SPIN TORIC MANIFOLDS ASSOCIATED TO FINITE GRAPHS

MIHO HATANAKA

Abstract. We describe a necessary and sufficient condition for a toric manifold to admit a spin structure. In particular, a toric manifold admits a spin structure if and only if its real part is orientable. It is known that a Delzant polytope can be constructed from a building set or pseudograph, so that one can associate a toric manifold to a building set or pseudograph. In this paper, we characterize building sets and pseudographs whose associated toric manifolds admit spin structures.

1. Introduction

A toric variety is a normal algebraic variety of complex dimension n with an effective algebraic $(\mathbb{C}^*)^n$-action having an open dense orbit. The family of toric varieties one-to-one corresponds to that of fans which are objects in combinatorics. In this paper, we deal with nonsingular toric varieties called toric manifolds.

As is well-known, a smooth manifold M admits a spin structure if and only if its first Stiefel-Whitney class $w_1(M)$ and second Stiefel-Whitney class $w_2(M)$ vanish. Using this condition, we can describe a necessary and sufficient condition for a toric manifold M to admit a spin structure in terms of the corresponding fan (Proposition 2.1). It turns out that this is equivalent to its real toric manifold $M_{\mathbb{R}}$ being orientable ([9]).

There are constructions of toric manifolds from two kinds of finite graphs (i.e. from simple graphs ([3]) and from pseudographs ([2])), and the construction from simple graphs is generalized to that from building sets. Using Proposition 2.1, we can characterize finite simple graphs and building sets whose associated toric manifolds admit spin structures (Theorem 3.8, Corollary 3.9). Moreover, this characterization of simple graphs is generalized to pseudographs which may have multiedges and loops (Theorem 4.13).

This paper is organized as follows. In section 2, we describe the necessary and sufficient condition for a toric manifold to admit a spin structure and for a real toric manifold to be orientable. In section 3, we review the construction of toric manifolds from finite simple graphs and building sets on $[n+1] := \{1, \ldots, n+1\}$, and characterize finite simple graphs and building sets whose associated toric manifolds admit spin structures by applying the necessary and sufficient condition above. In section 4, we similarly review the construction of toric manifolds from finite pseudographs, and characterize finite pseudographs whose associated toric manifolds admit spin structures.

2. Spin toric manifolds and the orientability of real toric manifolds

In this section, we give a necessary and sufficient condition for a projective toric manifold to admit a spin structure and for a real toric manifold to be orientable. Let P be a Delzant polytope of dimension n in $\mathbb{R}^n$ with m facets, λ be a function mapping each facet of P to its facet vector (i.e. a normal primitive vector to the facet), and λ' be the modulo 2 reduction

Date: July 28, 2015.

of λ. A toric manifold constructed from P is written by $M(P)$, and its real part (i.e. its real toric manifold) is written by $M_{\mathbb{R}}(P)$.

Proposition 2.1. *The followings are equivalent.*

(1) *The toric manifold $M(P)$ admits a spin structure.*
(2) *The real toric manifold $M_{\mathbb{R}}(P)$ is orientable.*
(3) *There is a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0,1\}$ such that $\epsilon(\lambda'(\mathbf{F})) = \{1\}$, where $\mathbf{F}$ is the set of facets of the Delzant polytope P.*

Proof. We prove the equivalence between (1) and (3). We can prove the equivalence between (2) and (3) similarly, so we omit the proof. The equivalence between (2) and (3) was proved by [9], however the following proof is different from their proof.

A manifold M admits a spin structure if and only if its first Stiefel-Whitney class $w_1(M)$ and second Stiefel-Whitney class $w_2(M)$ vanish. Since the cohomology group $H^1(M(P))$ of the projective toric manifold $M(P)$ is trivial, its first Stiefel-Whitney class $w_1(M(P))$ vanishes. So, it is enough to prove the equivalence between (3) and the vanishing of $w_2(M(P))$.

Let T^n be a compact torus $(S^1)^n$, $M = M(P)$, and $\pi : ET^n \times_{T^n} M \to BT^n$ be the Borel construction of M. Since the Serre spectral sequence of π degenerates at the E_2-level, we have the following exact sequence.

$$0 \longrightarrow H^2(BT^n;\mathbb{Z}_2) \xrightarrow{\pi^*} H^2_{T^n}(M;\mathbb{Z}_2) \xrightarrow{\rho^*} H^2(M;\mathbb{Z}_2) \longrightarrow 0, \tag{2.1}$$

where ρ^* is the surjection induced from an inclusion of the fiber $\rho : M \to ET^n \times_{T^n} M$.

Let $F_1, \dots, F_m$ be the facets of P and $\tau_1, \dots, \tau_m$ be elements in $H^2_{T^n}(M;\mathbb{Z}_2)$ which are Poincaré dual to the characteristic submanifolds of M corresponding to $F_1, \dots, F_m$. Then, $\pi^*(u)$ is written as a linear combination of $\tau_1, \dots, \tau_m$ as follows (see [8] for example):

$$\pi^*(u) = \sum_{i=1}^m v_i(u)\tau_i.$$

Here, v_i can be regarded as an element in $\mathrm{Hom}(H^2(BT^n);\mathbb{Z}_2) = H_2(BT^n;\mathbb{Z}_2)$. So, $\pi^*(u)$ is written as follows:

$$\pi^*(u) = \sum_{i=1}^m \langle u, v_i\rangle\tau_i,$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology. Let λ' be a homomorphism $\mathbf{F}$ to $H_2(BT^n;\mathbb{Z}_2)$ which maps F_i to v_i. Then,

$$\pi^*(u) = \langle u, \lambda'(F_1)\rangle\tau_1 + \cdots + \langle u, \lambda'(F_m)\rangle\tau_m.$$

It is known that the equivariant Stiefel-Whitney class $w^{T^n}(M)$ is of the form

$$w^{T^n}(M) = \prod_{i=1}^m (1 + \tau_i),$$

so we have $w_2^{T^n}(M) = \sum_{i=1}^m \tau_i$ ([4]). The second Stiefel-Whitney class $w_2(M)$ is the image of $w_2^{T^n}(M)$ by ρ^* in (2.1). Since (2.1) is an exact sequence, the equation $w_2(M) = 0$ is equivalent to the existence of an element u in $H^2(BT^n;\mathbb{Z}_2)$ such that $\pi^*(u) = w_2^{T^n}(M)$. So we have

$$\sum_{i=1}^m \langle u, \lambda'(F_i)\rangle\tau_i = \sum_{i=1}^m \tau_i.$$

Therefore, $w_2(M)$ vanishes if and only if $\langle u, \lambda'(F_i)\rangle$ is 1 for each $i = 1, \ldots, m$, which implies the equivalence between (1) and (3). □

Remark 2.2. The same proof as above shows that Proposition 2.1 holds for a toric manifold whose realization of the underlying simplicial complex of the corresponding fan is a disk ([1, 7]), for a quasitoric manifold ([4]) and for a topological toric manifold ([6]).

A truncation of a Delzant polytope P along a face corresponds to blowing-up along the submanifold of $M(P)$ corresponding to the face. To be precise, let F be a codimension k face which is an intersection of k facets $F_1, \ldots, F_k$ of a Delzant polytope P, and $\lambda(F_i)$ be the facet vector of the facet F_i for each i. A face truncation at F is to cut P along the face F in such a way that the facet vector of the new facet is $\lambda(F_1) + \cdots + \lambda(F_k)$ (Figure 1). The projective toric manifold corresponding to the truncated Delzant polytope is formed by blowing-up $M(P)$ along the submanifold corresponding to the face F.

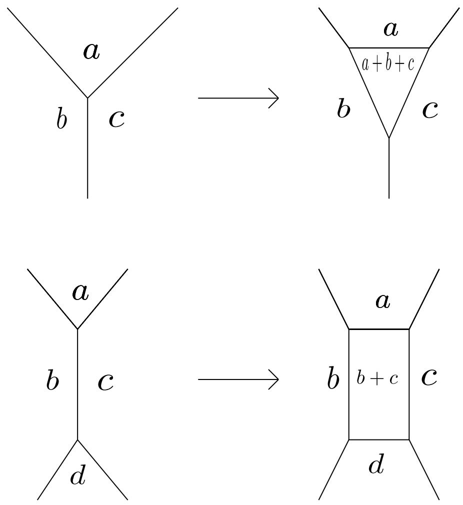


FIGURE 1. face truncations and new facet vectors corresponding to blowing-up

3. Spin toric manifolds associated to finite simple graphs and building sets

We set $[n+1] := \{1, \ldots, n+1\}$. In this section, we assume that a graph G is simple, and review the construction of a toric manifold $M(G)$ (resp. $M(B)$) from a finite simple graph G (resp. a building set B on $[n+1]$), and characterize a graph G (resp. a building set B) whose associated toric manifold $M(G)$ (resp. $M(B)$) admits a spin structure. There are two kinds of constructions of a Delzant polytope from G (resp. B). One is to realize a Delzant polytope in $\mathbb{R}^{n+1}$ by Minkowski sum, and the other is to truncate faces of a simplex in $\mathbb{R}^n$. In this section, we use the second construction.

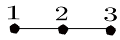
Let G be a simple graph with $n+1$ nodes, and its node set $V(G)$ be $[n+1]$. We set

$$B(G) := \{I \subset V(G) \mid G|I \text{ is connected}\},$$

where $G|I$ is a maximal subgraph of G with the node set I (i.e. the induced subgraph). The empty set $\emptyset$ is not in $B(G)$. We call $B(G)$ a *graphical building set* of G. A graphical building set $B(G)$ is a building set on $V(G)$, so we review the construction of a toric manifold from a building set.

Definition 3.1. A *building set* B on $[n+1]$ is a collection of nonempty subsets of $[n+1]$ such that
(1) B contains all singletons $\{i\}$ for every i,
(2) if $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.
If $[n+1] \in B$, then B is called a *connected* building set.

Example 3.2. We consider the following path graph P_3.
Then,

P_3


$$B(P_3) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\},$$
which we simply express as follows:
$$B(P_3) = \{1, 2, 3, 12, 23, 123\}.$$

When a building set B is connected, we construct a Delzant polytope P_B in $\mathbb{R}^n$ as follows ([3]). We take an n-simplex in $\mathbb{R}^n$ such that its facet vectors are $e_1, \ldots, e_n$, and $-e_1 - \cdots - e_n$, where $e_1, \ldots, e_n$ are the standard basis of $\mathbb{R}^n$. Each facet vector e_i $(1 \leq i \leq n)$ corresponds to an element i in B, and the facet vector $-e_1 - \cdots - e_n$ corresponds to an element $n+1$ in B, where i in B means the singleton $\{i\}$ in B. We truncate the n-simplex along faces in increasing order of dimension. Let F_i denote the facet corresponding to an element i in B. For every element $I = i_1 \ldots i_k$ in $B \setminus [n+1]$ we truncate the simplex along a face $F_{i_1} \cap \cdots \cap F_{i_k}$ in such a way that the facet vector of the new facet, denoted F_I, is the sum of the facet vectors of the facets $F_{i_1}, \ldots, F_{i_k}$. Then the resulting polytope, denoted P_B, is a Delzant polytope, and called a *nestohedron*. The set $B \setminus [n+1]$ one-to-one corresponds to the set of facets of P_B. Let $M(B)(M_{\mathbb{R}}(B))$ denote a (real) toric manifold corresponding to P_B. A nestohedron constructed from a graphical building set $B(G)$ is called a *graph associahedron*, and the associated (real) toric manifold is denoted by $M(G)(M_{\mathbb{R}}(G))$. When a building set B is disconnected, the corresponding nestohedron is defined as the product of nestohedrons associated to connected building sets in B. The corresponding (real) toric manifold is also defined as the product of (real) toric manifolds associated to connected building sets in B.

Remark 3.3. The size of an n-simplex is not important because the size does not affect the topology of the associated toric manifolds. The important data are a simple polytope and its facet vectors.

Example 3.4.
(1) When a graph G is a point, the associated (real) toric manifold is also a point. We understand that a point is orientable and admits a spin structure.
(2) When G is a connected graph with 2 nodes, the corresponding graph associahedron P_G in $\mathbb{R}$ is an 1-simplex (Figure 2), and the associated (real) toric manifold is diffeomorphic to $\mathbb{C}P^1$ $(\mathbb{R}P^1)$. $\mathbb{C}P^1$ admits a spin structure and $\mathbb{R}P^1$ is orientable.
(3) When G is a connected graph with 3 nodes, G is a path graph P_3 or cycle graph C_3. If G is the path graph P_3, then its graphical building set $B(P_3)$ is $\{1, 2, 3, 12, 23, 123\}$, and the corresponding graph associahedron P_{P_3} is a pentagon (Figure 2). So, the associated toric manifold is diffeomorphic to $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ and does not admit a spin structure. If G is the cycle graph C_3, then its graphical building set $B(C_3)$ is $\{1, 2, 3, 12, 23, 31, 123\}$, and the

corresponding graph associahedron P_{C_3} in $\mathbb{R}^2$ is a hexagon (Figure 2). So, the associated toric manifold is diffeomorphic to $\mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}$ and also does not admit a spin structure.

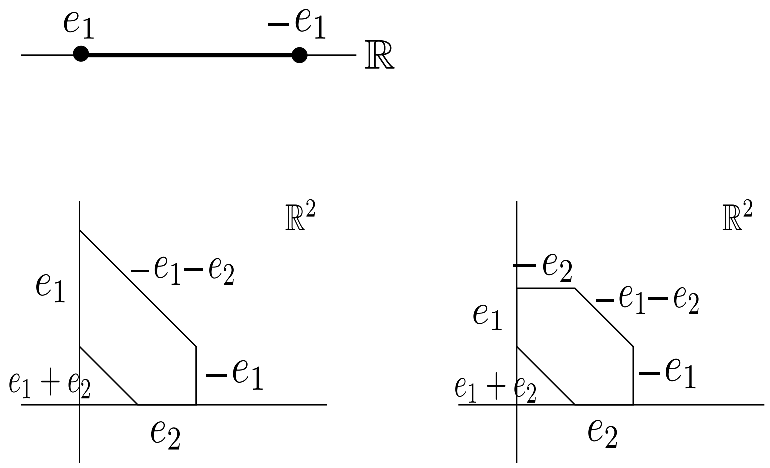


FIGURE 2. graph associahedrons and facet vectors in (2) and (3)

Example 3.5.
(1) A building set on [1] is only $\{1\}$, so the corresponding nestohedron $P_{\{1\}}$ is a point, and the associated (real) toric manifold is a point.
(2) Building sets on [2] are $\{1,2\}$ and $\{1,2,12\}$. If B is $\{1,2\}$, then its nestohedron P_B is a point, so the associated (real) toric manifold is a point. If B is $\{1,2,12\}$, then its nestohedron P_B is an 1-simplex, so the associated (real) toric manifold is diffeomorphic to $\mathbb{C}P^1$ ($\mathbb{R}P^1$).
(3) Building sets on [3] are essentially the following.

$$\{1,2,3\},\ \{1,2,3,12\},\ \{1,2,3,12,23,123\},\ \{1,2,3,12,23,31,123\},$$

$$\{1,2,3,123\},\ \{1,2,3,12,123\}.$$

Each nestohedron P_B is a point, 1-simplex, pentagon, hexagon, 2-simplex, and square. The last two are not constructed from any graph, and the corresponding Delzant polytopes are as in Figure 3. The toric manifolds $M(B)$ (resp, real toric manifolds $M_{\mathbb{R}}(B)$) associated to the building sets are respectively diffeomorphic to a point, $\mathbb{C}P^1, \mathbb{C}P^2\sharp 2\overline{\mathbb{C}P^2}, \mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}, \mathbb{C}P^2$, and $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ (resp, a point, $\mathbb{R}P^1, 3\mathbb{R}P^2, 4\mathbb{R}P^2, \mathbb{R}P^2$, and $2\mathbb{R}P^2$).

Lemma 3.6. *Let B be a connected building set on $[n+1]$. Then the following are equivalent.*
(1) *The toric manifold $M(B)$ admits a spin structure.*
(2) *The real toric manifold $M_{\mathbb{R}}(B)$ is orientable.*
(3) *$n+1$ is even and any element in $B\backslash\{[n+1]\}$ has odd order.*

Proof. Let $\mathbf{F}$ be the set of facets of the nestohedron P_B, λ be a function mapping each facet of P_B to its facet vector, and λ' be the modulo 2 reduction of λ. By Proposition 2.1, it is enough to show the equivalence between (3) and the existence of a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0,1\}$ satisfying $\epsilon(\lambda'(\mathbf{F})) = \{1\}$.

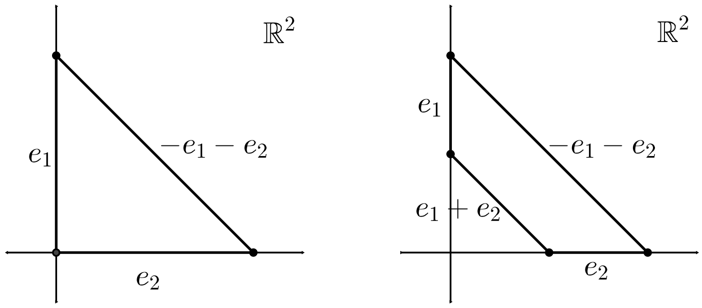


FIGURE 3. nestohedrons corresponding to {1,2,3,123} and {1,2,3,12,123}

The nestohedron P_B has $e_1, \dots, e_n$, and $e_1 + \cdots + e_n$ as facet vectors modulo 2, where the facets associated to these facet vectors correspond to the singletons in B, that is, $\lambda'(F_1) = e_1, \dots, \lambda'(F_n) = e_n, \lambda'(F_{n+1}) = e_1 + \cdots + e_n$. Suppose that there is a homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0, 1\}$ such that $\epsilon(\lambda'(\mathbf{F})) = \{1\}$. Then n is odd. We assume that there is an element I with an even order in $B \backslash \{[n+1]\}$, and let F_I be the facet of P_B corresponding to I. Then, since $\epsilon(\lambda'(F_1)) = \cdots = \epsilon(\lambda'(F_{n+1})) = 1$, we have $\epsilon(\lambda'(F_I)) = 0$. This is a contradiction.

If (3) holds, then we can take the homomorphism ϵ from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2 = \{0, 1\}$ mapping each e_i to 1. □

Lemma 3.7. *Suppose that a smooth manifold M is diffeomorphic to the product of smooth manifolds $M_1, \dots, M_k$. Then the followings hold.*
(1) *M is orientable if and only if each factor M_i is orientable.*
(2) *M admits a spin structure if and only if each factor M_i admits a spin structure.*

Proof. We use the following formula. Let ξ, η be vector bundles over base spaces B_1, B_2. Then the l-th Stiefel-Whitney class of the product bundle $\xi \times \eta$ over $B_1 \times B_2$ is

$$w_l(\xi \times \eta) = \sum_{i=0}^{l} w_i(\xi) \times w_{l-i}(\eta). \tag{3.1}$$

In particular
$$w_1(M) = w_1(M_1) + \cdots + w_1(M_k).$$
Therefore, $w_1(M) = 0$ if and only if $w_1(M_1) = \cdots = w_1(M_k) = 0$ since there is no relation among $w_1(M_1), \dots, w_1(M_k)$. This means (1).

If M admits a spin structure, then $w_1(M_1) = \cdots = w_1(M_k) = 0$ because of the orientability of each M_i. So, it follows from (3.1) that
$$w_2(M) = w_2(M_1) + \cdots + w_2(M_k).$$
Therefore $w_2(M) = 0$ if and only if since there is no relation among $w_2(M_1), \dots, w_2(M_k)$, $w_2(M_1) = \cdots = w_2(M_k) = 0$. This means (2). □

The following theorem follows from Lemmas 3.6 and 3.7.

Theorem 3.8. *Let B be an union of connected building sets $B_1, \dots, B_k$ on subsets $S_1, \dots, S_k$ in $[n+1]$. Then the following are equivalent.*
(1) *The toric manifold $M(B)$ admits a spin structure.*
(2) *The real toric manifold $M_{\mathbb{R}}(B)$ is orientable.*
(3) *Each building set B_i satisfies either of the following.*
(I) $|S_i| = 1$.
(II) $|S_i|$ *is even and any element in $B_i \backslash \{S_i\}$ has an odd order.*

Corollary 3.9. *Let G be a finite simple graph.*
(1) *The toric manifold $M(G)$ admits a spin structure if and only if $M(G)$ is diffeomorphic to $(\mathbb{C}P^1)^k$.*
(2) *The real toric manifold $M_{\mathbb{R}}(G)$ is orientable if and only if $M_{\mathbb{R}}(G)$ is diffeomorphic to $(\mathbb{R}P^1)^k$.*
Moreover, the corresponding graph is the disjoint union of k connected graphs with 2 nodes and finitely many points.

Proof. We assume that a graph G has k connected component $G_1, \dots, G_k$. Then we can take the graphical building set of G as B in Theorem 3.8, the graphical building set of G_i as B_i, and the node set of G_i as S_i. (3)(I) in Theorem 3.8 means that G_i is a point, and (3)(II) means that G_i is a connected graph with 2 nodes. In fact, G_i has even nodes because $|S_i|$ is even, and if G_i has more than or equal to 4 nodes, then G_i has a connected proper subgraph with 2 nodes, which gives an even order element in $B_i \setminus \{S_i\}$. □

Remark 3.10. A compact toric manifold M has trivial 1-st cohomology group ([5]), so that M admits only one spin structure if M admits a spin structure.

4. Spin toric manifolds associated to finite pseudographs

In this section, we construct a toric manifold $M(G)$ from a pseudograph G (i.e. a graph may have multiedges and loops) ([2]), and characterize a pseudograph G whose associated toric manifold $M(G)$ admits a spin structure.

Definition 4.1. Let G be a finite pseudograph.
(1) A *tube* G_t of G is a proper connected subgraph of G such that if a pair of nodes of G_t is connected by an edge of G, then G_t contains at least one edge connecting the pair.
(2) Two tubes are *compatible,* if one is included in the other, or they are disjoint and cannot be connected by an edge of G.
(3) A *tubing* of G is the set of pairwise compatible tubes and the union of such tubes is not G.

Example 4.2. (a) and (b) in Figure 4 are tubings. However, (c) in Figure 4 is not a tubing because two tubes are not compatible. (d) in Figure 4 is also not a tubing because the union of the tubes is the whole graph.

Definition 4.3. Let G be a pseudograph.
(1) Suppose that a pair of nodes is connected by at least two edges. Then the set of all edges connecting the pair of nodes is called a *bundle.*
(2) The *underlying simple graph* G_s of G is the graph obtained by deleting all loops and replacing each bundle to an edge.

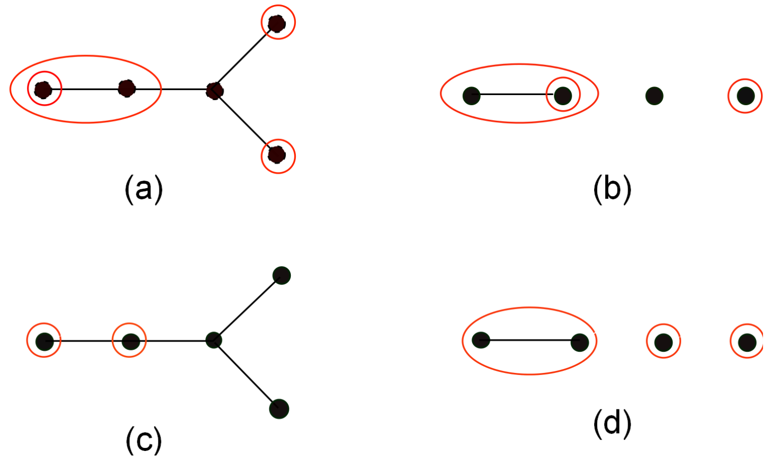


FIGURE 4. tubings and non-tubings

Example 4.4. The underlying simple graph of the left pseudograph in Figure 5 is the right simple graph. Here, B_1 and B_2 are bundles.

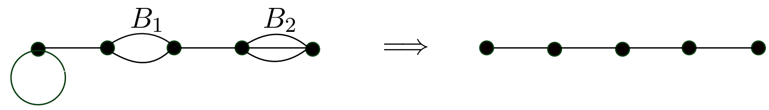


FIGURE 5. underlying simple graph

For each tube G_t of a pseudograph G, we define a set S as follows.
(1) All nodes of G_t are in S.
(2) All edges of G_t except for edges not contained in bundles and all loops of G_t are in S.
(3) All edges in bundles of G not containing edges of G_t are in S.
(4) All loops not incident to any node of G_t are in S.
We call S a *label* of G_t.

Definition 4.5. A tube G_t is called *full,* if it is a subgraph that consists of some of the nodes of the original graph and all of the edges that connect them in the original graph (i.e. an induced subgraph of G).

Example 4.6. Figure 6 shows examples of full tubes of a graph and their associated labeling. Here, $3abcd$ means the set $\{3, a, b, c, d\}$.

Let G be a pseudograph with $n+1$ nodes and l loops, $B_1, \ldots, B_k$ be bundles of G with $b_1+1, \ldots, b_k+1$ edges, Δ^s be an s-simplex, and ρ be a ray. We define

$$\Sigma_G := \Delta^n \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l,$$

and label every face in Σ_G as follows.
(1) Each facet of Δ^n corresponds to a node of G. Each face of Δ^n corresponds to a proper subset of the node set of G and is the intersection of the facets associated to nodes in that subset.

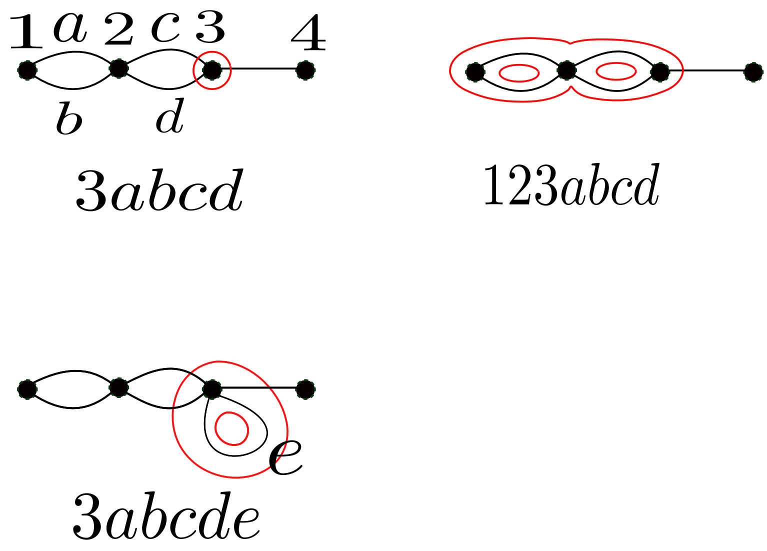


FIGURE 6. full tubes and corresponding labels

(2) Each vertex of Δ^{b_i} corresponds to an edge of the bundle B_i. Each face of Δ^{b_i} corresponds to a subset of an edge set of B_i defined by the vertices spanning the face.
(3) Each ρ corresponds to a loop of G.
Each face of Σ_G is labeled by the product of each factor naturally.

Remark 4.7. Let G_t be a tube of G. Suppose that the label of G_t contains k nodes of G and does not contain l edges in bundles and m loops. Then the face of Σ_G corresponding to G_t is of codimension $k + l + m$ by the way of labeling faces of Σ_G.

Facets of Σ_G are

$$(\text{facets of } \Delta^n) \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l,$$

$$\Delta^n \times (\text{facets of } \Delta^{b_j}) \times \prod_{i=1, i\neq j}^{k} \Delta^{b_i} \times \rho^l \quad (j = 1, \dots, k), \text{ and}$$

$$\Delta^n \times \prod_{i=1}^{k} \Delta^{b_i} \times (\text{facets of } \rho^l).$$

The number of facets in each line above is $n+1, \sum_{j=1}^{k}(b_j+1)$, and l respectively. We embed Σ_G in an Euclidean space such that a facet vector of each facet is respectively

$$
\begin{aligned}
&e_1, \dots, e_n, -e_1 - \cdots - e_n,\\
&e_{n+1}, \dots, e_{n+b_1}, -e_{n+1} - \cdots - e_{n+b_1},\\
&e_{n+b_1+1}, \dots, e_{n+b_1+b_2}, -e_{n+b_1+1} - \cdots - e_{n+b_1+b_2},\\
&\qquad\vdots\\
&e_{n+b_1+\cdots+b_{k-1}+1}, \dots, e_{n+b_1+\cdots+b_k}, -e_{n+b_1+\cdots+b_{k-1}+1} - \cdots - e_{n+b_1+\cdots+b_k},\\
&e_{n+b_1+\cdots+b_k+1}, \dots, e_{n+b_1+\cdots+b_k+l}.
\end{aligned}
$$

Here, $\{e_i\}_i$ is the standard basis in the Euclidean space of the dimension of Σ_G.

Example 4.8. We consider the pseudograph G drawn below. We embed Σ_G in $\mathbb{R}^3$ in such a way that each facet vector is

$$1ab \to e_1, \quad 2ab \to e_2, \quad 3ab \to -e_1 - e_2, \quad 123a \to e_3, \quad 123b \to -e_3.$$

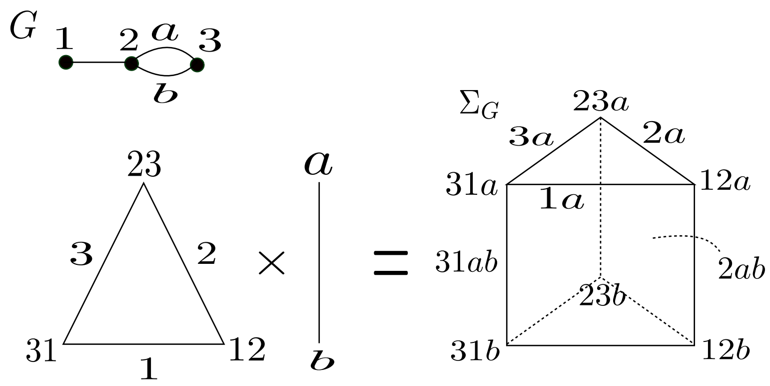


FIGURE 7. Σ_G and labels of faces

Then, we construct a pseudograph associahedron KG by truncating Σ_G along some faces. At first, one truncates Σ_G along faces with labels corresponding to full tubes as follows. If a face F of Σ_G with a label corresponding to a full tube is denoted by $F_1 \cap \cdots \cap F_k$, where each F_i is a facet of Σ_G, then truncate Σ_G along the face F in such a way that the facet vector of the new facet is the sum of the facet vectors of $F_1, \dots, F_k$. We repeat this truncation from low dimensional faces to high dimensional faces. The label corresponding to a full tube is (nodes of this full tube)(every edge in bundles and every loop in G). Therefore, if we truncate Σ_G along all faces with labels corresponding to full tubes, then Σ_G turns to

$$P_{G_s} \times \prod_{i=1}^{k} \Delta^{b_i} \times \rho^l, \tag{4.1}$$

where P_{G_s} is the graph associahedron corresponding to the underlying simple graph G_s of G. Next, one truncates (4.1) along faces with labels corresponding to non-full tubes in the same way as full tubes.

Proposition 4.9. ([2]) *Let G be a pseudograph, and KG be the pseudograph associahedron constructed from G. If G does not have any loop, then KG is a Delzant polytope and if G has a loop, then KG is a simple polyhedral cone. Its face poset is isomorphic to the set of tubings of G, ordered under the reverse subset containment. In particular, there is a one-to-one correspondence between facets of KG and tubes of G.*

We denote the (real) toric manifold corresponding to KG by $M(G)(M_{\mathbb{R}}(G))$.

Example 4.10. We shall observe the pseudograph associahedron KG for the pseudograph G in Example 4.8. Figure 8 indicates all tubes of G and the corresponding labels. The first line indicates full tubes, and the second line indicates non-full tubes. Truncating Σ_G along

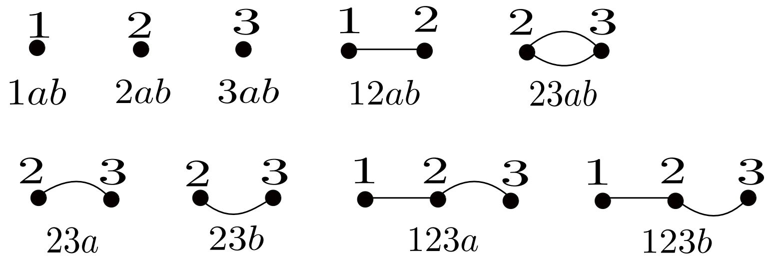


FIGURE 8. tubes and corresponding labels

faces with labels corresponding to the full tubes, Σ_G turns into the left in Figure 9. This is the product of 1-simplex and the graph associahedron constructed from the underlying simple graph of G. Moreover, truncating the left in Figure 9 along faces with labels corresponding to non-full tubes, the left turns into the right in Figure 9. This is the pseudograph associahedron KG associated to G. Each facet vector is as follows:

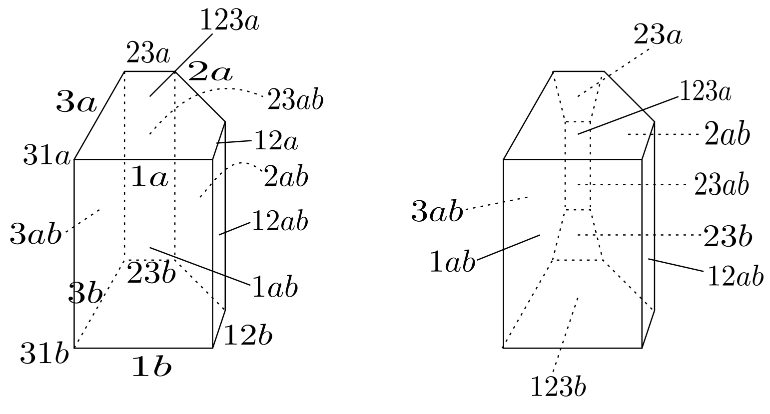


FIGURE 9. pseudograph associahedron

$1ab \to e_1,$	$2ab \to e_2,$	$3ab \to -e_1 - e_2,$	$12ab \to e_1 + e_2,$	$23ab \to -e_1$
$23a \to -e_1 + e_3,$	$23b \to -e_1 - e_3,$	$123a \to e_3,$	$123b \to -e_3$	

Example 4.11. When G is the disjoint union of $n+1$ nodes, the pseudograph associahedron KG is as follows. The polytope Σ_G is an n-simplex, and the nodes of G correspond to the $n+1$ facets of the n-simplex. Every tube of G is 1 node and full. Suppose that the tube G_i of G is the node i of G, then the label of G_i is i. So, KG is an n-simplex since KG is a polytope obtained by truncating the n-simplex along $n+1$ facets. Therefore, the associated toric manifold $M(G)$ is diffeomorphic to $\mathbb{C}P^n$.

Remark 4.12. The graph associahedron P_G of G above is a point. If a simple graph G is not connected, then the associated pseudograph associahedron KG is different from the graph associahedron P_G.

Theorem 4.13. *Let G be a finite pseudograph.*
(1) *The toric manifold $M(G)$ admits a spin structure if and only if $M(G)$ is diffeomorphic to one of $\mathbb{C}P^{k-1}(k:1$ or even$), \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1$, and $\mathbb{C}$.*
(2) *The real toric manifold $M_{\mathbb{R}}(G)$ is orientable if and only if $M_{\mathbb{R}}(G)$ is diffeomorphic to one of $\mathbb{R}P^{k-1}(k:1$ or even$), \mathbb{R}P^1, \mathbb{R}P^1 \times \mathbb{R}P^1$, and $\mathbb{R}$.*
Moreover, the associated pseudograph is respectively the disjoint union of k nodes, a connected simple graph with 2 nodes, a connected pseudograph with 2 nodes and 2 multiedges, and 1 node with 1 loop.

Remark 4.14. If G is a pseudograph with loops, then the realization of the underlying simplicial complex which is dual to the boundary complex of KG is a disk. Because truncating Σ_G along faces preserves the homeomorphic type of a realization of the underlying simplicial complex. So, by Remark 2.2, Proposition 2.1 can be applied even if G has loops.

Proof. If $M(G)$ is diffeomorphic to one of $\mathbb{C}P^{k-1}(k:1 \text{ or even}), \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1$, and $\mathbb{C}$, then $M(G)$ admits a spin structure.

The toric manifold $M(G)$ does not admit any spin structure unless the following two conditions are satisfied:

(4.2) The cardinality of the node set $V(G)$ is 1 or even.

(4.3) The number of multiedges in any bundle is even.

Because if the cardinality $n+1$ of the node set $V(G)$ is more than one, then KG has facet vectors $e_1, \ldots, e_n, -e_1 - \cdots - e_n$, so (3) in Proposition 2.1 implies (4.2) if $M(G)$ admits a spin structure. A similar argument implies (4.3). If Σ_G is truncated along a codimension 2 face, then (3) in Proposition 2.1 is not satisfied. Therefore, it is enough to consider G which satisfies (4.2) and (4.3) and whose associated pseudograph associahedron KG is constructed without truncating Σ_G along any codimension 2 faces.

Suppose that G contains a proper full tube shown in Figure 10. The label of this full tube is ij(all edges in all bundles and all loops), so this tube corresponds to a codimension 2 face of Σ_G by Remark 4.7. Therefore, G does not contain the proper full tube in Figure 10 if $M(G)$ admits a spin structure.

(1) Assume that G is a connected pseudograph in Figure 10 with the node set $\{1,2\}$ and has more than or equal to 2 loops (Figure 11). Labels of two full tubes are $1a_1 \ldots a_k l_1 \ldots l_{s_1} l'_1 \ldots l'_{s_2}$ and $2a_1 \ldots a_k l_1 \ldots l_{s_1} l'_1 \ldots l'_{s_2}$, and corresponding faces of Σ_G are two facets. Since truncating Σ_G along facets does not change Σ_G, a non-full tube obtained by removing 2 loops from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

(2) Assume that G is a pseudograph with the node set $\{1,2\}$, edges $a_1, \ldots, a_k$ (k is 1 or even) and a loop l incident to the node 1 (Figure 12). Labels of two full tubes of G are $1a_1 \ldots a_k l$

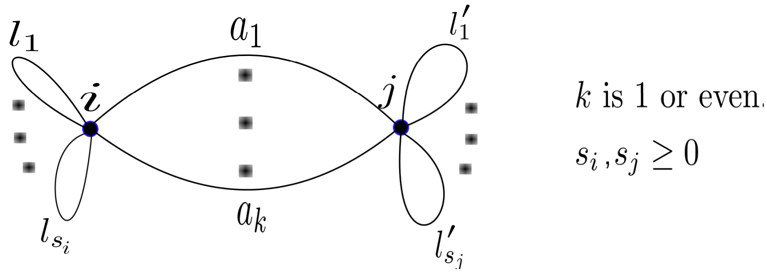


Figure 10. proper full tube

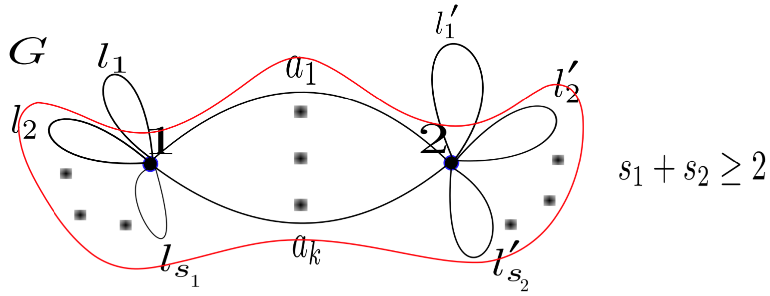


Figure 11. pseudograph and non-full tube in (1)

and $2a_1 \dots a_k l$, and corresponding faces of Σ_G are two facets. Similarly to (1), a non-full tube which is the node 1 corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

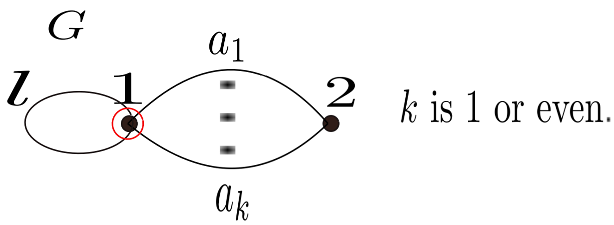


Figure 12. pseudograph and non-full tube in (2)

(3) Assume that G is a pseudograph with the node set $\{1, 2\}$ and multiedges $a_1, \dots, a_k$ ($k \geq 4$, even) (Figure 13). Labels of full tubes are $1a_1 \dots a_k$ and $2a_1 \dots a_k$, and corresponding faces of Σ_G are two facets. So, a non-full tube obtained by removing 2 edges from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

(4) If G is a pseudograph with the node set $\{1, 2\}$ and has 1 or 2 multiedges but does not have loops (Figure 14), then the associated toric manifolds $\mathbb{C}P^1$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ admit spin structures.

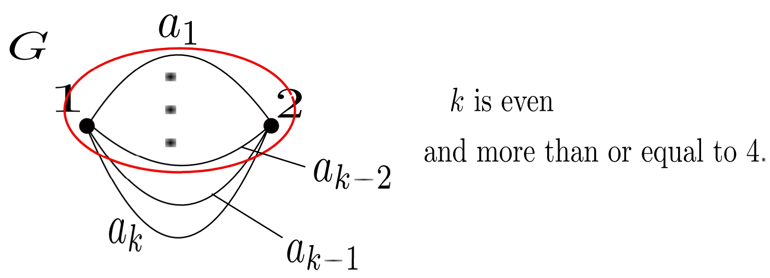


FIGURE 13. pseudograph and non-full tube in (3)

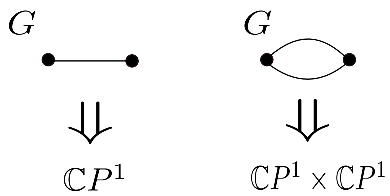


FIGURE 14. (4)

(5) Assume that G is a pseudograph with 1 node and s loops ($s \geq 2$) (Figure 15). There is no full tube, so a non-full tube obtained by removing 2 loops from G corresponds to a codimension 2 face of Σ_G. So, $M(G)$ does not admit a spin structure.

(6) If G is a pseudograph with 1 node and 1 loop, then the associated toric manifold $\mathbb{C}$ admits a spin structure. If G is 1 node, then the associated toric manifold is a point and admits a spin structure (Figure 16).

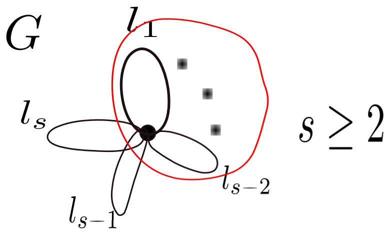


FIGURE 15. pseudograph and non-full tube in (5)

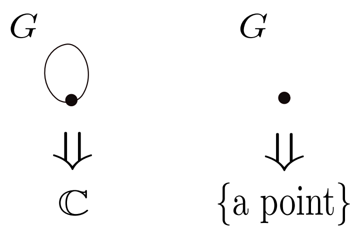


FIGURE 16. (6)

The above observation shows that if G is connected, then the associated toric manifold admits a spin structure if and only if G is 1 node, 1 node with 1 loop, a path graph with 2 nodes, or a pseudograph with 2 nodes and 2 multiedges.

Suppose that G is not connected. Then each connected component of G has only 1 node since G does not contain a proper full tube in Figure 10. If a connected component of G has s loops ($s \geq 1$), then a tube obtained by removing 1 loop from the connected component corresponds to a codimension 2 face of Σ_G. So, if G is not connected, then each connected component of G is 1 node if $M(G)$ admits a spin structure. $\square$

Acknowledgement. I would like to thank Professors Mikiya Masuda and Seonjeong Park for many interesting and fruitful discussions on this subject.

References

[1] G. Barthel, J.-P. Basselet, K.-H. Fieseler, and L. Kaup, *Combinatorial intersection cohomology for fans*, Tohoku Math. J. 54, 1-41 (2002).

[2] M. Carr and S. L. Devadoss and S. Forcey, *Pseudograph associahedra*, Journal of Combinatorial Theory, Series A 118 (2011), 2035-2055.

[3] S. Choi and H. Park, *A new graph invariant arises in toric topology*, to appear in J. Math. Soc. of Japan, arXiv:1210.3776v1.

[4] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991), 417-451.

[5] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Studies, vol.131, Princeton Univ. Press, Princeton, N.J., 1993.

[6] H. Ishida, Y. Fukukawa, and M. Masuda, *Topological toric manifolds*, Moscow Math. J. 13 (2013), no. 1, 57–98; arXiv:1012.1786.

[7] M. Masuda, *Cohomology of open torus manifolds*, Proceedings of the Steklov Institute of Mathmatics, 2006, Vol. 252, pp. 1-9.

[8] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. 51 (1999), 237-265.

[9] H. Nakayama and Y. Nishimura, *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. 42 (2005), 243-256.

Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan.
E-mail address: hatanaka.m.123@gmail.com