overline{\Omega})$ with $0 < \gamma < 1$ as

$$\|u\|_{2,\gamma} := \max_{|l|\le 2} \sup_{x\in\Omega} \left| D^l u(x) \right| + \max_{|l|=2} \sup_{x,y\in\Omega} \frac{|D^l u(x) - D^l u(y)|}{|x-y|^\gamma}.$$

If X is a Banach space with its norm $\|\cdot\|_X$, we consider the norm in $\mathbb{R}\times X$ as $\|(\lambda,u)\|_{\mathbb{R}\times X} := \left(|\lambda|^2 + \|u\|_X^2\right)^{1/2}$. For the convergence of sequences $\{(\lambda_n, u_n)\} \subset$

$\mathbb{R} \times X$, we write $(\lambda_n, u_n) \to (\lambda_0, u_0)$ in $\mathbb{R} \times X$ as $n \to \infty$ for $(\lambda_0, u_0) \in \mathbb{R} \times X$. Even when $\lambda_0 = \infty$ or $u_0 = \infty$, we write as above and regard it as $\lambda_n \to \lambda_0$ and $u_n \to u_0$ in X respectively. For p as in introduction, we define the usual Sobolev constant,

$$S_{p+1} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{2/(p+1)}}.$$

We write $S := S_6$ when $N = 3$. Furthermore, note that we denote ϕ_* as the principle eigenfunction of $-\Delta$ on Ω which is normalized by $L^{p+1}(\Omega)$ norm, that is, $\|\phi_*\|_{p+1}^{p+1} = \int_\Omega \phi_*^{p+1} dx = 1$. Finally, we use a same character $C > 0$ to denote several positive values when there is no confusion.

2. The structure of the solution set

In this section, using Proposition 1.1 in Section 1, we show the structure of the solution set of (P) with $a(t) = 1 + \alpha t$ and $\alpha > 0$. This type of the nonlocal coefficient is considered in many works [10][21]-[28][30][34] and the typical example of the general settings in [6][7][14]-[16][20] and references therein. Here, we define

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R} \times C^{2,\gamma}(\overline{\Omega}) \ ; \ \text{satisfies (P) and } u \neq 0\}$$

and

$$\mathcal{T} := \left\{(\mu, v) \in \mathbb{R} \times C^{2,\gamma}(\overline{\Omega}) \ ; \ \text{satisfies (P}_0\text{) and } v \neq 0\right\}.$$

Furthermore, we often use the notation as,

$$\left[\|v\|^2 < (=, >) c\right] := \{(\mu, v) \in \mathbb{R} \times C^{2,\gamma}(\overline{\Omega}) \mid \|v\|^2 < (=, >, resp.) c\},$$

with some constants $c \in \mathbb{R}$. Notice that the following result is divided to three parts, i.e., the 3-superliner case; $p > 3$ [25][34], the asymptotically 3-linear case; $p = 3$ [30][22] and the 3-sublinear case; $1 < p < 3$ which has not been treated yet to our best knowledge.

Theorem 2.1 (The structure of the solution set). *Let $a(t) = 1 + \alpha t$ with $\alpha > 0$. Then the next assertions are true.*

(i) If $p > 3$, $\mathcal{S} \cup \{(\lambda_*, 0)\}$ is homeomorphic to $\mathcal{T} \cup \{(\lambda_*, 0)\}$, write,

$$\mathcal{S} \cup \{(\lambda_*, 0)\} \approx \mathcal{T} \cup \{(\lambda_*, 0)\}.$$

(ii) If $p = 3$, we have

$$\mathcal{S} \cup \{(\lambda_*, 0)\} \approx (\mathcal{T} \cup \{(\lambda_*, 0)\}) \cap \left[\|v\|^2 < \alpha^{-1}\right].$$

(iii) Suppose $1 < p < 3$, then we get

$$(\mathcal{S} \cup \{(\lambda_*, 0)\}) \cap [\|u\|^2 < c_p/\alpha] \approx (\mathcal{T} \cup \{(\lambda_*, 0)\}) \cap [\|v\|^2 < c_p'/\alpha],$$

$$\mathcal{S} \cap [\|u\|^2 > c_p/\alpha] \approx \mathcal{T} \cap [\|v\|^2 < c_p'/\alpha]$$

and

$$\mathcal{S} \cap [\|u\|^2 = c_p] \approx \mathcal{T} \cap [\|v\|^2 = c_p'/\alpha]$$

where

$$c_p = \frac{p-1}{3-p} \text{ and } c_p' = \left(\frac{p-1}{2}\right)\left(\frac{3-p}{2}\right)^{\frac{3-p}{p-1}}.$$

Remark 2.2. *In this theorem, we regard both $\mathcal{T} \cup \{(\lambda_*, 0)\}$ and $\mathcal{S} \cup \{(\lambda_*, 0)\}$ as subsets in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$.*

Remark 2.3. *Using this theorem, we construct unbounded continua of solutions of (P) in Section 3.*

Proof. Set $\mathcal{X} := \mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. We first formally define a map $F : \mathcal{X} \to \mathcal{X}$ so that

$$F(\mu, v) = \begin{cases} (\lambda_*, 0) \text{ if } (\mu, v) = (\lambda_*, 0), \\ \big(\mu a(t), a(t)^{1/(p-1)} v\big) \text{ if } (\mu, v) \in \mathcal{T} \end{cases}$$

where $t > 0$ is a solution of an equation for $\tau > 0$;

$$\tau = a(\tau)^{2/(p-1)} \beta \tag{1}$$

with $\beta = \|v\|^2$.

Proof for (i). Note that since $p > 3$, the equation (1) has the unique solution for each $\beta > 0$. Thus F is well defined on $\mathcal{T} \cup \{(\lambda_*, 0)\}$. In addition, Proposition 1.1 implies $F(\mathcal{T} \cup \{(\lambda_*, 0)\}) = \mathcal{S} \cup \{(\lambda_*, 0)\}$. Now we shall show that F is a homeomorphism onto $\mathcal{S} \cup \{(\lambda_*, 0)\}$. To this end, we first claim that F is continuous on $\mathcal{T} \cup \{(\lambda_*, 0)\}$. Fix $(\mu_0, v_0) \in \mathcal{T}$. For any $\varepsilon > 0$, take sufficiently small $\delta > 0$ which is determined later. For all $(\mu, v) \in \mathcal{T}$ with $\|(\mu_0, v_0) - (\mu, v)\|_{\mathcal{X}} < \delta$, we may assume $\|v_0 - v\|$ is small. This implies $|t_0 - t|$ is also small, where t_0, $t > 0$ are the unique solutions of (1) with $\beta = \|v_0\|^2, \|v\|^2$ respectively. Thus, we can select $\delta > 0$ sufficiently small so that

$$\|F(\mu_0, v_0) - F(\mu, v)\|_{\mathcal{X}} < \varepsilon.$$

Similarly we can conclude that F is continuous at $(\lambda_*, 0)$. This proves the claim. Next we show that F is one to one. To do this, assume

$$F(\mu_1, v_1) = F(\mu_2, v_2), \tag{2}$$

for some $(\mu_i, v_i) \in \mathcal{T}$ with $i = 1, 2$. Then we get

$$t_1 = a(t_1)^{\frac{2}{p-1}} \|v_1\|^2 = a(t_2)^{\frac{2}{p-1}} \|v_2\|^2 = t_2$$

where $t_1, t_2 > 0$ is the unique solutions of (1) with $\beta = \|v_1\|^2, \|v_2\|^2$ respectively. Put $t^* := t_1 = t_2$. Then (2) implies

$$\left(\mu_1 a(t^*), a(t^*)^{\frac{1}{p-1}} v_1\right) = \left(\mu_2 a(t^*), a(t^*)^{\frac{1}{p-1}} v_2\right).$$

Since $a(t) \geq 1 > 0$ for all $t \geq 0$, we conclude $(\mu_1, v_1) = (\mu_2, v_2)$. This confirms the claim. Lastly we show that the inverse F^{-1} of F is continuous. To do this, fix $(\lambda_0, u_0) \in \mathcal{S}$. For $\varepsilon > 0$, take $\delta > 0$ which is determined later. Consider any $(\lambda, u) \in \mathcal{S}$ with $\|(\lambda_0, u_0) - (\lambda, u)\|_{\mathcal{X}} < \delta$. Note $F^{-1}(\lambda_0, u_0) = (\lambda_0/a(t_0), a(t_0)^{1/(1-p)} u_0)$ and $F^{-1}(\lambda, u) = (\lambda/a(t), a(t)^{1/(1-p)} u))$ where $t_0 = \|u_0\|^2$ and $t = \|u\|^2$. We may assume $|\lambda_0 - \lambda|$, $|t_0 - t|$ and $\|u_0 - u\|_{2,\gamma}$ are small if we take δ small enough. Thus we can select a constant $\delta > 0$ so that

$$\|F^{-1}(\lambda_0, u_0) - F^{-1}(\lambda, u)\|_{\mathcal{X}} < \varepsilon.$$

Similarly we get the continuity of F^{-1} at $(\lambda_*, 0)$. This completes the proof.

Proof for (ii). Assume $p = 3$. Notice that for all $\alpha > 0$, there exists a solution of (1) if and only if $\beta < \alpha^{-1}$ and it's unique. Thus F is well-defined on $(\mathcal{T} \cup \{(\lambda_*, 0)\}) \cap \left[\|v\|^2 < \alpha^{-1}\right]$. Then similarly to the proof for (i), we conclude that F is homeomorphism onto $\mathcal{S} \cup \{(\lambda_*, 0)\}$. This finishes the proof for (ii).

Proof for (iii). Let $1 < p < 3$. Then, there exist just two solutions of (1) if $\beta < c_p' \alpha^{-1}$, the unique one if $\beta = c_p' \alpha^{-1}$ and no solution for all $\beta > c_p' \alpha^{-1}$. Noting this, we define three maps $F_i : \mathcal{X} \to \mathcal{X}$ for $i = 1, 2, 3$,

$$F_1(\mu, v) = \begin{cases} (\lambda_*, 0) \text{ if } (\mu, v) = (\lambda_*, 0), \\ (\mu a(t_{\min}), a(t_{\min})^{1/(p-1)} v) \text{ if } (\mu, v) \in \mathcal{T} \cap [\|v\|^2 < c_p' \alpha^{-1}], \end{cases}$$

$$F_2(\mu, v) = (\mu a(t_{\max}), a(t_{\max})^{1/(p-1)} v) \text{ for all } (\mu, v) \in \mathcal{T} \cap [\|v\|^2 < c_p' \alpha^{-1}]$$

and

$$F_3(\mu, v) = (\mu a(t_0), a(t_0)^{1/(p-1)} v) \text{ for all } (\mu, v) \in \mathcal{T} \cap [\|v\|^2 = c_p' \alpha^{-1}]$$

where $t_{\max} = \max\{t > 0 \mid t = a(t)^{2/(p-1)} \|v\|^2\}$, $t_{\min} = \min\{t > 0 \mid t = a(t)^{2/(p-1)} \|v\|^2\}$ and $t_0 > 0$ is the unique solution of (1) for $\beta = \|v\|^2$. Then, from the facts stated above, F_i are well-defined for $i = 1, 2, 3$. It is not difficult to conclude that F_i for $i = 1, 2, 3$ are homeomorphisms onto $F_1((\mathcal{T} \cup \{(\lambda_*, 0)\}) \cap [\|v\|^2 < c_p'/\alpha]) = (\mathcal{S} \cup \{(\lambda_*, 0)\}) \cap [\|u\|^2 < c_p/\alpha]$, $F_2(\mathcal{T} \cap [\|v\|^2 < c_p'/\alpha]) = \mathcal{S} \cap [\|u\|^2 > c_p/\alpha]$ and $F_3(\mathcal{T} \cap [\|v\|^2 = c_p'/\alpha]) = \mathcal{S} \cap [\|u\|^2 = c_p/\alpha]$ respectively. This proves the theorem. □

3. The bifurcation results

In this section, we construct unbounded continua of positive solutions (λ, u) of (P). To do this, we utilize Proposition 1.1, Theorem 2.1 and the unbounded continuum of solutions of (P_0) which meets the trivial solution $(\lambda_*, 0)$ [31]. Note that, we mainly consider the very simple case, when Ω is a ball. Then, the solution set $\mathcal{T} \cup \{(\lambda_*, 0)\}$ is a simple curve since the positive solutions admit the uniqueness and non-degeneracy assertions. See preliminaries below. This makes our arguments and results much clearer and would be enough to demonstrate how drastically the nonlocal coefficient affects the bifurcation phenomena on (P). Although it is possible to apply our arguments and get some global results on the general case, we leave it for interested readers for the simplicity of this paper. We limit ourselves only to remarking that for the case, we need to be much more careful about a priori bounds, the treatments of the continua and so on. Throughout this section, we always consider the positive solutions. To this aim, we define $\mathcal{S}_+ := \{(\lambda, u) \in \mathcal{S} \mid u > 0 \text{ in } \Omega\}$ and $\mathcal{T}_+ := \{(\mu, v) \in \mathcal{T} \mid v > 0 \text{ in } \Omega\}$. Since the relation between $\mathcal{S}_+$ and $\mathcal{T}_+$ is described as in Theorem 2.1, we rewrite them as $\mathcal{S}$ and $\mathcal{T}$ again for the simplicity.

3.1. Preliminaries

We first put some preliminaries for the main argument later. Recall that when Ω is a ball, every positive solution is radially symmetric [17], non-degenerate and unique (see [2][5][29][32] and references therein). The implicit function theorem yields the following.

Proposition 3.1. *Let $1 < p < \infty$ if $N = 1, 2$, $1 < p < 5$ if $N = 3$ and Ω be a ball. Then there exists a continuous map $f : (-\infty, \lambda_*] \to C^{2,\gamma}(\overline{\Omega})$ such that $\mathcal{T} \cup \{(\lambda_*, 0)\} = \{(\mu, f(\mu)) \mid \mu \in (-\infty, \lambda_*]\}$. In particular, $\mathcal{T} \cup \{(\lambda_*, 0)\}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$.*

In addition, we put the following a priori bound.

Lemma 3.2. *Let $1 < p < \infty$ if $N = 1, 2$, $1 < p < 5$ if $N = 3$ and v be a solution of (P_0). Then we have*

$$\|v\|^2 \begin{cases} > C_1 \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{2}{p-1}} & \text{if } 0 < \mu < \lambda_* \\ \geq C_1 \ \text{if } \mu = 0 \\ > C_1 \ \text{if } \mu < 0 \end{cases} \tag{3}$$

with a constant $C_1 = S_{p+1}^{\frac{p+1}{p-1}}$. In addition, suppose that v is the least energy solution of the Brezis-Nirenberg type [8] of (P_0), that is,

$$v = S(\mu)^{\frac{1}{p-1}} w_\mu$$

where

$$\begin{aligned} S(\mu) :&= \inf \left\{ \|w\|^2 - \mu \int_\Omega w^2 dx \;\middle|\; \int_\Omega |w|^{p+1} dx = 1 \right\} \\ &= \|w_\mu\|^2 - \mu \int_\Omega w_\mu^2 dx \end{aligned}$$

with $\int_\Omega w_\mu^{p+1}dx = 1$*. Then, we obtain*

$$\|v\|^2 \begin{cases} < C_2 \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{2}{p-1}} & \textit{if } 0 < \mu < \lambda_* \\ \leq C_3 \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{p+1}{p-1}} & \textit{if } \mu \leq 0 \end{cases} \tag{4}$$

with constants $C_2 = \|\phi_*\|^{4/(p-1)}(S_{p+1} + \lambda_*|\Omega|^{(p-1)/(p+1)})$ *and* $C_3 := \|\phi_*\|^{\frac{2(p+1)}{p-1}}$ *where* $|A|$ *is the* N*-dimensional Lebesgue measure of* $A \subset \mathbb{R}^N$ *and* $\phi_* > 0$ *in* Ω *is the principle eigenfunction of* $-\Delta$ *on* Ω *with* $\int_\Omega \phi_*^{p+1}dx = 1$*.*

Proof. Let v be a solution of (P_0). Then the Poincaré and Sobolev inequalities yield

$$\begin{aligned} 0 &= \|v\|^2 - \mu \int_\Omega v^2 dx - \int_\Omega v^{p+1} dx \\ &\begin{cases} > \left(1 - \frac{\mu}{\lambda_*}\right) \|v\|^2 - S_{p+1}^{-\frac{p+1}{2}} \|v\|^{p+1} \text{ if } 0 < \mu < \lambda_*, \\ \geq \|v\|^2 - S_{p+1}^{-\frac{p+1}{2}} \|v\|^{p+1} dx \text{ if } \mu \leq 0. \end{cases} \end{aligned}$$

In addition, if $\mu < 0$, the inequality is strict. Solving this with respect to $\|v\|^2$, we obtain the first assertion. Next assume $v := S(\mu)^{1/(p-1)} w_\mu$. Then from the definition, we get

$$\|v\|^2 < \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{2}{p-1}} \|\phi_*\|^{\frac{4}{p-1}} \|w_\mu\|^2.$$

Here, the definition and the Hölder inequality imply,

$$\begin{aligned} \|w_\mu\|^2 &= S(\mu) + \mu \int_\Omega w_\mu^2 dx \\ &\begin{cases} < S_{p+1} + \lambda_* |\Omega|^{\frac{p-1}{p+1}} \text{ if } 0 < \mu < \lambda_*, \\ \leq \|\phi_*\|^2 \left(1 - \frac{\mu}{\lambda_*}\right) \text{ if } \mu \leq 0. \end{cases} \end{aligned}$$

This completes the proof. □

After this, assume that $a(t) = 1 + \alpha t$ with $\alpha > 0$.

3.2. The 3*-superlinear and subcritical case*

First, we consider the 3-superlinear and subcritical case. For this case, we may put $\alpha = 1$ for the simplicity.

Theorem 3.3. *Assume that* $a(t) = 1 + t$*,* $3 < p < \infty$ *if* $N = 1, 2$ *and* $3 < p < 5$ *if* $N = 3$*. Then we have a priori bound,*

> *there exists a constant* $C > 0$ *such that* $\|u\| \geq C$ *for all* $(\lambda, u) \in \mathcal{S}$ *with* $\lambda \leq \lambda_*$ *where* C *is independent of* λ*.*

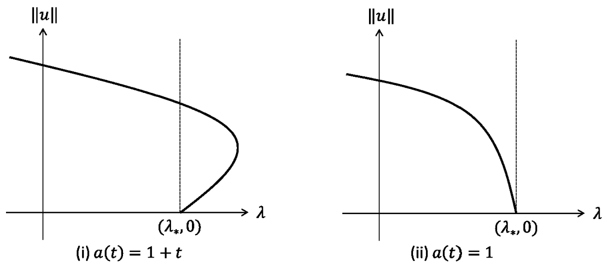

Figure 1: The global diagram for the case $p > 3$(subcritical)

Furthermore, assume that Ω is a ball. Then $\mathcal{S} \cup \{(\lambda_*, 0)\}$ is an unbounded simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$ which starts from $(\lambda_*, 0)$. More precisely, there exists a homeomorphism $\mathcal{F} : (0, 1] \to \mathcal{S} \cup \{(\lambda_*, 0)\}$, put $(\lambda(s), u(s)) := \mathcal{F}(s)$ for $s \in (0, 1]$, such that $\mathcal{S} \cup \{(\lambda_*, 0)\} = \mathcal{F}((0, 1])$, $(\lambda(1), u(1)) = (\lambda_*, 0)$ and $\lim_{s\to 0} \lambda(s) = -\infty$.

From Theorem 3.3, the diagram looks as Figure 1-(i) (compare with that for the semilinear case (ii)). Because of a priori bound in the theorem above, the branch must emanate to the right, which is actually the effect of the nonlocal coefficient. In view of the existence, we get the following.

Corollary 3.4. *Let a, p and Ω be as in Theorem 3.3. Then there exists a constant $\Lambda > \lambda_*$ such that the next assertions are true.*

(i) (P) has at least one solution if $\lambda \le \lambda_*$,
(ii) (P) has at least two solutions if $\lambda_* < \lambda < \Lambda$,
(iii) (P) poses at least one solution if $\lambda = \Lambda$,
(iv) (P) admits no solution if $\lambda > \Lambda$.

Recall that if $a(t) = 1$, there exists a positive solution if and only if $\lambda < \lambda_1$. Here we can observe that the nonlocal coefficient induces the existence of a solution for all $\lambda_* \le \lambda < \Lambda$ and further, multiple solutions for all $\lambda_* < \lambda < \Lambda$. We will put an interpretation of this result from the variational point of view in Section 4.

Proof of Theorem 3.3. We first prove a priori bound. If u is a solution of (P) with $\lambda \le \lambda_*$, the Poincaré and Sobolev inequalities imply

$$\begin{aligned} 0 &= \|u\|^2 + \|u\|^4 - \lambda \int_\Omega u^2 dx - \int_\Omega u^{p+1} dx \\ &\ge \left(1 - \frac{\lambda}{\lambda_*}\right) \|u\|^2 + \|u\|^4 - C\|u\|^{p+1} \\ &\ge \|u\|^4 - C\|u\|^{p+1} \end{aligned}$$

for some constant $C > 0$. Since $p > 3$, we get the desired bound. Next from Theorem 2.1, $\mathcal{S} \cup \{(\lambda_*, 0)\}$ is homeomorphic to $\mathcal{T} \cup \{(\lambda_*, 0)\}$. Thus noting Proposition 3.1, $\mathcal{S} \cup \{(\lambda_*, 0)\}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. Let F

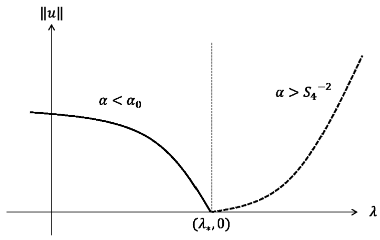

Figure 2: The global diagram for the case $p = 3$ (with small and large $\alpha > 0$)

be the homeomorphism defined in the proof of Theorem 2.1. Put $\mathcal{F}(\mu) := F(\mu, f(\mu))$ and $(\lambda(\mu), u(\mu)) := \mathcal{F}(\mu)$ for all $\mu \in (-\infty, \lambda_*]$. Then $\mathcal{S} \cup \{(\lambda_*, 0)\} = \mathcal{F}((-\infty, \lambda_*])$. Furthermore, from the definition, $(\lambda(\lambda_*), u(\lambda_*)) = (\lambda_*, 0)$. Since $\lambda(\mu) = \mu a(\|u(\mu)\|^2)$ and $a \geq 1$, $\lambda(\mu) \to -\infty$ as $\mu \to -\infty$. We trivially change the parameter and conclude the proof. $\square$

3.3. Asymptotically 3-linear case; $p = 3$

Next we deal with the asymptotically 3-linear case. The related existence result is given in [22] and [30]. We have the following.

Theorem 3.5. *Assume $a(t) = 1 + \alpha t$ with $\alpha > 0$, $p = 3$ and Ω is a ball. Then there exists an unbounded simple curve $\mathcal{C} \subset \mathcal{S} \cup \{(\lambda_*, 0)\}$ in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$ which contains $(\lambda_*, 0)$. More precisely, there exists a homeomorphism $\mathcal{F} : (0, 1] \to \mathcal{S} \cup \{(\lambda_*, 0)\}$ onto $\mathcal{F}((0, 1])(=: \mathcal{C})$, write $(\lambda(s), u(s)) := \mathcal{F}(s)$ for $s \in (0, 1]$, such that $(\lambda(1), u(1)) = (\lambda_*, 0)$ and satisfies the following.*

(i) There exists a constant $\alpha_0 < S_4^{-2}(= C_1^{-1})$ such that if $\alpha < \alpha_0$, $\lambda < \lambda_*$ for all $(\lambda, u) \in \mathcal{S}$ and $\lim_{s\to 0} \lambda(s) = -\infty$.

(ii) If $\alpha \geq S_4^{-2}$, $\lambda > \lambda_*$ for all $(\lambda, u) \in \mathcal{S}$ and further, if $\alpha > S_4^{-2}$, $(\lambda(s), u(s)) \to (\infty, \infty)$ in $\mathbb{R} \times H_0^1(\Omega)$ and also in $\mathbb{R} \times L^\infty(\Omega)$ as $s \to 0$.

Remark 3.6. *In view of the theorem above, the diagram looks as Figure 2. When $\alpha > S_4^{-2}$, the bifurcation diagram drastically different from the semilinear case. Obviously, it ensures the existence of one solution for all $\lambda > \lambda_*$. The related result on the general situation is Theorem 1.1 in [22].*

In Theorem 3.5, we avoid the detail for the intermediate case; $\alpha_0 < \alpha \leq S_4^{-2}$. For this case, we have a variant which ensures, surprisingly enough, the multiplicity of solutions. We put a remark on this phenomenon after the proof of Theorem 3.5.

Proof of Theorem 3.5. Let f be as in Proposition 3.1 and take a value $\mu_0 < \lambda_*$ such that $\|f(\mu_0)\|^2 = \alpha^{-1}$ and $\|f(\mu)\|^2 < \alpha^{-1}$ for all $\mu_0 < \mu \leq \lambda_*$. If $\|f(\mu_0)\|^2 < \alpha^{-1}$ for all $\mu < \lambda_*$, we regard $\mu_0 = -\infty$. Clearly, $\{(\mu, f(\mu)) \mid \mu \in (\mu_0, \lambda_*]\}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. Let F be the homeomorphism defined in the proof of Theorem 2.1. Put $\mathcal{F}(\mu) := F(\mu, f(\mu))$, write $(\lambda(\mu), u(\mu)) := \mathcal{F}(\mu)$ for $\mu \in (\mu_0, \lambda_*]$ and define $\mathcal{C} := \mathcal{F}((\mu_0, \lambda_*]) \subset \mathcal{S} \cup \{(\lambda_*, 0)\}$. Then from Theorem

2.1-(ii), $\mathcal{C}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$ and $(\lambda(\lambda_*), u(\lambda_*)) = (\lambda_*, 0)$. If $\mu_0 > -\infty$, $\|f(\mu)\|^2 \to \alpha^{-1}$ as $\mu \to \mu_0$. This implies $u(\mu) \to \infty$ in $H_0^1(\Omega)$ and also in $L^\infty(\Omega)$ as $\mu \to \mu_0$ by the definition. On the other hand, if $\mu_0 = -\infty$, $\lambda(\mu) \to -\infty$ as $\mu \to \mu_0$ again from the definition. Thus $\mathcal{C}$ is unbounded. Now we prove (i). To do this, assume $(\lambda, u) \in \mathcal{S}$ with $\lambda \geq \lambda_*$. Then from Proposition 1.1, there exists an element $(\mu, v) \in \mathcal{T}$ such that $\lambda = \mu a(\|u\|^2)$ and $\|u\|^2 = a(\|u\|^2)\|v\|^2$. Noting the uniqueness and (4) in Lemma 3.2, we get

$$\begin{aligned} \|u\|^2 &< C_2 \left(\alpha \|u\|^2 + 1 - \frac{\lambda}{\lambda_*} \right) \\ &\leq \alpha C_2 \|u\|^2 \end{aligned}$$

since $\lambda \geq \lambda_*$. Thus we have a contradiction, if $\alpha > 0$ is small enough. Furthermore, we get $\mu_0 < 0$ if $\alpha > 0$ is small enough again by (4). It follows that $\lambda(\mu) \to -\infty$ as $\mu \to \mu_0$ by the definition and the argument as above. This concludes (i) with a appropriate change of the parameter. To prove (ii), we assume $\alpha \geq S_4^{-2}$ and there exists $(\lambda, u) \in \mathcal{S}$ with $\lambda \leq \lambda_*$. Then by Proposition 1.1, we get $(\mu, v) \in \mathcal{T}$ similarly to the one above. It follows from (3) that,

$$\begin{aligned} \|u\|^2 &= a(\|u\|^2)\|v\|^2 \\ &\begin{cases} > C_1 \left(\alpha \|u\|^2 + 1 - \frac{\lambda}{\lambda_*} \right) \text{ if } 0 < \lambda \leq \lambda_*, \\ \geq C_1(1 + \alpha \|u\|^2) \text{ if } \lambda \leq 0. \end{cases} \end{aligned}$$

Noting $\lambda \leq \lambda_*$ we clearly get a contradiction. Furthermore, if $\alpha > S_4^{-2}$, (3) implies that $\mu_0 > 0$. Recalling the definition and the argument above, we conclude the proof. $\square$

Lastly, we put a remark on the intermediate case. To this aim, we give a priori bound which determines the direction of the bifurcation from the trivial solution $(\lambda_*, 0)$.

Proposition 3.7. *Let a, p, α and Ω be as in Theorem 3.5. Furthermore, assume $\alpha > \|\phi_*\|^{-4}$. Then it follows that,*

> *there exists a constant $C \geq 0$ such that $\|(\lambda, u) - (\lambda_*, 0)\|_{\mathbb{R} \times H_0^1(\Omega)} \geq C$ for all $(\lambda, u) \in \mathcal{S}$ with $\lambda \leq \lambda_1$.*

Remark 3.8. *Because of this bound, the branch from the trivial solution $(\lambda_*, 0)$ emerges to the right.*

Proof. If the assertion fails, we have a sequence $(\lambda_n, u_n) \subset \mathcal{S}$ such that $\lambda_n \leq \lambda_*$ and $(\lambda_n, u_n) \to (\lambda_*, 0)$ in $\mathbb{R} \times H_0^1(\Omega)$. Proposition 1.1 implies that there exits an element $(\mu_n, v_n) \in \mathcal{T}$ such that $(\lambda_n, u_n) = (\mu_n a(\|u_n\|^2), a(\|u_n\|^2)^{1/2} v_n)$. We can put $v_n = S(\mu_n)^{1/2} w_{\mu_n}$ as in Lemma 3.2 by the uniqueness. Then we have by the definition, the Poincaré inequality and our choice $\lambda_n \leq \lambda_*$,

$$\begin{aligned} \|u_n\|^2 &= a(\|u_n\|^2) S_\mu \|w_\mu\|^2 \\ &\geq \left(a\left(\|u_n\|^2\right) - \frac{\lambda_n}{\lambda_*} \right) \|w_n\|^4 \\ &\geq \alpha \|u_n\|^2 \|w_n\|^4. \end{aligned}$$

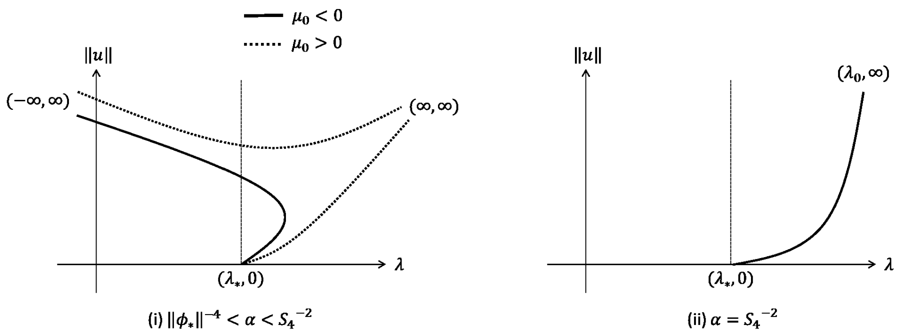

Figure 3: $p = 3$, the intermediate case $\|\phi_*\|^{-4} < \alpha \leq S_4^{-2}$

Here, from the assumption, we have a constant $\varepsilon > 0$ such that $\alpha = (\|\phi_*\|^4 - \varepsilon)^{-1}$. Take $0 < \varepsilon' < \varepsilon$. Note that since $\lambda_n \to \lambda_*$, $\|w_n\| \to \|\phi_*\|$ as $n \to \infty$. Therefore, we may assume $\|w_n\|^4 > \|\phi_*\|^4 - \varepsilon'$. Consequently, noting the inequality above, we get for large n,

$$\begin{aligned} 0 &\leq (1 - \alpha\|w_n\|^4)\|u_n\|^2 \\ &\leq \left(1 - \frac{\|\phi_*\|^4 - \varepsilon'}{\|\phi_*\|^4 - \varepsilon}\right)\|u_n\|^2 \\ &< 0, \end{aligned}$$

a contradiction. This concludes the proof. □

Now let us construct unbounded simple curves for this case. The construction is similar but we should be slightly more careful. First, if $\alpha < S_4^{-2}$, take μ_0 as in the proof of Theorem 3.5. In view of Lemma 3.2, we have two possibilities, either $\mu_0 < 0$ or $\mu_0 > 0$. If $\mu_0 < 0$, $F(\{(\mu, f(\mu)) \mid \mu \in (\mu_0, \lambda_*)\})$ is the desired unbounded simple curve and the diagram looks as the solid line in Figure 3-(i). The proofs of the bifurcation at $(\lambda_*, 0)$ and infinity are by Proposition 3.7 and the similar argument in the proof of Theorem 3.5. On the other hand, if $\mu_0 > 0$, we still have constants $\mu_2 < 0 < \mu_1 < \mu_0$ such that $\|f(\mu_i)\|^2 = \alpha^{-1}$ for $i = 1, 2$ and $\|f(\mu)\|^2 < \alpha^{-1}$ for all $\mu \in (\mu_2, \mu_1)$. Consequently, we can construct two disjoint unbounded simple curves $F(\{(\mu, f(\mu)) \mid \mu \in (\mu_0, \lambda_*]\})$ and $F(\{(\mu, f(\mu)) \mid \mu \in (\mu_1, \mu_2)\})$. Put $(\lambda(\mu), u(\mu)) := F(\mu, f(\mu))$ for μ in the regions above. Then, noting $\|u(\mu)\| \to \infty$ as $\mu \to \mu_i$ for $i = 0, 1, 2$ and $\lambda(\mu) = \mu a(\|u(\mu)\|^2)$, we have $\lambda(\mu) \to \infty$ (and $-\infty$) as $\mu \to \mu_0, \mu_1$ (and μ_2 respectively). Therefore, the diagram looks as the dotted line in Figure 3-(i). Notice that, for both cases, (P) admits at least two solutions for $\lambda > \lambda_*$ if λ is not too large. Lastly, if $\alpha = S_4^{-2}$, we have either $\mu_0 = 0$ or $\mu_0 > 0$. Then similarly, $F(\{(\mu, f(\mu)) \mid \mu \in (\mu_0, \lambda_*]\})$ is the desired simple curve. As above, put $(\lambda(\mu), u(\mu)) := F(\mu, f(\mu))$ for $\mu \in (\mu_0, \lambda_*]$ and then $\lambda(\mu) = \mu a(\|u(\mu)\|^2)$. Consequently, we get $\liminf_{\mu \to \mu_0} \lambda(\mu) =: \underline{\lambda}$ and $\limsup_{\mu \to \mu_0} \lambda(\mu) \to \overline{\lambda}$ where

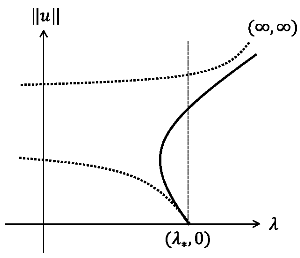

Figure 4: The global diagram for the case $1 < p < 3$

$\lambda_* \leq \underline{\lambda} \leq \overline{\lambda} \leq \infty$ by Theorem 3.5-(ii). In particular, if $\mu_0 > 0$, $\underline{\lambda} = \overline{\lambda} = \infty$ and if $\mu_0 = 0$, $\overline{\lambda}$ can be finite. Thus, one may expect the diagram as Figure 3-(ii). For this case, (P) poses at least one solution if $\lambda > \lambda_*$ is not too large.

3.4. The 3*-sublinear case;* $1 < p < 3$

In this subsection, we show the diagram for the 3-sublinear case. To our best knowledge, there is no previous work on (P) for this case. We can observe that the bifurcation diagram is different from all the previous cases and get the new multiplicity results. We remark that for the concave-convex problem with this nonlinearity, they prove the existence of three positive solutions [10].

Theorem 3.9. *Suppose $a(t) = 1 + \alpha t$ with $\alpha > 0$, $1 < p < 3$ and Ω is a ball. Then it follows that,*

> *there exists a constant $C > 0$ which is independent of λ such that $\|u\| \geq C$ for all $(\lambda, u) \in \mathcal{S}$ with $\lambda \geq \lambda_*$.*

Furthermore, if $\sup\{\|v\|^2 \mid (\mu, v) \in \mathcal{T}\} > c_p\alpha^{-1}$, there exists an unbounded simple curve $\mathcal{C} \subset \mathcal{S} \cup \{(\lambda_*, 0)\}$ which meets $(\lambda_*, 0)$. More precisely, there exists a homeomorphism $\mathcal{F} : (0, 1] \to \mathcal{S} \cup \{(\lambda_*, 0)\}$ onto $\mathcal{F}((0, 1])(=: \mathcal{C})$, write $(\lambda(s), u(s)) := \mathcal{F}(s)$ for $s \in (0, 1]$, such that $(\lambda(1), u(1)) = (\lambda_*, 0)$ and $(\lambda(s), u(s)) \to (\infty, \infty)$ in $\mathbb{R} \times H^1_0(\Omega)$ as $s \to 0$.

Remark 3.10. *From a priori bound in the theorem, the branch from the trivial solution $(\lambda_*, 0)$ emerges to the left. But, it finally bends back to the right and goes to (∞, ∞). The diagram can be described by the solid line in Figure 4.*

Remark 3.11. *Notice that we do not consider the diagram for the case* $\sup\{\|v\|^2 \mid (\mu, v) \in \mathcal{T}\} \leq c_p\alpha^{-1}$ *here. This may be the case if the solutions of (P_0) is uniformly bounded, i.e.,* $\sup\{\|v\|^2 \mid (\mu, v) \in \mathcal{T}\} < \infty$*. In this case, in particular, if the inequality is strict, there exist two disjoint continua like the dotted line in Figure 4. The proof is similar to that for Theorem 3.9 and previous arguments. For the simplicity, we omit the statement and its proof here.*

Remark 3.12. *As a consequence of the theorem and the remark above, we observe the multiplicity of solutions for every $\lambda < \lambda_*$ which is sufficiently closed to λ_*. In addition, we confirm the existence of a solution even for all $\lambda \geq \lambda_*$*

which is impossible in the semilinear case. For these cases, we actually observe a global minimum solution in addition to the usual mountain pass type solution by the variational argument. See Section 4.

Proof of Theorem 3.9. First, we assume $(\lambda, u) \in \mathcal{S}$ and $\lambda \geq \lambda_*$. Then noting the uniqueness, we have the least energy solution v of (P_0) with $\mu = \lambda/a(\|u\|^2)$ such that $\|u\|^2 = a(\|u\|^2)^{2/(p-1)}\|v\|^2$. Then (4) in Lemma 3.2 implies

$$\begin{aligned} 0 &= \|u\|^2 - a(\|u\|^2)^{\frac{2}{p-1}}\|v\|^2 \\ &> \|u\|^2 - C_2\left(\alpha\|u\|^2 - \frac{\lambda}{\lambda_*} + 1\right)^{\frac{2}{p-1}} \\ &\geq \|u\|^2 - \alpha^{\frac{2}{p-1}}C_2\|u\|^{\frac{4}{p-1}}. \end{aligned} \tag{5}$$

Since $1 < p < 3$, this concludes the first assertion. Next, similarly to the proof of Theorem 3.5, take $-\infty < \mu_0 < \lambda_*$ such that $\|f(\mu_0)\|^2 = c_p'/\alpha$ and $\|f(\mu)\|^2 < c_p'/\alpha$ for all $\mu_0 < \mu \leq \lambda_*$. Then define $\mu_1(s) = 2(\lambda_* - \mu_0)s + 2\mu_0 - \lambda_*$ for all $s \in (1/2, 1]$, $\mu_2(s) := -2(\lambda_* - \mu_0)s + \lambda_*$ for $s \in [0, 1/2)$ and the homeomorphism $\mathcal{F}$ defined on $(0, 1]$ onto $\mathcal{C} := \mathcal{F}((0, 1]) \subset \mathcal{S} \cup \{(\lambda_*, 0)\}$ so that

$$\mathcal{F}(s) := \begin{cases} F_1(\mu_1(s), f(\mu_1(s)) \text{ if } s \in (1/2, 1], \\ F_2(\mu_2(s), f(\mu_2(s)) \text{ if } s \in (0, 1/2), \\ F_3(\mu_0, f(\mu_0)) \text{ if } s = 1/2, \end{cases}$$

where F_i for $i = 1, 2, 3$ are as in the proof of Theorem 2.1-(iii). Then by the theorem, $\mathcal{C}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. Put $(\lambda(s), u(s)) := \mathcal{F}(s)$ for $s \in (0, 1]$. Then from the definition, clearly we have $(\lambda(1), u(1)) = (\lambda_*, 0)$. In addition, since $(\mu_2(s), f(\mu_2(s))) \to (\lambda_*, 0)$ in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$ as $s \to 0$, we get $(\lambda(s), u(s)) \to (\infty, \infty)$ in $\mathbb{R} \times H_0^1(\Omega)$ as $s \to 0$ again by the definition. This concludes the proof. $\square$

Remark 3.13. *We put a comment on the asymptotic behavior of $\mathcal{C}$ or $\mathcal{S} \cup \{(\lambda_*, 0)\}$. It is clear from (4) that $\mu_0 = -\infty$ for small $\alpha > 0$ or otherwise, $\mu_0 \to -\infty$ as $\alpha \to 0$. This implies*

$$\inf\{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{C}\} \begin{cases} = -\infty \textit{ for small } \alpha > 0, \textit{ or otherwise} \\ \leq \mu_0 a(c_p/\alpha) \to -\infty \textit{ as } \alpha \to 0. \end{cases}$$

On the other hand, it follows that

$$\lambda_* > \inf\{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{C}\} \geq \inf\{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{S}\} \to \lambda_* \textit{ as } \alpha \to \infty.$$

In fact, if $(\lambda, u) \in \mathcal{S}$ with $\lambda < \lambda_*$, $(\mu, v) := (\lambda/a(t), a(t)^{1/(1-p)}u) \in \mathcal{T}$ where $t = \|u\|^2$ as usual. Then we get from (3),

$$\begin{aligned} t &= a(t)^{\frac{2}{p-1}}\|v\|^2 \\ &\geq C_1\left(\alpha t + 1 - \frac{\lambda}{\lambda_*}\right)^{\frac{2}{p-1}}. \end{aligned}$$

Since $1 < p < 3$, for every λ, we get a contradiction if $\alpha > 0$ is large enough. This proves the claim. In addition, we claim that the lower part of $\mathcal{S} \cup \{(\lambda_*, 0)\}$, i.e. $\mathcal{S} \cap [\|u\|^2 < c_p/\alpha]$, tends to the solution set $\mathcal{T} \cup \{(\lambda_*, 0)\}$ of the semilinear elliptic problem as $\alpha \to 0$, in the following sense. First, by the observation above, for all $\lambda < \lambda_*$, there exists an element $(\lambda, u_\alpha) \in \mathcal{S} \cap [\|u\|^2 < c_p/\alpha]$ if $\alpha > 0$ is small enough. Then we can show that $(\lambda, u_\alpha) \to (\lambda, f(\lambda)) \in \mathcal{T}$ as $\alpha \to 0$ here f is as in Proposition 3.1. Actually, note that there exists an element $(\mu_\alpha, v_\alpha) = (\mu_\alpha, f(\mu_\alpha)) = (\lambda a(\|u_\alpha\|^2), a(\|u_\alpha\|^2)^{1/(1-p)} u_\alpha) \in \mathcal{T} \cap [\|v\|^2 < c'_p/\alpha]$ by Proposition 1.1. Since $\|u_\alpha\|^2 < c_p/\alpha$, $a(\|u_\alpha\|^2)$ is uniformly bounded from above with respect to α. Then so are μ_α and $\|v_\alpha\|^2 = \|f(\mu_\alpha)\|^2$. Consequently, so is $\|u_\alpha\|^2$. This implies $a(\|u_\alpha\|^2) \to 1$ as $\alpha \to 0$. Therefore $\mu_\alpha = \lambda a(\|u_\alpha\|^2) \to \lambda$ and then, $u_\alpha = a(\|u_\alpha\|^2)^{1/(1-p)} f(\mu_\alpha) \to f(\lambda)$ as $\alpha \to 0$. This proves the claim. On the other hand, for the upper part, it clearly follows that $\inf\{\|u\|^2 \mid (\lambda, u) \in \mathcal{S} \cap [\|u\|^2 > c_p/\alpha]\} \geq c_p/\alpha \to \infty$ as $\alpha \to 0$.

3.5. The critical case; $N = 3$ and $p = 5$

Finally, we consider the critical case. An existence result via the variational argument is in [25] (see Theorem 5.1). Define

$$\underline{\lambda} := \inf\{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{S}\}$$

and

$$\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{S}\}.$$

It is well known that if $\alpha = 0$, (P) has a solution u_λ if and only if $\lambda_*/4 < \lambda < \lambda_*$, in particular, $\underline{\lambda} = \lambda_*/4$ and $\overline{\lambda} = \lambda^*$ [8]. Furthermore, $u_\lambda \to \infty$ in $L^\infty(\Omega)$ as $\lambda \to \lambda_*/4$ [9]. Our interest is in what happens for the case $\alpha > 0$.

Theorem 3.14. *Let $a(t) = 1 + \alpha t$ with $\alpha > 0$, $N = 3$ and $p = 5$. Furthermore assume Ω is a ball. Then we have the following.*

(i) *For all $\alpha > 0$, $\underline{\lambda} > \lambda_*/4$. In addition, if $\alpha > 0$ is small enough, $\lambda_*/4 < \underline{\lambda} < \lambda_*$. On the other hand, if $\alpha > 0$ is large enough, $\lambda > \lambda_*$ for all $(\lambda, u) \in \mathcal{S}$ and $\underline{\lambda} = \lambda_*$.*

Furthermore, $\mathcal{S} \cup \{(\lambda_*, 0)\}$ is a simple curve in $\mathbb{R} \times C^{2,\alpha}(\overline{\Omega})$, more precisely, there exists a homeomorphism $\mathcal{G} : (0, 1] \to \mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$ such that $\mathcal{S} \cup \{(\lambda_*, 0)\} = \mathcal{G}((0, 1])$ and if we put $(\lambda(s), u(s)) := \mathcal{G}(s)$ for $s \in (0, 1]$, the following assertions are true.

(ii) $(\lambda(1), u(1)) = (\lambda_*, 0)$.

(iii) *$u(s) \to \infty$ in $L^\infty(\Omega)$ as $s \to 0$ and*

$$\frac{\lambda^*}{4} + C_-(\alpha) \leq \liminf_{s \to 0} \lambda(s), \ \limsup_{s \to 0} \lambda(s) \leq \frac{\lambda^*}{4} + C_+(\alpha)$$

where

$$C_\pm(\alpha) := \frac{\alpha\lambda_*}{8} \left(\frac{3\alpha C_{i_\pm}^2}{4} + \sqrt{\frac{9\alpha^2 C_{i_\pm}^4}{16} + 3C_{i_\pm}^2} \right),$$

with $i_- = 1$ and $i_+ = 2$.

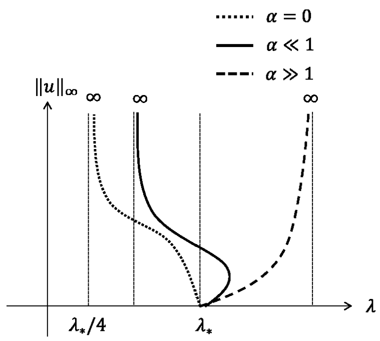

Figure 5: The global diagram for the critical case $p = 5$

Remark 3.15. *In this case, a priori bound in Theorem 3.3 still holds. That is,* $\|u\| \geq C$ *for all* $(\lambda, u) \in \mathcal{S}$ *with* $\lambda \leq \lambda_*$ *where* $C > 0$ *is independent of* $\lambda \leq \lambda_*$*. Because of this bound, the branch from* $(\lambda_*, 0)$ *must emanate to the right. Consequently, we get* $\overline{\lambda} > \lambda_*$ *for all* $\alpha > 0$*.*

Remark 3.16. *Notice that* $C_\pm(\alpha) \to 0$ *as* $\alpha \to 0$ *and* $C_\pm(\alpha) \to \infty$ *as* $\alpha \to \infty$*. In this point of view, we may say that the blow up point (or region) with respect to* λ *shifts from* $\lambda = \lambda_*/4$ *and goes up to infinity depending on* $\alpha > 0$*.*

Remark 3.17. *The diagram looks as Figure 5. But notice that, in view of the theorem above, there still remain possibilities of additional multiplicities around the blow up region* $(\lambda_*/4 + C_-(\alpha), \lambda_*/4 + C_+(\alpha))$*.*

Proof of Theorem 3.14. First, note the following. Similarly to Proposition 3.1, from the non-degeneracy and the uniqueness of the solutions [32], we get a continuous map $g : (\lambda_*/4, \lambda_*] \to C^{2,\gamma}(\overline{\Omega})$ such that

$$\mathcal{T} \cup \{(\lambda_*, 0)\} = \{(\mu, g(\mu)) \in \mathbb{R} \times C^{2,\gamma}(\overline{\Omega}) \mid \mu \in (\lambda_*/4, \lambda_*]\}.$$

In particular, $\mathcal{T} \cup \{(\lambda_*, 0)\}$ is a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. In addition, from Theorem 2.1-(i), $\mathcal{S} \cup \{(\lambda_1, 0)\}$ is also a simple curve in $\mathbb{R} \times C^{2,\gamma}(\overline{\Omega})$. Now we confirm (i). We first claim that for all $\alpha > 0$, $\underline{\lambda} > \lambda_*/4$. For this, consider a set for every $\lambda \in \mathbb{R}$,

$$C_\lambda := \left\{ \left(t, a(t)^{1/2} \|g(\lambda/a(t))\|^2\right) \;\middle|\; \lambda/a(t) \in (\lambda_*/4, \lambda_*] \right\}.$$

Notice that C_λ is the graph of the continuous function $\beta_\lambda(t) := a(t)^{1/2}\|g(\lambda/a(t))\|^2$ defined on the interval,

$$\left[\frac{1}{\alpha} \left(\frac{\lambda}{\lambda_*} - 1 \right), \frac{4}{\alpha} \left(\frac{\lambda}{\lambda_*} - \frac{1}{4} \right) \right). \tag{6}$$

In view of Proposition 1.1, we only have to show that $C_\lambda \cap \{(t,t) \in \mathbb{R}^2 \mid t > 0\} = \emptyset$ if $\lambda > \lambda_*/4$ is sufficiently closed to $\lambda_*/4$. Actually, set $\lambda_*/4 < \lambda < \lambda_*$

which is determined later so as to $|\lambda_*/4 - \lambda|$ is sufficiently small. Note that, similarly to Lemma 3.2, we get

$$C_1 \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{1}{2}} < \|g(\mu)\|^2 < C_2 \left(1 - \frac{\mu}{\lambda_*}\right)^{\frac{1}{2}}, \tag{7}$$

for all $\mu \in (\lambda_*/4, \lambda_*)$. Then, we obtain

$$t - a(t)^{\frac{1}{2}} \|g(\lambda/a(t))\|^2 < t - C_1 \left(\alpha t + 1 - \frac{\lambda}{\lambda_*}\right)^{\frac{1}{2}}.$$

Notice that the right hand side is strictly less than 0 for all

$$0 < t < \frac{1}{2} \left(\alpha S^3 + \sqrt{\alpha^2 S^6 + 4S^3 \left(1 - \frac{\lambda}{\lambda_*}\right)}\right).$$

Moreover, for all $\alpha > 0$, we have

$$\frac{4}{\alpha} \left(\frac{\lambda}{\lambda_*} - \frac{1}{4}\right) < \frac{1}{2} \left(\alpha S^3 + \sqrt{\alpha^2 S^6 + 4S^3 \left(1 - \frac{\lambda}{\lambda_*}\right)}\right)$$

if $|\lambda - \lambda_*/4|$ is sufficiently small. Therefore, (6) shows the conclusion. Furthermore, if $\alpha > 0$ is large enough, we get the same inequality above for all $\lambda \leq \lambda_*$. This concludes that $\lambda > \lambda_*$ for all $(\lambda, u) \in \mathcal{S}$ and $\underline{\lambda} = \lambda_*$ if $\alpha > 0$ is sufficiently large. It will be proved by (iii) that $\underline{\lambda} < \lambda_*$ for sufficiently small $\alpha > 0$. Thus we proceed to the proof of (ii) and (iii). To show them, we take the homeomorphism F as in the proof of Theorem 2.1. As usual, define a homeomorphism $\mathcal{G}(\mu) := F(\mu, g(\mu))$ and put $(\lambda(\mu), u(\mu)) := \mathcal{G}(\mu)$ for $\mu \in (\lambda_*/4, \lambda_*]$. Note that $(\lambda(\mu), u(\mu)) = (\mu a(t(\mu)), a(t(\mu))^{1/(p-1)} g(\mu))$ where $t(\mu) > 0$ is the unique solution of (1) with $\beta = \|g(\mu)\|^2$ for $\mu \in (\lambda_*/4, \lambda_*)$ and $(\lambda(\lambda_*), u(\lambda_*)) = (\lambda_*, 0)$. Since $g(\mu) \to \infty$ in $L^\infty(\Omega)$ as $\mu \to \lambda_*/4$ (see Theorem 3 in [9]) and $a \geq 1$, we have $u(\mu) \to \infty$ in $L^\infty(\Omega)$ as $\mu \to \lambda_*/4$. Finally, (7) implies

$$\frac{\sqrt{3} C_1}{2} \leq \liminf_{\mu \to \frac{\lambda_*}{4}} \|g(\mu)\|^2 \text{ and } \limsup_{\mu \to \frac{\lambda_*}{4}} \|g(\mu)\|^2 \leq \frac{\sqrt{3} C_2}{2}.$$

In addition, as $\|u(\mu)\|^2 = a(\|u(\mu)\|^2)^{1/2} \|g(\mu)\|^2$, we get

$$\|u(\mu)\|^2 = \frac{1}{2} \left\{\alpha \|g(\mu)\|^4 + \sqrt{\alpha^2 \|g(\mu)\|^8 + 4\|g(\mu)\|^4}\right\}.$$

Consequently, we have

$$\frac{\lambda_*}{4} + C_-(\alpha) \leq \liminf_{\mu \to \lambda_*/4} \left(\mu a(\|u(\mu)\|^2)\right) = \liminf_{\mu \to \lambda_*/4} \lambda(\mu)$$

and

$$\limsup_{\mu \to \lambda_*/4} \lambda(\mu) = \limsup_{\mu \to \lambda_*/4} \left(\mu a(\|u(\mu)\|^2)\right) \leq \frac{\lambda_*}{4} + C_+(\alpha),$$

where $C_\pm(\alpha) > 0$ is defined as in the statement of the theorem. We can trivially perform a change of parameter and conclude the proof. $\square$

4. Remarks on the multiplicity

In this last section, we put some remarks on the multiplicity of solutions induced by the nonlocal coefficient. As we see in the previous sections, the nonlocal coefficient can induce the multiplicity of positive solutions even if Ω is a ball where the uniqueness assertion holds for the semilinear problem. This is the important and typical effect of the nonlocal coefficient as observed in the homogeneous case in [12], [13] and references therein. Actually, it can induce up to uncountably many solutions also for our nonlinear problem (P).

4.1. Construction of infinitely many solutions

There would be several ways to construct infinitely many solutions using Proposition 1.1. Here, as in the previous section, we first consider the simple case, Ω is a ball and give the following.

Proposition 4.1. *Let Ω be a ball. In addition, we assume $1 < p < \infty$ if $N = 1, 2$ and $1 < p < 5$ if $N = 3$. Then there exists a continuous function $a : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that (P) has infinitely many solutions for all $\lambda \leq \lambda_1$.*

Proof. In view of Proposition 1.1, for every $\lambda \leq \lambda_*$, it is enough if we get the infinitely many solutions for the equation for $t > 0$,

$$t = a(t)^{\frac{2}{p-1}} \left\| f\left(\lambda/a(t)\right)\right\|^2 \tag{8}$$

where f is chosen from Proposition 3.1. We put the right hand side as $\beta_\lambda(t)$ for $t > 0$. Take $a(t) = 1 + (t|\sin t|/C_1)^{(p-1)/2}$ for instance. Then for $s_k := k\pi$, it follows that

$$s_k - \beta_\lambda(s_k) > 0$$

if $k \in \mathbb{N}$ is sufficiently large. For this $k \in \mathbb{N}$, put $t_k := (k + 1/2)\pi$. Then we get from (3),

$$t_k - \beta_\lambda(t_k) < 0.$$

Since β_λ is continuous, the intermediate value theorem implies that there exists a solution $s_k < \tau_k < t_k$ of (8). As this holds for any sufficiently large $k \in \mathbb{N}$, we conclude the proof. $\square$

Remark 4.2. *With this choice of the nonlocal coefficient, for every $\lambda \leq \lambda_*$, we have a sequence of positive solutions (u_k) such that $u_k \to \infty$ in $H_0^1(\Omega)$ as $k \to \infty$.*

Lastly we remark that if $\lambda = 0$, (P) with an appropriate nonlocal coefficient may have a continuum of solutions. Set $1 < p < \infty$ if $N = 1, 2$ and $1 < p < 5$ if $N = 3$. For example, take,

$$a(t) := \begin{cases} 1 \text{ if } 0 \leq t \leq S_{p+1}^{\frac{p+1}{p-1}}, \\ \left(S_{p+1}^{-\frac{p+1}{p-1}} t\right)^{\frac{p-1}{2}} \quad \text{if } t > S_{p+1}^{\frac{p+1}{p-1}}. \end{cases} \tag{9}$$

Then, choose $(0, v_0) \in \mathcal{T}$ with $v_0 = S_{p+1}^{1/(p-1)} w_0$ where $\|w_0\|^2 = S_{p+1}$ with $\int_\Omega w_0^{p+1} dx = 1$. Clearly, $\|v_0\|^2 = S_{p+1}^{\frac{p+1}{p-1}}$. Then, there exists a continuum of solutions $t \geq S_{p+1}^{(p+1)/(p-1)}$ of the equation for $\tau > 0$,

$$\tau = a(\tau)^{\frac{2}{p-1}} \|v_0\|^2.$$

This completes the claim. The nonlocal coefficient can induce an arbitrary number of solutions.

4.2. Comments from the variational point of view

Finally, we give some remarks from the variational point of view. This will give us natural interpretations for the results in previous sections. Note that since the proof is standard, we only put the observations here. Let Ω be a bounded domain with smooth boundary. We define the C^1 functional on $H_0^1(\Omega)$,

$$I(u) := \frac{1}{2} A\left(\|u\|^2\right) - \frac{\lambda}{2} \int_\Omega u^2 dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} dx$$

where $A(t) := \int_0^t a(s) ds$. As is well-known, every critical point of I corresponds to a solution of (P) [7].

4.2.1. The 3*-superlinear and subcritical case*

As in Subsection 3.2, assume $a(t) = 1 + t$ and $p > 3$ is subcritical. The existence result is already given by Corollary 3.4 for a ball. For the general bounded domain, using the variational argument, we get the following.

(i) (P) has at least one solution for all $\lambda \leq \lambda_*$.
(ii) There exists a constant $\Lambda_0 > \lambda_*$ such that (P) admits at least two solutions for all $\lambda_* < \lambda < \Lambda_0$.

(i) is proved by using the Poincaré inequality and the mountain pass theorem. Note that it is valid even for $\lambda = \lambda_*$ thanks to the nonlocal effect. For the proof, see [34]. Here, let us see the new case, $\lambda > \lambda_*$. In order to understand the energy structure, for every $u \in H_0^1(\Omega)$, we define the fibering map $f_u(t) := I(tu)$ for $t > 0$. Then we get the next lemma.

Lemma 4.3. *There exists a constant $\Lambda_0 > \lambda_*$ such that for all $\lambda_* < \lambda < \Lambda_0$, the next assertions are true.*

(i) For all $u \in H_0^1(\Omega) \setminus \{0\}$, there exists a unique maximum point $t_{max} > 0$ of f_u such that $f_u''(t_{max}) < 0$.
(ii) If $\|u\|^2 - \lambda \int_\Omega u^2 dx < 0$, there exists a unique local minimum point $0 < t_{min} < t_{max}$ of f_u such that $f_u''(t_{min}) > 0$.

The proof is easy. Actually, (ii) is the case when we choose $u := \phi_*$. This structure is induced by the nonlocal term and allows us to obtain two solutions.

In fact, for example, the proof is given by the method of the Nehari manifold [35]. Define

$$\mathcal{N} := \{u \in H_0^1(\Omega) \mid f_u'(1) = 0\}$$

and split it into three parts,

$$\begin{aligned} \mathcal{N}^+ &:= \{u \in \mathcal{N} \mid f_u''(1) > 0\}, \\ \mathcal{N}^0 &:= \{u \in \mathcal{N} \mid f_u''(1) = 0\}, \\ \mathcal{N}^- &:= \{u \in \mathcal{N} \mid f_u''(1) < 0\}. \end{aligned}$$

Similarly to the original argument in [35], we get the critical points $u^\pm$ of I which attains the minimums on $\mathcal{N}^\pm$ respectively. We can also refer to the proof for the critical case in [25]. Note that u^+ attains the local minimum of I on $H_0^1(\Omega)$ and $I(u^+) < 0$.

Remark 4.4. *The local minimum solutions bifurcate from the trivial solution at $\lambda = \lambda_*$ and the branch emanates to the right. In fact, from the argument above, we have a solution $u_\lambda \in \mathcal{N}^+$ with $I(u_\lambda) < 0$ for all $\lambda \in (\lambda_*, \Lambda_0)$. Then we get by the Poincaré inequality,*

$$\begin{aligned} 0 &> I(u_\lambda) - \frac{1}{p+1}\langle I'(u_\lambda), u_\lambda\rangle \\ &= -\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\lambda}{\lambda_*} - 1\right)\|u_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\|u_\lambda\|^4. \end{aligned}$$

Since $p > 3$, it follows that

$$\|u_\lambda\|^2 < C\left(\frac{\lambda}{\lambda_*} - 1\right),$$

where $C > 0$ is some constant which is independent of λ. Thus we have $u_\lambda \to 0$ in $H_0^1(\Omega)$ as $\lambda \to \lambda_*$. This concludes the claim.

These observations support the validity of Figure 1-(i). It seems that the lower branch corresponds to the local minimum solution and the upper branch does the mountain pass type solution.

4.2.2. The asymptotically 3-linear case

Let a as above and $p = 3$. We mainly put a remark on the existence results related to Theorem 3.5. If $\alpha < S_4^{-2}$, we have a mountain pass solution for all $\lambda < \lambda_1$. For the proof, readers can refer to the proof of Theorem 1.2 in [22] or also that of Theorem 1.2 in [26]. (Notice that the boundedness of PS sequences are obtained as usual for our nonlinearity.) On the other hand, if $\alpha > S_4^{-2}$, we easily get that I is coercive i.e., $I(u) \to \infty$ as $\|u\| \to \infty$, by using the Poincaré and Sobolev inequality. Furthermore, $I(t\phi_*) < 0$ for sufficiently small $t > 0$ if $\lambda > \lambda_*$. This leads us to get a global minimum critical point of I for all $\lambda > \lambda_1$. The multiplicity result on the case $\|\phi_*\|^{-4} < \alpha \le S_4^{-2}$ (see the last paragraph in Subsection 3.3) is more delicate. We leave it for the paper in preparation [27].

4.2.3. The 3-sublinear case

Assume $1 < p < 3$. For this case (recall Remark 3.12), we have the following.

Lemma 4.5. *Let $a(t) = 1 + \alpha t$ with $\alpha > 0$ and $1 < p < 3$. Then I is coercive, i.e. $I(u) \to \infty$ as $\|u\| \to \infty$. Furthermore, there exits a constant $\lambda_0 < \lambda_*$ such that the next assertions are true.*

(i) If $\lambda < \lambda_*$, there exist constants $d, \rho > 0$ such that $I(u) \geq d$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. In addition, if $\lambda_0 < \lambda < \lambda_*$, there exists a function $u_0 \in H_0^1(\Omega)$ such that $\|u_0\| > \rho$ and $I(u_0) < 0$.

(ii) If $\lambda \geq \lambda_*$, $\inf_{u \in H_0^1(\Omega)} I(u) < 0$.

Proof. Noting the Poincaré inequality, the first assertion in (i) clearly follows. In particular, note that $0 \in H_0^1(\Omega)$ is the local minimum of I if $\lambda < \lambda_*$. For the rest of the assertions, it is easy to show

$$\inf_{t>0} I(t\phi_*) < 0$$

for all $\lambda > \lambda_0$ if we take $\lambda_0 < \lambda_*$ so that $|\lambda_* - \lambda_0|$ is sufficiently small. This completes the proof. $\square$

Consequently, we get both a mountain pass type solution with the positive energy and a global minimum solution with the negative energy for all $\lambda_0 < \lambda < \lambda_*$ and a global minimum one for all $\lambda \geq \lambda_*$. This is consistent with the result in Subsection 3.4. For the related challenging problem, see the critical problem in high dimension [28].

4.2.4. On the infinitely many solutions

Lastly, we put comments on the infinitely many solutions obtained in the former part of this section. First, we consider the case $a(t) = 1+(|t \sin t|/C_1)^{(p-1)/2}$ where $C_1 := S_{p+1}^{(p+1)/(p-1)}$. Put $f_u(t) := I(tu)$. Take any nontrivial function u and set $\lambda \leq \lambda_*$. Then the direct calculation shows

$$f_u'(t) := \left\{ 1 + \left(\frac{|\|tu\|^2 \sin \|tu\|^2|}{C_1} \right)^{\frac{p-1}{2}} \right\} t\|u\|^2 - t\lambda \int_\Omega u^2 dx - t^p \int_\Omega |u|^{p+1} dx.$$

Thus if $t = \sqrt{k\pi}/\|u\| =: \tau_k$ where $k \in \mathbb{N}$, we have

$$f_u'(\tau_k) = \tau_k \left(\|u\|^2 - \lambda \int_\Omega u^2 dx \right) - \tau_k^p \int_\Omega |u|^{p+1} dx.$$

This implies $f_u'(\tau_k) < 0$ for sufficiently large $k \in \mathbb{N}$. Fix this $k \in \mathbb{N}$. Then, the Poincaré and Sobolev inequalities imply

$$f_u'(t) > \left(\left(|\|tu\|^2 \sin \|tu\|^2|\right)^{(p-1)/2} t\|u\|^2 - t^p \|u\|^{p+1} \right) S_{p+1}^{-\frac{p+1}{2}}.$$

Thus if $t = \sqrt{(k+1/2)\pi}/\|u\| =: \tau_k'$, we get

$$f'(\tau_k') > 0.$$

The intermediate value theorem suggests that there exists a constant $\tau_k < \tau_k^0 < \tau_k'$ such that $f'(\tau_k^0) = 0$. This confirms that f_u has infinitely many critical points. Next, choose $\lambda = 0$ and set a as in (9). Now let v_0 be as in Subsection 4.1. Then, for all $t \geq S_{p+1}^{(p+1)(p-1)}$, we get

$$f'_{v_0}(t) = \left(S_{p+1}^{-\frac{p+1}{p-1}} t^2 \|v_0\|^2 \right)^{\frac{p-1}{2}} t\|v_0\|^2 - t^p \int_\Omega v_0^{p+1} dx = 0.$$

In this situation, the fibering map f_{v_0} has a continuum of critical points.

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