Root systems and graph associahedra

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Abstract : It is known that a connected simple graph G associates a simple polytope P_G called a graph associahedron in Euclidean space. In this paper we show that the set of facet vectors of P_G forms a root system if and only if G is a cycle graph and that the root system is of type A.
Key words : graph associahedron; facet vector; root system.

1. Introduction. Let G be a connected simple graph with $n+1$ nodes and its node set $V(G)$ be $[n+1] = \{1, 2, \dots, n+1\}$. We can construct the graph associahedron P_G in $\mathbb{R}^n$ from G ([3]). We call a primitive (inward) normal vector to a facet of P_G a *facet vector* and denote by $F(G)$ the set of facet vectors of P_G. One can observe that when G is a complete graph, $F(G)$ agrees with the primitive edge vectors of the fan formed by the Weyl chambers of a root system of type A ([1]), in other words, $F(G)$ is *dual* to a root system of type A when G is a complete graph. Motivated by this observation, we ask whether $F(G)$ itself forms a root system for a connected simple graph G. It turns out that $F(G)$ forms a root system if and only if G is a cycle graph (Theorem 2). On the way to prove it, we show that $F(G)$ is centrally symmetric (this is the case when $F(G)$ forms a root system) if and only if G is a cycle graph or a complete graph.

2. Construction of graph associahedra. We set

$$B(G) := \{I \subset V(G) \mid G|I \text{ is connected}\},$$

where $G|I$ is a maximal subgraph of G with the node set I (i.e. the induced subgraph). The empty set $\emptyset$ is not in $B(G)$. We call $B(G)$ a *graphical building set* of G. We take an n-simplex in $\mathbb{R}^n$ such that its facet vectors are $e_1, \dots, e_n$, and $-e_1 - \cdots - e_n$, where $e_1, \dots, e_n$ are the standard basis of $\mathbb{R}^n$. Each facet vector e_i $(1 \le i \le n)$ corresponds to an element $\{i\}$ in $B(G)$, and the facet vector $-e_1 - \cdots - e_n$ corresponds to an element $\{n+1\}$ in $B(G)$. We truncate the n-simplex along faces in increasing order of dimension. Let F_i denote the facet of the simplex corresponding to $\{i\}$ in $B(G)$. For every element $I = \{i_1, \dots, i_k\}$ in $B(G) \setminus [n+1]$ we truncate the simplex along the face $F_{i_1} \cap \cdots \cap F_{i_k}$ in such a way that the facet vector of the new facet, denoted F_I, is the sum of the facet vectors of the facets $F_{i_1}, \dots, F_{i_k}$. Then the resulting polytope, denoted P_G, is called a *graph associahedron.* We denote by $F(G)$ the set of facet vectors of P_G.

3. Facet vectors associated to complete graphs. As mentioned in the Introduction, $F(G)$ is *dual* to a root system of type A when G is a complete graph. We shall explain what this means. If G is a complete graph K_{n+1} with $n+1$ nodes, then the graphical building set $B(K_{n+1})$ consists of all subsets of $[n+1]$ except for $\emptyset$ so that the graph associahedron $P_{K_{n+1}}$ is a permutohedron obtained by cutting all faces of the n-simplex with facet vectors $e_1, \dots, e_n, -(e_1 + \cdots + e_n)$. It follows that

$$(1) \qquad F(K_{n+1}) = \Big\{ \pm \sum_{i \in I} e_i \mid \emptyset \neq I \subset [n] \Big\}.$$

On the other hand, consider the standard root system $\Delta(A_n)$ of type A_n given by

$$(2) \qquad \Delta(A_n) = \{\pm(e_i - e_j) \mid 1 \le i < j \le n+1\}$$

which lies on the hyperplane H of $\mathbb{R}^{n+1}$ with $e_1 + \cdots + e_{n+1}$ as a normal vector. Take $e_1 - e_2, e_2 - e_3, \dots, e_n - e_{n+1}$ as a base of $\Delta(A_n)$ as usual. Then their dual base with respect to the standard inner product on $\mathbb{R}^{n+1}$ is what is called the fundamental dominant weights given by

$$\lambda_i = (e_1 + \cdots + e_i) - \tfrac{i}{n+1}(e_1 + \cdots + e_{n+1})$$
$$(i = 1, 2, \dots, n)$$

which also lie on the hyperplane H. The Weyl group action permutes $e_1, \dots, e_{n+1}$ so that it preserves H. We identify H with the quotient vector space H^* of $\mathbb{R}^{n+1}$ by the line spanned by $e_1 + \cdots + e_{n+1}$ using the inner product, namely put the condition $e_1 + \cdots + e_{n+1} = 0$. Then the set of elements obtained from the orbits of $\lambda_1, \dots, \lambda_n$ by the Weyl group action is

$$\Big\{ \sum_{j \in J} e_j \mid \emptyset \neq J \subset [n+1] \Big\} \quad \text{in } H^*.$$

This set agrees with $F(K_{n+1})$ in (1) because $e_{n+1} = -(e_1 + \cdots + e_n)$. In this sense $F(K_{n+1})$ is *dual* to $\Delta(A_n)$.

4. Main theorem. We note that $F(K_{n+1})$ itself forms a root system (of type A_n) when $n = 1$ or 2. However the following holds.

Lemma 1. *If $n \geq 3$, then $F(K_{n+1})$ does not form a root system.*

Proof. Suppose that $F(K_{n+1})$ forms a root system for $n \geq 3$. Then $F(K_{n+1})$ is of rank n and the number of positive roots is $2^n - 1$ by (1). On the other hand, no irreducible root system of rank $n(\geq 3)$ has $2^n - 1$ positive roots (see [2, Table 1 in p.66]). Therefore, it suffices to show that $F(K_{n+1})$ is irreducible if it forms a root system.

Let V be an m-dimensional linear subspace of $\mathbb{R}^n$ such that $E = F(K_{n+1}) \cap V$ is a root subsystem of $F(K_{n+1})$. We consider the mod 2 reduction map

$$\varphi\colon \mathbb{Z}^n \cap V \to (\mathbb{Z}^n \cap V) \otimes \mathbb{Z}/2$$

where $\mathbb{Z}/2 = \{0, 1\}$. Since $(\mathbb{Z}^n \cap V) \otimes \mathbb{Z}/2$ is a vector space over $\mathbb{Z}/2$ of dimension $\leq m$, it contains at most $2^m - 1$ nonzero elements. On the other hand, since the coordinates of an element in $F(K_{n+1})$ are either in $\{0, 1\}$ or $\{0, -1\}$ by (1), the number of elements in $\varphi(E)$ is exactly equal to the number of positive roots in E.

Now suppose that the root system $F(K_{n+1})$ decomposes into the union of two nontrivial components E_i for $i = 1, 2$. Then there are m_i-dimensional linear subspaces V_i of $\mathbb{R}^n$ such that $E_i = F(K_{n+1}) \cap V_i$ and $m_1 + m_2 = n$, where $m_i \geq 1$. Since the number of positive roots in E_i, denoted by p_i, is at most $2^{m_i} - 1$ by the observation above, we have

$$p_1 + p_2 \leq (2^{m_1} - 1) + (2^{m_2} - 1) < 2^n - 1.$$

However, since $F(K_{n+1}) = E_1 \cup E_2$ and the number of positive roots in $F(K_{n+1})$ is $2^n - 1$ as remarked before, we must have $2^n - 1 = p_1 + p_2$. This is a contradiction. Therefore, $F(K_{n+1})$ must be irreducible if it forms a root system. □

The following is our main theorem.

Theorem 2. *Let G be a connected finite simple graph with more than two nodes. Then the set $F(G)$ of facet vectors of the graph associahedron associated to G forms a root system if and only if G is a cycle graph. Moreover, the root system associated to the cycle graph with $n+1$ nodes is of type A_n.*

The rest of this paper is devoted to the proof of Theorem 2. We begin with the following lemma.

Lemma 3. *Let C_{n+1} be the cycle graph with $n+1$ nodes. Then $F(C_{n+1})$ forms a root system of type A_n.*

Proof. An element I in the graphical building set $B(C_{n+1})$ different from the entire set $[n+1]$ is one of the following:
(I) $\{i, i+1, \ldots, j\}$ where $1 \leq i \leq j \leq n$,
(II) $\{i, i+1, \ldots, n+1\}$ where $2 \leq i \leq n+1$,
(III) $\{i, i+1, \ldots, n+1, 1, \ldots, j\}$ where $1 \leq j < i \leq n+1$ and $i - j \geq 2$.
Therefore the facet vector of the facet corresponding to I is respectively given by

$$\sum_{k=i}^{j} e_k, \qquad -\sum_{k=1}^{i-1} e_k, \qquad -\sum_{k=j+1}^{i-1} e_k$$

according to the cases (I), (II), (III) above. Hence

$$F(C_{n+1}) = \Big\{ \pm \sum_{k=i}^{j} e_k \mid 1 \leq i < j \leq n \Big\}. \tag{3}$$

This set forms a root system of type A_n. Indeed, an isomorphism from $\mathbb{Z}^n$ to the lattice

$$\big\{ (x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0 \big\}$$

sending e_i to $e_i - e_{i+1}$ for $i = 1, 2, \ldots, n$ maps $F(C_{n+1})$ to the standard root system $\Delta(A_n)$ of type A_n in (2). □

The following lemma is a key observation.

Lemma 4. *Let G be a connected simple graph. Then $F(G)$ is* centrally symmetric, *which means that $\alpha \in F(G)$ if and only if $-\alpha \in F(G)$ (note that $F(G)$ is centrally symmetric if $F(G)$ forms a root system) if and only if the following holds:*

$$I \in B(G) \implies V(G) \setminus I \in B(G). \tag{4}$$

Proof. Let $V(G) = [n+1]$ as before and let I be an element in $B(G)$ and α_I be the facet vector of the facet of P_G corresponding to I. If we set $e_{n+1} := -(e_1 + \cdots + e_n)$, then $\alpha_I = \sum_{i \in I} e_i$. Since $\alpha_I + \sum_{i \in [n+1] \setminus I} e_i = \sum_{i \in [n+1]} e_i = 0$, we obtain $-\alpha_I = \sum_{i \in [n+1] \setminus I} e_i$ and this implies the lemma. □

Using Lemma 4, we prove the following.

Lemma 5. *Let G be a connected finite simple graph. Then $B(G)$ satisfies (4) if and only if G is a cycle graph or a complete graph.*

Proof. If G is a cycle or complete graph, then $F(G)$ is centrally symmetric by (1) or (3) and hence $B(G)$ satisfies (4) by Lemma 4. So the "if" part is proven.

We shall prove the "only if" part, so we assume that $B(G)$ satisfies (4). Suppose that G is not a complete graph. Then there are $i \neq j \in V(G)$ such that $\{i, j\}$ is not contained in $B(G)$. By (4), $V(G) \setminus \{i, j\}$ is not contained in $B(G)$, which means that the induced subgraph $G|(V(G) \setminus \{i, j\})$ is not connected. On the other hand, since $B(G)$ contains $\{i\}$ and $\{j\}$, $B(G)$ contains $V(G) \setminus \{i\}$ and $V(G) \setminus \{j\}$ by (4). Hence

$$G|(V(G) \setminus \{i\}) \text{ and } G|(V(G) \setminus \{j\}) \text{ are connected.} \tag{5}$$

Let k be the number of connected components of $G|(V(G) \setminus \{i, j\})$ and we denote its k components by $G_1, \ldots, G_k$ (Figure 1). By (5), the nodes i and j are respectively joined to every connected component by at least one edge. Since $G|(V(G_1) \cup \{i, j\})$ is connected, $G|(V(G_2) \cup \cdots \cup V(G_k))$ is also connected by (4). However, $G|(V(G_2) \cup \cdots \cup V(G_k))$ is the disjoint union of the connected subgraphs $G_2, \ldots, G_k$. Therefore we have $k = 2$.

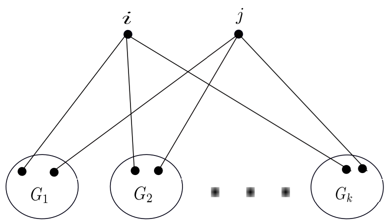

FIGURE 1

If G_1 and G_2 are both path graphs and the node i is joined to one end node of G_1, G_2 respectively and the node j is joined to the other end node of G_1, G_2, then G is a cycle graph (Figure 2).

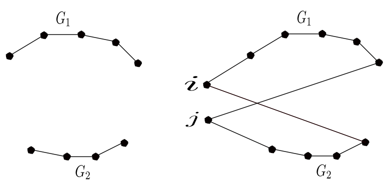

FIGURE 2. the case of cycle graph

We consider the other case, that is, either

(I) G_1 or G_2 is not a path graph, or
(II) both G_1 and G_2 are path graphs but the nodes i and j are not joined to the end points of G_1 and G_2 (see Figure 3, left).

Then there exist nodes $i_1, j_1 \in V(G_1)$ and $i_2, j_2 \in V(G_2)$ such that

- i_1 and i_2 are joined to i,
- j_1 and j_2 are joined to j, and
- either the shortest path P_1 from i_1 to j_1 in G_1 is not the entire G_1 or the shortest path P_2 from i_2 to j_2 in G_2 is not the entire G_2.

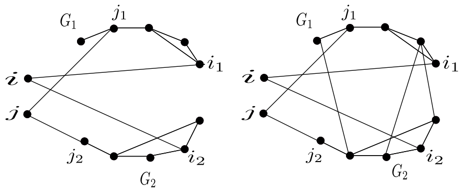

FIGURE 3. the other case

Without loss of generality we may assume that $P_1 \neq G_1$. Since $G|(V(P_1) \cup \{i, j, i_2, j_2\})$ is connected, so is $G|(V(G) \setminus (V(P_1) \cup \{i, j, i_2, j_2\}))$ by (4). This means that there is at least one edge joining G_1 and G_2 (Figure 3, right), and hence $G|(V(G) \setminus \{i, j\})$ is connected. This contradicts that $G|(V(G) \setminus \{i, j\})$ consists of two connected components. □

Now Theorem 2 follows from Lemmas 1, 3, 4 and 5.

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