Asymptotic behavior of the least-energy solutions of a semilinear elliptic equation with the Hardy-Sobolev critical exponent

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Abstract

We investigate the existence, the non-existence and the asymptotic behavior of the least-energy solutions of a semilinear elliptic equation with the Hardy-Sobolev critical exponent. In the boundary singularity case, it is known that the mean curvature of the boundary at origin plays a crucial role on the existence of the least-energy solutions. In this paper, we study the relation between the asymptotic behavior of the solutions and the mean curvature at origin.

Keywords: asymptotic behavior, boundary singularity, Hardy-Sobolev inequality, minimization problem

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1. Introduction

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ bounded domain with smooth boundary, $0 < s < 2$, $2^*(s) = 2(N-s)/(N-2)$ and λ be a positive parameter. In this paper we assume that $0 \in \partial\Omega$. We study the existence, the non-existence and the asymptotic behavior as $\lambda \to \infty$ of the least-energy solutions of

$$\begin{cases} -\Delta u + \lambda u = \frac{u^{2^*(s)-1}}{|x|^s}, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

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The existence of the least-energy solution of (1) is equivalent to the existence of the minimizer for the corresponding minimization problem

$$\mu^N_{s,\lambda}(\Omega) = \inf\left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2)dx \middle| u \in H^1(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1 \right\}. \quad (2)$$

Actually, if the minimizer u_λ for $\mu^N_{s,\lambda}(\Omega)$ exists then $v_\lambda := \mu^N_{s,\lambda}(\Omega)^{(N-2)/(4-2s)}u_\lambda$ is a least-energy solution of (1) and vise versa.

Minimization problems and semilinear elliptic equations on the Hardy-Sobolev type inequality has been studied extensively by many authors. The Dirichlet case, that is, concerning the attainability for

$$\mu^D_s(\Omega) = \inf\left\{ \int_\Omega |\nabla u|^2 dx \middle| u \in H^1_0(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1 \right\}$$

is studied in [9]-[12], [14], [17]. In the interior singularity case, the remainder term of the Hardy-Sobolev inequality is studied by [18]. The optimal Hardy-Sobolev inequality on compact Riemannian manifold is also studied due to [15].

In the Neumann case, we have obtained some results. In the interior singularity case, the existence and non-existence results of the minimizer for $\mu^N_{s,\lambda}(\Omega)$ are obtained by [13]. In the boundary singularity case, some results are due to [5], [9] and [13]. Due to these results, the attainability for $\mu^N_{s,\lambda}(\Omega)$ is different for each situation. In both the Dirichlet case and the Neumann case, the position of 0 on Ω affects the attainability for the best constant.

There are many results on the least-energy solutions of the Neumann problem

$$\begin{cases} -d\Delta u + u = u^p, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

where $d > 0$ is a constant. It is shown that the least-energy solution of (3) exists by [1], [25] and so on. Moreover, by for instance [3], [4], [26], [27] Lin-Ni's conjecture is studied, that is, they investigate that for d sufficiently large whether the solution of (3) is only constant or not.

The asymptotic behavior of the least-energy solutions as $d \to 0$ is studied particularly by [2], [19]-[23]. In the subcritical case $1 < p < (N+2)/(N-2)$, the least-energy solution has only one maximum point and this point lies on the boundary. Moreover, this maximum point approaches the boundary point of maximum mean curvature as $d \to 0$ and the peak is bounded from above uniformly with respect to d. On the other hand, in the critical case $p = (N+2)/(N-2)$, it is proved that peak is at most one and blows up on a boundary point. By [23] we know that the asymptotic behavior of the best constant for the embedding $H^1(\Omega) \subset L^{2N/(N-2)}(\Omega)$, that is,

$$S_d^N(\Omega) = \inf \left\{ \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{d}\|u\|^2_{L^2(\Omega)} \middle| u \in H^1(\Omega), \|u\|^2_{L^{\frac{2N}{N-2}}(\Omega)} = 1 \right\}$$

as $d \to 0$. On the asymptotic behavior of the least-energy solutions of (3) and S_d^N the mean curvature of $\partial\Omega$ plays a crucial role.

Our main purpose of this paper is to investigate the asymptotic behavior of the least-energy solutions of (1) as $\lambda \to \infty$. In [5] and [9], the existence of the least energy solutions of (1) is guaranteed for any $\lambda > 0$ if the mean curvature of $\partial\Omega$ at 0 is positive. Thus it is natural that we investigate the asymptotic behavior of the least-energy solutions of (1). However in the case when the mean curvature at 0 is non-positive, the existence of the least-energy solutions of (1) is not studied so far. As our second purpose of this paper we obtain the answer of this problem through the investigation into the asymptotic behavior.

This paper is organized as follows. In Section 2 we prepare the useful facts and some lemmas. In Section 3 we consider the asymptotic behavior of the least-energy solution of (1). In Section 4 we consider the behavior of $\mu^N_{s,\lambda}(\Omega)$ as $\lambda \to \infty$. Throughout this two sections we assume the existence of the least-energy solutions of (1) for any Ω. In section 5 we show some results on the minimization problem of $\mu^N_{s,\lambda}(\Omega)$.

Remark 1.1. Since the nonlinear term in (1) has a singularity at 0, solutions are not classical solutions. Indeed, if $u \in H^1(\Omega)$ is a weak solution of (1) by the elliptic regularity theory $u \in C^2_{loc}(\overline{\Omega} \setminus \{0\})$ and $u \in C^{0,\alpha}(\overline{\Omega})$ (see [5], [10]). Therefore we should regard $\partial/\partial\nu$ as the bounded linear operator from $W^{2,p}(\Omega)$ to $L^p(\partial\Omega)$ at 0.

2. Preliminaries

In this section we prepare some useful facts.

We recall that some facts about a diffeomorphism straightening a boundary portion around a point $P \in \partial\Omega$, which was introduced in [19]-[22]. Through translation and rotation of the coordinate system we may assume that P is the origin and inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. In a neighborhood $\mathcal{N}$ around P, there exists a smooth function $\psi(x'),\ x' = (x_1, \dots, x_{N-1})$ such that $\partial\Omega \cap \mathcal{N}$ can be represented by

$$x_N = \psi(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$$

where $\alpha_1, \dots, \alpha_{N-1}$ are the principal curvatures of $\partial\Omega$ at P. For $y \in \mathbb{R}^N$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y) = (\Phi_1(y), \dots, \Phi_N(y))$ by

$$\Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial \psi}{\partial x_j}(y') & j = 1, \dots, N-1 \\ y_N + \psi(y') & j = N. \end{cases}$$

The differential map $D\Phi$ is

$$D\Phi(y) = \begin{pmatrix} \delta_{ij} - \dfrac{\partial^2 \psi}{\partial x_i \partial x_j}(y') y_N & -\dfrac{\partial \psi}{\partial x_i}(y') \\ \dfrac{\partial \psi}{\partial x_j}(y') & 1 \end{pmatrix}_{1 \le i,j \le N-1}$$

and near $y = 0$

$$|J\Phi(y)| = |\mathrm{det} D\Phi(y)| = 1 - (N-1)H(P)y_N + O(|y|^2).$$

We write as $\Psi(x) = (\Psi_1(x), \dots, \Psi_N(x))$ instead of the inverse map $\Phi^{-1}(x)$. $B_r(a)$ denotes a open ball with center a and radius r. In addition, suppose $B_r = B_r(0)$ and $B_r^+ = \{y \in B_r | y_N > 0\}$.

We set a function as

$$U(x) = \left(1 + \frac{|x|^{2-s}}{(N-s)(N-2)}\right)^{-\frac{N-2}{2-s}}. \tag{4}$$

Note that $U(0) = 1$ and U is a minimizer for

$$\mu_s := \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \,\middle|\, u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s} dx = 1 \right\} \tag{5}$$

which is the best constant for the Hardy-Sobolev inequality. For U define the scaling function by

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x}{\varepsilon}\right).$$

We have the following lemma regarding $\mu_{s,\lambda}^N(\Omega)$.

Lemma 2.1. *(i)* $\mu^N_{s,\lambda}(\Omega)$ *is continuous and non-decreasing with respect to* λ.

(ii) For any $\lambda > 0$, $\mu^N_{s,\lambda}(\Omega) \le \mu_s/2^{(2-s)/(N-s)}$.

(iii) $\lim_{\lambda\to 0} \mu^N_{s,\lambda}(\Omega) = 0$.

Proof. We show only part (ii).

For given $\phi \in C^1(\Omega\cap\mathcal{N}_0)$ we set $\tilde{\phi}(y) = \phi(\Phi(y))$, where $\mathcal{N}_0$ is a neighborhood around 0 such that $\Omega\cap\mathcal{N}_0 = \Phi(B^+_\delta)$. If $\tilde{\phi}(y)$ is a radially symmetric function, we have

$$\begin{aligned}\int_{\Omega\cap\mathcal{N}_0} |\nabla\phi(x)|^2 dx \;=\;& \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-1}|\tilde{\phi}'|^2(r)dr \\ &-\frac{(N-1)\pi^{\frac{N-1}{2}}}{(N+1)\Gamma(\frac{N+1}{2})}H(0)\int_0^\delta r^N|\tilde{\phi}'|^2(r)dr \\ &+\int_0^\delta O(r^{N+1})|\tilde{\phi}'|^2(r)dr, \end{aligned} \tag{6}$$

$$\int_{\Omega\cap\mathcal{N}_0} |\phi(x)|^2 dx = \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-1}\tilde{\phi}^2(r)dr + \int_0^\delta O(r^N)|\phi^2|(r)dr, \tag{7}$$

$$\begin{aligned}\int_{\Omega\cap\mathcal{N}_0} \frac{\phi^{2^*(s)}}{|x|^s}dx \;=\;& \frac{\omega_{N-1}}{2}\int_0^\delta r^{N-s-1}\tilde{\phi}^{2^*(s)}(r)dr \\ &-(N-1)\left[1-\frac{s}{2(N+1)}\right]\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})}H(0)\int_0^\delta r^{N-s}\phi^{2^*(s)}dr \\ &+\int_0^\delta O(r^{N-s+1})\tilde{\phi}^{2^*(s)}dr, \end{aligned} \tag{8}$$

where ω_{N-1} is the surface area of a unit sphere. Set a cut-off function $\eta(y) = \eta(|y|)$ such that support of η is in B_δ and $\eta = 1$ in $B_{\delta/2}$. Choosing $\eta(y)U_\varepsilon(y)$

as $\tilde{\phi}$ in (6), (7) and (8) and hence we obtain

$$\frac{\int_\Omega (|\nabla(\eta U_\varepsilon)|^2 dx + \lambda|\eta U_\varepsilon|^2) dx}{\left(\int_\Omega \frac{|\eta U_\varepsilon|^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)}}$$

$$= \begin{cases} \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - c_1 H(0)\varepsilon + [\lambda (c_2 + O(\varepsilon|\log\varepsilon|)) + O(\varepsilon)]\,\varepsilon^2 & (N \geq 5) \\ \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - c_1 H(0)\varepsilon + \left[\lambda \left(c_2 + O\left(|\log\varepsilon|^{-1}\right)\right) + O(1)\right] \varepsilon^2 |\log\varepsilon| & (N = 4) \\ \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - c_1 H(0)\varepsilon|\log\varepsilon| + [\lambda (c_2 + O(\varepsilon)) + O(1)]\,\varepsilon & (N = 3) \end{cases}$$

where c_1, c_2 are positive constants which depend only on N. Tending ε to 0 and we obtain the estimate of part (ii). □

Lemma 2.2. *We have either*

(i) There exist $\tilde{\lambda}$ such that for $\lambda \geq \tilde{\lambda}$

$$\mu^N_{s,\lambda}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s, \tag{9}$$

or

(ii) For all λ the equality (9) does not hold and

$$\lim_{\lambda\to\infty} \mu^N_{s,\lambda}(\Omega) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s. \tag{10}$$

where μ_s is defined by (5). To prove this lemma, we prepare one proposition.

Proposition 2.3. *Fix $\varepsilon > 0$ sufficiently small. Then there exists a positive constant $C = C(\varepsilon)$ such that for $u \in H^1(\Omega)$*

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_\Omega \frac{u^{2^*(s)}}{|x|^s}\right)^{2/2^*(s)} \leq (1+\varepsilon)\int_\Omega |\nabla u|^2 dx + C\int_\Omega u^2 dx. \tag{11}$$

Proof of Proposition 2.3. We choose small constant $\delta > 0$, $r > 0$ and V which is a neighborhood around 0 such that

$$x_N = \psi_0(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2), \quad |\nabla \psi_0(x')| \le \delta \quad \text{on} \quad \partial\Omega \cap V,$$

and $\{(x', x_N - \psi_0) | (x', x_N) \in \Omega \cap V\} = B_r^+$.

Due to [13] there exists a positive constant $C = C(B_r)$ such that

$$\mu_s \left(\int_{B_r} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)} \le \int_{B_r} |\nabla u|^2 dx + C \int_{B_r} u^2 dx. \tag{12}$$

By the transformation $y' = x'$, $y_N = x_N - \psi_0(x')$ and the inequality (12), it follows that

$$\begin{aligned} & \mu_s \left(\int_{\Omega \cap V} \frac{|u|^{2^*(s)}}{(|x'|^2 + |x_N - \psi_0|^2)^{s/2}} dx \right)^{2/2^*(s)} \\ = \; & \mu_s \left(\frac{1}{2} \int_{B_r^+} \frac{|\hat{u}|^{2^*(s)}}{|y|^s} dy \right)^{2/2^*(s)} \\ \le \; & \left(\frac{1}{2} \right)^{2/2^*(s)} \int_{B_r^+} (|\nabla_y \hat{u}|^2 + C\hat{u}^2) dy \\ \le \; & 2^{\frac{2-s}{N-s}} \left(1 + (N-1)\delta + \delta^2\right) \int_{\Omega \cap V} |\nabla_x u|^2 + C\hat{u}^2 dx \end{aligned}$$

where $\hat{u}(y) = u(y', y_N + \psi_0)$. On the other hand, if $|x|$ sufficiently small

$$(|x'|^2 + |x_N - \psi_0|^2)^{s/2} = (|x|^2 - 2\psi_0 x_N + \psi_0^2)^{s/2} \le (1 + C_0|x|)|x|^s.$$

Now, we may assume that $\mathrm{diamV} < \mathrm{C}_1\delta$ for some C_1. Consequently taking ε such that

$$1 + \varepsilon = \frac{1 + (N-2)\delta + \delta^2}{1 + C_0 C_1 \delta}$$

and we obtain

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_{\Omega \cap V} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)} \le (1 + \varepsilon) \int_{\Omega \cap V} |\nabla u|^2 dx + C \int_{\Omega \cap V} u^2 dx.$$

In $\overline{\Omega \setminus V}$, taking into account that $|x|^{-s}$ has not a singularity and we have

$$\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \left(\int_{\Omega\setminus V} \frac{u^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)} \le (1+\varepsilon) \int_{\Omega\setminus V} |\nabla u|^2 dx + C \int_{\Omega\cap V} u^2 dx.$$

The detail of calculations is in [13]. Hence we obtain (11). □

Proof of Lemma 2.2. If there exist $\tilde{\lambda}$ such that (9) holds, then by part (i) and part (ii) of Lemma 2.1 we can prove part (i).

Assume that for all $\lambda > 0$, the equality (9) does not hold. For any $\varepsilon > 0$ and $\lambda > 0$, there exist $u_{\lambda,\varepsilon}$ such that

$$\mu_{s,\lambda}^N(\Omega) \ge \int_\Omega |\nabla u_{\lambda,\varepsilon}|^2 dx + \lambda \int_\Omega u_{\lambda,\varepsilon}^2 dx - \varepsilon$$

We choose $\lambda = \lambda(\varepsilon)$ such that $\lambda \to \infty$ as $\varepsilon \to 0$ and $\lambda \ge C$ where C is given in Proposition 2.3. From the above inequality and (11) we have

$$0 \le \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - \mu_{s,\lambda}^N(\Omega) \le \varepsilon \int_\Omega |\nabla u_{\lambda,\varepsilon}|^2 dx + \varepsilon \le \varepsilon(1+\mu_s).$$

Hence tending ε to 0 and we obtain the equality (10). □

By the next lemma we can see the relation between the value of $\mu_{s,\lambda}^N(\Omega)$ and the existence of the minimizer of $\mu_{s,\lambda}^N(\Omega)$.

Lemma 2.4. *(i) If $\mu_{s,\lambda}^N(\Omega) < \mu_s$ then $\mu_{s,\lambda}^N(\Omega)$ is attained.*

(ii) If there exist a positive constant $\tilde{\lambda}$ such that $\mu_{s,\tilde{\lambda}}^N(\Omega) = \mu_s$ then $\mu_{s,\lambda}^N(\Omega)$ is not attained for all $\lambda > \tilde{\lambda}$.

Proof. (i) proved by the proof of Proposition 2.1 in [5].

We prove (ii). Let $\lambda > \tilde{\lambda}$ and u_λ be a minimizer of $\mu_{s,\lambda}^N(\Omega)$. Then we have

$$\mu_s = \mu_{s,\tilde{\lambda}}^N(\Omega) \le \int_\Omega (|\nabla u_\lambda|^2 + \tilde{\lambda} u_\lambda^2) dx < \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2) dx = \mu_{s,\lambda}^N(\Omega) \le \mu_s.$$

This is a contradiction. □

3. Asymptotic behavior I

In this section and the next section we assume that the least-energy solution of (1) exists.

We investigate the asymptotic behavior of the least-energy solution of (1) as $\lambda \to \infty$. In order to prove Theorem 3.1, we apply the strategy in [19]-[22] to the equation (1). We assume v_λ is a least-energy solution of (1) and define α_λ and β_λ as

$$\alpha_\lambda = \|v_\lambda\|_{L^\infty(\Omega)} = v_\lambda(x_\lambda), \quad \beta_\lambda = \alpha_\lambda^{-\frac{2}{N-2}}.$$

Theorem 3.1. *We obtain the following results;*

(i) For all $x \in \Omega$, $v_\lambda(x) \to 0$,

(ii) $\alpha_\lambda^{\frac{4}{N-2}}/\lambda = (\lambda\beta^2)^{-1} \to \infty$,

(iii) $|x_\lambda| = o(\beta_\lambda)$

as $\lambda \to \infty$. *For any* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a positive constant* λ_0 *such that for all* $\lambda > \lambda_0$

(iv) $\left|\dfrac{v_\lambda(x)}{\alpha_\lambda} - U\left(\dfrac{\Psi_\lambda(x)}{\beta_\lambda}\right)\right| < \varepsilon$ *in* $\Omega \cap B_{\beta_\lambda \delta}$,

(v) $v_\lambda \le 2\varepsilon\lambda^{\frac{N-2}{(4-2s)}}\exp(-\gamma_0\xi(x)\lambda^{\frac{1}{2}})$ *in* $\Omega \setminus B_\delta$,

where U *is defined in (4),* $\xi(x) = \min\{\eta_0, \mathrm{dist}(x, \partial\Omega \cap B_\delta)\}$, $\eta_0 = \eta_0(\Omega)$ *and* $\gamma_0 = \gamma_0(\Omega, \varepsilon)$ *are positive constants.*

Lemma 3.2. *There exist a positive constant* C *which is independent of* λ *such that*

$$\frac{\alpha_\lambda^{\frac{4}{N-2}}}{\lambda} \ge C.$$

Proof. For simplicity, we write $v = v_\lambda$ and $\alpha = \alpha_\lambda$ for each. C_0, C_1, C_2, C_3 are positive constants which depends only on domain Ω. We have

$$\int_\Omega \nabla v \nabla \phi dx + \lambda \int_\Omega v\phi dx \le \alpha^{2^*(s)-2} \int_\Omega \frac{v\phi}{|x|^s} dx \tag{13}$$

for all $\phi \in H^1(\Omega)$ satisfying $\phi \ge 0$. For $\beta \ge 1$, we define a function $H \in C^1([0,\infty))$ by setting $H(t) = t^\beta$ and $G(t) := \int_0^t |H'(s)|^2 ds = \frac{\beta^2}{2\beta-1} t^{2\beta-1}$. We easily find that

$$vG'(v) \ge G(v). \tag{14}$$

Replacing ϕ in (13) by $G(v)$ we have

$$\int_\Omega \nabla v \nabla G(v) dx + \lambda \int_\Omega vG(v)dx \le \alpha^{2^*(s)-2} \int_\Omega \frac{vG(v)}{|x|^s} dx.$$

The chain rule, the definition of G and (14) yield

$$\int_\Omega |\nabla H(v)|^2 dx + \lambda \frac{\beta^2}{2\beta - 1} \int_\Omega H(v)^2 dx \le \alpha^{2^*(s)-2} \int_\Omega \frac{|vH'(v)|^2}{|x|^s} dx \tag{15}$$

For $\lambda \frac{\beta^2}{2\beta-1} \ge 1$, by the Hardy-Sobolev inequality it follows that

$$\mu_s^N(\Omega) \left(\int_\Omega \frac{H(v)^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla H(v)|^2 dx + \lambda \frac{\beta^2}{2\beta - 1} \int_\Omega H(v)^2 dx \tag{16}$$

where $\mu_s^N(\Omega) := \inf \left\{ \int_\Omega (|\nabla u|^2 + u^2) dx | u \in H^1(\Omega), \int_\Omega |u|^{2^*(s)}/|x|^s dx = 1 \right\}$. Since

$$H(v) = v^\beta, \quad vH'(v) = \beta v^\beta \tag{17}$$

Combining (15), (16) and (17) we have

$$\|v\|^2_{L^{2^*(s)\beta}(\Omega,|x|^{-s}dx)} \le C_0^{\frac{1}{\beta}} \alpha^{(2^*(s)-2)\frac{1}{\beta}} \beta^{\frac{2}{\beta}} \|v\|^2_{L^{2\beta}(\Omega,|x|^{-s}dx)}$$

For $m = 0, 1, 2, \cdots$ we define $\beta_{m+1} = (2^*(s)/2)^m$, then we have

$$\begin{aligned} & \|v\|^2_{L^{2^*(s)\beta_{m+1}}(\Omega,|x|^{-s}dx)} \\ \le \; & C_0^{\frac{1}{\beta_{m+1}}} \alpha^{(2^*(s)-2)\frac{1}{\beta_{m+1}}} \beta_{m+1}^{\frac{2}{\beta_{m+1}}} \|v\|_{L^{2\beta_{m+1}}(\Omega,|x|^{-s}dx)} \\ = \; & \prod_{l=0}^{m} C_0^{\frac{1}{2(2^*(s)/2)^l}} \alpha^{(2^*(s)-2)\frac{1}{(2^*(s)/2)^l}} \left(\frac{2^*(s)}{2} \right)^{l \frac{1}{(2^*(s)/2)^l}} \|v\|_{L^2(\Omega,|x|^{-s}dx)}. \end{aligned} \tag{18}$$

Note that

$$\sum_{l=0}^{\infty} \left(\frac{2^*(s)}{2} \right)^{-l} = \lim_{m \to \infty} \frac{1 - \left(\frac{2^*(s)}{2} \right)^{-m-1}}{1 - \left(\frac{2^*(s)}{2} \right)^{-1}} = \frac{2^*(s)}{2^*(s) - 2},$$

$$\sum_{l=0}^{\infty} l \left(\frac{2^*(s)}{2} \right)^{-l} \le \sum_{l=0}^{\infty} (l+1) \left(\frac{2^*(s)}{2} \right)^{-l} \le \frac{2^*(s)}{(2^*(s) - 2)^2}.$$

Tending $m \to \infty$ in (18), and thus

$$\|v\|_\infty^2 \le C_1 \alpha^{(2^*(s)-2)\frac{2^*(s)}{2^*(s)-2}} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)} = C_1 \alpha^{2^*(s)} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)}.$$

Using the Hölder inequality we have

$$\begin{aligned} \|v\|^2_{L^2(\Omega,|x|^{-s}dx)} &= \int_\Omega \frac{v^2}{|x|^s} dx \le \left(\int_\Omega \frac{v^2}{|x|^2}\right)^{s/2} \left(\int_\Omega v^2 dx\right)^{1-s/2} \\ &< C_2 \left(\int_\Omega v^2 dx\right)^{1-s/2}. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{1}{C_1 C_2} &\le \alpha^{2^*(s)-2} \left(\int_\Omega v^2 dx\right)^{1-s/2} \\ &= \left(\alpha^{\frac{4}{N-2}} \int_\Omega v^2 dx\right)^{1-s/2} \\ &\le C_3 \left(\frac{\alpha^{\frac{4}{N-2}}}{\lambda}\right)^{1-s/2}. \end{aligned}$$

Therefore we obtain

$$\frac{\alpha^{\frac{4}{N-2}}}{\lambda} > C.$$

□

Lemma 3.3.

$$|x_\lambda| = O(\beta_\lambda)$$

Proof. Step 1. First of all, we show that $d(x_\lambda, \partial\Omega) = O(\beta_\lambda)$. We assume that

$$\lim_{\lambda\to\infty} \frac{d(x_\lambda, \partial\Omega)}{\beta_\lambda} = \infty \tag{19}$$

and derive a contradiction. Assume that λ_k is positive increasing sequence such that $\lambda_k \to \infty$ as $k \to \infty$. By the assumption of (19) we may take a positive constant R such that

$$|B_R(0)| > \frac{1}{2} S_N(\Omega)^{-1} \mu_s^{\frac{N-s}{2-s}} \quad \text{and} \quad x_{\lambda_k} + \beta_{\lambda_k} z \in \Omega \quad \text{for all} \quad z \in B_{3R}(0) \tag{20}$$

where $|B_R(0)|$ is N-dimensional volume of $B_R(0)$ and

$$S^N(\Omega) = \inf\left\{ \int_\Omega (|\nabla u|^2 + u^2)dx \middle| \int_\Omega |u|^{\frac{2N}{N-2}}dx = 1 \right\}$$

is the best constant of the critical Sobolev embedding. We set

$$w_k(z) := \frac{v_{\lambda_k}(x_{\lambda_k} + \beta_{\lambda_k} z)}{\alpha_{\lambda_k}} \quad z \in B_{3R}(0).$$

Since $v_{\lambda_k} \in C^2_{loc}(\overline{\Omega} \setminus \{0\})$ $\tilde{v}_k$ satisfies

$$-\Delta w_k + \lambda\beta^2 w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{x_{\lambda_k}}{\beta_{\lambda_k}} + z\right|^s} \quad \text{in } B_{3R}(0).$$

Note that from (19) and Lemma 3.2

$$\lambda_k\beta_{\lambda_k} \to C, \quad \left|\frac{x_{\lambda_k}}{\beta_{\lambda_k}} + z\right|^{-s} = o(1) \quad \text{as} \quad k \to \infty \quad \text{for} \quad z \in B_{3R(0)}. \tag{21}$$

By using the elliptic regularity theory there exists w such that

$$w \in C^2(B_R(0)), \quad w_k \to w \quad \text{in} \quad C^2(B_R(0))$$

and

$$-\Delta w + Cw = 0 \quad \text{in} \quad B_R(0).$$

In addition $0 \le w(z) \le 1$ in $B_R(0)$ and $w(0) = 1$ since $\tilde{v}_k(0) = 1$. By the strong maximum principle $w \equiv 1$. But

$$\begin{aligned}
|B_R(0)| &= \int_{B_R(0)} w^{\frac{2N}{N-2}}dz = \lim_{k\to\infty}\int_{B_R(0)} w_k^{\frac{2N}{N-2}}dz = \lim_{k\to\infty}\int_{B_{\beta_{\lambda_k}R}(x_{\lambda_k})} v_{\lambda_k}^{\frac{2N}{N-2}}dx \\
&\le \lim_{k\to\infty}\int_\Omega v_{\lambda_k}^{\frac{2N}{N-2}}dx \le \lim_{k\to\infty} S_N(\Omega)^{-1}\int_\Omega \left(|\nabla v_{\lambda_k}^2 + v_{\lambda_k}^2\right)dx \\
&= \frac{1}{2}S_N(\Omega)^{-1}\mu_s^{\frac{N-s}{2-s}}
\end{aligned}$$

which contradicts the choice of R in (20).

Step 2. To end of the proof of this lemma we show that

$$x_\lambda \not\to x \quad \text{for all} \quad x \in \partial\Omega \setminus \{0\}.$$

We assume that there exists a point $x_0 \in \partial\Omega\setminus\{0\}$ such that $|x_\lambda - x_0| = O(\beta_\lambda)$ and derive a contradiction.

By translation and rotation of the coordinate system we may consider the equation

$$\begin{cases} -\Delta v_\lambda + \lambda v_\lambda = \frac{v_\lambda^{2^*(s)-1}}{|a_0+x|^s} & \text{in } \Omega \\ \frac{\partial v_\lambda}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

and $x_\lambda \to 0$, where $a_0 \in \partial\Omega \setminus \{0\}$. Set $\lambda_k \to \infty$ and $x_{\lambda_k} \to 0$ as $k \to \infty$. For δ small sufficiently put $\hat{v}_{\lambda_k}(y) = v_{\lambda_k}(\Phi(y))$ for $y \in \overline{B}_{2\delta}^+$ and

$$\tilde{v}_{\lambda_k} = \begin{cases} \hat{v}_{\lambda_k}(y) & y \in \overline{B}_{2\delta}^+ \\ \hat{v}_{\lambda_k}(y', -y_N) & (y', -y_N) \in \overline{B}_{2\delta}^+. \end{cases}$$

We define a function w_k $(k = 1, 2, \ldots)$ by

$$w_k(z) = \frac{\tilde{v}_{\lambda_k}(Q_{\lambda_k} + \beta_{\lambda_k} z)}{\alpha_\lambda} \qquad z \in B_{\delta/\beta_{\lambda_k}}$$

where $Q_{\lambda_k} = \Psi(x_{\lambda_k}) = (q'_{\lambda_k}\beta_{\lambda_k}, q^N_{\lambda_k}\beta_{\lambda_k})$, $Q_{\lambda_k}/\beta_{\lambda_k} \to Q_\infty = (q'_\infty, q^N_\infty)$ as $k \to \infty$. By Step 1, $|Q_\infty| < \infty$.

We take a positive constant R such that

$$|B_R(0)| > S_N(\Omega)^{-1}\mu^{\frac{N-s}{2-s}}$$

in the same way as Step 1. Set a function ξ_k as

$$\xi_k(z) = \begin{cases} \Phi(Q_{\lambda_k} + \beta_{\lambda_k} z) & (z_N \geq -q^N_{\lambda_k}) \\ \Phi((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q^N_{\lambda_k} + z_N)\beta_{\lambda_k}) & (z_N < -q^N_{\lambda_k}). \end{cases}$$

Then w_k satisfies

$$-\sum_{i,j=1}^N a^k_{ij}(z)\frac{\partial^2 w_k}{\partial z_i \partial z_j} + \beta_{\lambda_k}\sum_{j=1}^N b^k_j(z)\frac{\partial w_k}{\partial z_j} + \lambda_k \beta^2_{\lambda_k} w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{a_0+\xi_k}{\beta_{\lambda_k}}\right|^s}$$

in $B_R(0) \setminus \{z_N = -q^N_{\lambda_k}\}$, where a^k_{ij}, b^k_j is defined as follows (there definitions is same as those in Step 2 in the section 4 in [22]):

$$a_{ij}(y) = \sum_{k=1}^N \frac{\partial \Psi_i}{\partial x_k}(\Phi(y))\frac{\partial \Psi_j}{\partial x_k}(\Phi(y)) \quad 1 \leq i, j \leq N \tag{23}$$

$$b_j(y) = (\Delta\Psi_j)(\Phi(y)) \quad 1 \leq j \leq N. \tag{24}$$

Then define

$$a_{ij}^k(z) = \begin{cases} a_{ij}(Q_{\lambda_k} + \beta_{\lambda_k} z) & z_N \geq -q_{\lambda_k}^N, \\ (-1)^{\delta_{iN}+\delta_{jN}} a_{ij}((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q_{\lambda_k}^N + z_N)\beta_{\lambda_k}) & z_N < q_{\lambda_k}^N, \end{cases}$$

$$b_j^k(z) = \begin{cases} b_j(Q_{\lambda_k} + \beta_{\lambda_k} z) & z_N \geq -q_{\lambda_k}^N, \\ (-1)^{\delta_{jN}} b_j((q'_{\lambda_k} + z')\beta_{\lambda_k}, -(q_{\lambda_k}^N + z_N)\beta_{\lambda_k}) & z_N < -q_{\lambda_k}. \end{cases}$$

By applying the elliptic regularity theory in [22] and arguing in the same manner as in Step 1 we have

$$w \in C^2(B_R(0)), \quad w_k \to w \quad \text{in} \quad C^2(B_R(0))$$

and $w \equiv 1$. It follows that

$$|B_R(0)| \;=\; \int_{B_R} w^{\frac{2N}{N-2}} dz \leq \lim_{k\to\infty} 2 \int_\Omega v_{\lambda_k}^{\frac{2N}{N-2}} dz \leq S_N(\Omega)^{-1} \mu_s^{\frac{N-s}{2-s}}.$$

This contradicts the choice of R. □

Proof of Theorem 3.1 (ii), (iii), (iv). We can see $x_\lambda \to 0$ from Lemma 3.3. Put $k \to \infty$ and define λ_k, x_{λ_k}, $\hat{v}_{\lambda_k}$, $\tilde{v}_{\lambda_k}$, Q_{λ_k}, w_k and ξ_k respectively as those in Step 2 of the proof of Lemma 3.3. w_k satisfies

$$-\sum_{i,j=1}^N a_{ij}^k(z) \frac{\partial^2 w_k}{\partial z_i \partial z_j} + \beta_{\lambda_k} \sum_{j=1}^N b_j^k(z) \frac{\partial w_k}{\partial z_j} + \lambda_k \beta_{\lambda_k}^2 w_k = \frac{w_k^{2^*(s)-1}}{\left|\frac{\xi_k}{\beta_{\lambda_k}}\right|^s}$$

in $B_{2\delta/\beta_{\lambda_k}}(0) \setminus \{z_N = -q_{\lambda_k}^N\}$. By the definition of ξ_k we have $|\xi_k/\beta_{\lambda_k}| \to |Q_\infty + z|$.

For any $L > 0$ and some $r > N/2$ by the Hölder inequality we have

$$\int_{B_L(-Q_\infty)} \left(\frac{w_k^{2^*(s)-1}}{\left|\frac{\xi_k}{\beta_{\lambda_k}}\right|^s} \right)^r dz < C(L) < \infty. \tag{25}$$

By applying the elliptic regularity theory in [22] there exists a function w such that

$$w \in C^2(B_L(-Q_\infty) \backslash \{-Q_\infty\}), \quad w_k \to w \quad \text{in} \quad C^{0,\alpha}(B_L(-Q_\infty)) \cap H^1(B_L(-Q_\infty)).$$

Moreover, w satisfies $w(0) = 1$ and $w \in D^{1,2}(\mathbb{R}^N)$. In fact

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w|^2 dz &\leq \int_{\mathbb{R}^N} (|\nabla w|^2 + Cw^2) dz \\
&= \lim_{L\to\infty} \int_{B_L} (|\nabla w|^2 + Cw^2) dz \\
&\leq \lim_{L\to\infty} \lim_{k\to\infty} 2 \int_\Omega (|\nabla v_k|^2 + \lambda_k v_k^2) dx \\
&\leq \mu_s^{\frac{N-s}{2-s}}
\end{aligned}$$

where C is defined in (21). Thus

$$w \in C^2_{loc}(\mathbb{R}^N \setminus \{-Q_\infty\}), \quad w_k \to w \quad \text{in} \quad C^{0,\alpha}_{loc}(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N).$$

If $C \neq 0$ w is a weak solution of

$$-\Delta w + Cw = \frac{w^{2^*(s)-1}}{|(Q_\infty + z)|^s} \quad \text{in} \quad \mathbb{R}^N.$$

Define the function $f : \mathbb{R}^N \setminus \{-Q_\infty\} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, u) = \frac{|u|^{2^*(s)-2} u}{|Q_\infty + z|^s} - Cu.$$

Then we can see w and f satisfy the all conditions of Claim 5.3 in [8] and hence from the claim we can $C = 0$. Furthermore we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{w^{2^*(s)}}{|(Q_\infty + z)|^s} dz &\leq \lim_{k\to\infty} 2 \int_\Omega \frac{v_{\lambda_k}^{2^*(s)}}{|x|^s} dx \\
&= \lim_{k\to\infty} \mu^N_{s,\lambda_k}(\Omega)^{\frac{N-s}{2-s}} \\
&= \mu_s^{\frac{N-s}{2-s}}.
\end{aligned}$$

Hence w is a minimizer of μ_s. Since $0 \leq w \leq 1$ and $w(0) = 1$, we obtain $w = U$ and $Q_\infty = 0$. Therefore part (ii) and (iii) is proved.

For $z \in B_{\delta/\beta_{\lambda_k}}$ we set

$$\tilde{w}_k(z) = \frac{\tilde{v}_{\lambda_k}(\beta_{\lambda_k} z)}{\alpha_{\lambda_k}}. \tag{26}$$

Then since $Q_{\lambda_k}/\beta_{\lambda_k} \to 0$ as $k \to \infty$ we have

$$\tilde{w}_k \to U \quad \text{in} \quad C^{0,\alpha}_{loc}(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N)$$

as $k \to \infty$. Hence part (iv) is obtained. □

Lemma 3.4. *We assume that $u \in H^1(\Omega)$ satisfy that $u \geq 0$ and*

$$\begin{cases} -\Delta u \leq \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Then for any $r > 0$ there exist positive constants $\mu = \mu(\Omega)$ and $C = C(\Omega, r)$ such that for any $Q \in \mathbb{R}^N$ we have

$$\sup_{x \in \Omega \cap B_r(Q)} v_\lambda(x) \leq C \left(\int_{\Omega \cap B_{2r}(Q)} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{1}{2^*(s)}} \tag{28}$$

provided that

$$\int_{\Omega \cap B_{4r}(Q)} \frac{u^{2^*(s)}}{|x|^s} dx \leq \mu.$$

Proof. We prove Lemma 3.4 in the same way as the strategy of the proof of Lemma 2.13 in [20] . □

Proof of Theorem 3.1 (i). From Lemma 2.1, if u_λ is a minimizer for $\mu^N_{s,\lambda}(\Omega)$ then $\|u_\lambda\|_{L^2(\Omega)} = O(1/\lambda)$. Thus we have $u_\lambda(x) \to 0$ a.e. in Ω. Since $v_\lambda = \mu^N_{s,\lambda}(\Omega)^{(N-2)/(4-2s)} u_\lambda$ we have $v_\lambda(x) \to 0$ a.e. in Ω.

For all $x \in \Omega$, there exists a positive constant κ such that $0 \notin \overline{\Omega \cap B_{4\kappa}(x)}$. We have

$$\lim_{\lambda \to \infty} \int_{\Omega \cap B_{4\kappa}(x)} \frac{v_\lambda^{2^*(s)}}{|x|^s} dx = 0.$$

By Lemma 3.4 we obtain

$$v_\lambda(x) \leq \sup_{x \in \Omega \cap B_\kappa(x)} u(x) \leq C \int_{\Omega \cap B_{2\kappa}(x)} \frac{v_\lambda^{2^*(s)}}{|x|^s} dx \leq C \int_{\Omega \cap B_{4\kappa}(x)} \frac{v_\lambda^{2^*(s)}}{|x|^s} dx \to 0$$

as $\lambda \to \infty$. □

Proof of Theorem 3.1 (iv). For all $\varepsilon > 0$ and $\delta > 0$ by part (i) there exists $\lambda_0 > 0$ such that $v_\lambda(x) < \varepsilon$ in $\Omega \setminus B_\delta$ for all $\lambda > \lambda_0$. We set $w_\lambda = \lambda^{-(N-2)/(4-2s)} v_\lambda$, then w_λ satisfies

$$\begin{cases} -\frac{1}{\lambda}\Delta w_\lambda + w_\lambda = \frac{w_\lambda^{2^*(s)-1}}{|x|^s} & \text{in} \quad \Omega \\ \frac{\partial w_\lambda}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

For w_λ, we use the strategy in the proof of Theorem 2.3 (iii) in [22]. Hence Theorem 3.1 (v) is proved. $\square$

4. Asymptotic behavior II

In this section, we consider the asymptotic behavior of $\mu^N_{s,\lambda}(\Omega)$. Suppose v_λ is a least-energy solution of (1). Define for $f \in H^1(\Omega)$

$$Q_\lambda(f) = \frac{\int_\Omega (|\nabla f|^2 + \lambda f^2) dx}{\left(\int_\Omega \frac{|f|^{2^*(s)}}{|x|^s} dx\right)^{2/2^*(s)}}.$$

Theorem 4.1. *Assume that $N \geq 5$. There exist positive constants C_1 and C_2 such that as $\lambda \to \infty$*

$$\mu^N_{s,\lambda}(\Omega) = Q_\lambda(v_\lambda) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - C_1 H(0)\varepsilon + C_2 \varepsilon^2 \lambda + o(\varepsilon^2 \lambda).$$

where $\varepsilon = O(1/\lambda)$ and $H(0)$ is the mean curvature at 0.

Proof. The approaches to prove Theorem 4.1 is very close to those in [23]. Therefore we omit the proof of Lemma 4.2 and Lemma 4.6.

Suppose that $\mathcal{N}_0$ is a neighborhood around 0 satisfying $\Omega \cap \mathcal{N}_0 = \Phi(B^+_{2\delta})$. For $y \in B^+_{2\delta}$ put $\hat{v}_\lambda$ and $\tilde{v}_\lambda$ as in Step 2 of the proof of Lemma 3.3. By using (23) and (24) we define an elliptic operator L by

$$L = \sum_{i,j=1}^N \tilde{a}_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{j=1}^N \tilde{b}_j(y) \frac{\partial}{\partial y_j},$$

where

$$\tilde{a}_{ij}(z) = \begin{cases} a_{ij}(z) & z_N \geq 0, \\ (-1)^{\delta_{iN} + \delta_{jN}} a_{ij}(z', -z_N)) & z_N < 0, \end{cases}$$

$$\tilde{b}_j(z) = \begin{cases} b_j(z) & z_N \geq 0, \\ (-1)^{\delta_{jN}} b_j(z', -z_N)\beta_{\lambda_k}) & z_N < 0. \end{cases}$$

Since v_λ is a least-energy solution of (1) $\tilde{v}_\lambda$ satisfies

$$-L\tilde{v}_\lambda + \lambda \tilde{v}_\lambda = \frac{\tilde{v}_\lambda^{2^*(s)-1}}{|\Phi(y)|^s} \tag{29}$$

a.e. in $B_{2\delta}$. Set

$$\langle \nabla\phi, \nabla\psi \rangle_g = \sum_{i,j=1}^{N} \int_{B_\delta(0)} a_{ij}(y) \left(\frac{\partial \phi}{\partial y_j}(y) \frac{\partial \psi}{\partial y_k}(y) \right) |J\Phi| dy,$$

$$\langle \phi, \psi \rangle_\lambda = \langle \nabla\phi, \nabla\psi \rangle_g + \lambda \int_{B_\delta(0)} \phi\psi |J\Phi| dy,$$

$$\|\nabla\phi\|_g^2 = \langle \nabla\phi, \nabla\phi \rangle_g, \quad \|\phi\|_\lambda^2 = \langle \phi, \phi \rangle_\lambda.$$

From Theorem 3.1, we have

$$\lim_{\lambda\to\infty} \|\nabla\tilde{v}_\lambda\|_g^2 = \mu_s^{\frac{N-s}{2-s}}, \quad \lim_{\lambda\to\infty} \lambda \int_{B_\delta} \tilde{v}_\lambda^2 |J\Phi| dy = 0, \quad \lim_{\lambda\to\infty} \|\nabla\tilde{v}_\lambda - \nabla U_{\beta_\lambda}\|_g = 0.$$

Define the projection $P : H^1(B_\delta) \to H_0^1(B_\delta)$ by $u = Pv$ such that

$$Lu = Lv.$$

By the definition of L if $v(y', y_N) = v(y', -y_N)$ then $u(y', y_N) = u(y', -y_N)$. We set

$$h_\lambda = v_\lambda - Pv_\lambda, \quad \phi_\varepsilon = U_\varepsilon - PU_\varepsilon$$

and we can see by part (v) of Theorem 3.1 and the maximum principle

$$0 < h_\lambda = O(\varepsilon^{-\gamma\sqrt{\lambda}}) \quad \text{in } \overline{B}_\delta.$$

We can see

$$\phi_\varepsilon = \varepsilon^{\frac{N-2}{2}} \left(\varepsilon^{2-s} + \frac{\delta^{2-s}}{(N-s)(N-2)} \right)^{-\frac{N-2}{2-s}}.$$

Let

$$M = \{cPU_\varepsilon | c \in \mathbb{R}_+, 0 < \varepsilon \leq 1\}, \quad \mathrm{dist}(u, M) = \inf_{\phi\in M} \|u - \phi\|_\lambda,$$

and

$$\mathcal{E}(\varepsilon,\lambda)=\left\{\phi\in H_0^1(\Omega)\middle|\langle\phi,PU_\varepsilon\rangle_\lambda=\left\langle\phi,\frac{\partial}{\partial\varepsilon}PU_\varepsilon\right\rangle_\lambda=0\right\}.$$

We obtain the following lemma.

Lemma 4.2. *Suppose that $N\geq 5$. Then for λ sufficiently large* $\operatorname{dist}(Pv_\lambda,M)$ *is attained by $c_\lambda PU_\varepsilon$, where $\varepsilon=\varepsilon(\lambda)$. Moreover,*

$$\frac{\varepsilon}{\beta_\lambda}\to 1\quad \textit{and}\quad c_\lambda\to 1$$

as $\lambda\to\infty$.

By this lemma we may write

$$Pv_\lambda=c_\lambda PU_\varepsilon+\omega_\lambda$$

where $\omega_\lambda\in\mathcal{E}(\varepsilon,\lambda)$ satisfying $\|\omega_\lambda\|_\lambda=o(1)$, $\|Pv_\lambda\|_\lambda^2=c_\lambda^2\|PU_\varepsilon\|_\lambda^2+\|\omega_\lambda\|_\lambda^2$. Thus

$$v_\lambda=c_\lambda PU_\varepsilon+\omega_\lambda+h_\lambda.$$

We investigate the detail of the estimates for ω_λ.

Lemma 4.3. *We assume that $N\geq 5$ and $\varepsilon=\varepsilon(\lambda)$ is given in Lemma 4.2. Then there exists $\sigma>0$ and λ_0 such that for all $\omega\in\mathcal{E}(\varepsilon,\lambda)$ and $\lambda>\lambda_0$ we have*

$$(2^*(s)-1+\sigma)\int_{B_\delta}\frac{U_\varepsilon^{2^*(s)-2}\omega^2}{|\Phi(y)|^s}|J\Phi|dy\leq\|\omega\|_\lambda^2.$$

Proof. Suppose the above lemma does not hold. Then there exist sequences $\lambda_n\to\infty$, $\{\omega_n\}\subset\mathcal{E}(\varepsilon_n,\lambda_n)$ such that

$$(2^*(s)-1+o(1))\int_{B_\delta}\frac{U_\varepsilon^{2^*(s)-2}\omega_n^2}{|\Phi(y)|^s}|D\Phi|dy\geq\|\omega_n\|_{\lambda_n}^2$$

where $\varepsilon_n=\varepsilon(\lambda_n)$. We may assume that $\|\omega_n\|_{\lambda_n}=1$ without loss of generality. Define $\psi_n(z)=\varepsilon_n^{(N-2)/2}\omega_n(\varepsilon_n z)$ for $z\in B_\delta/\varepsilon_n$. Then we have

$$1\leq(2^*(s)-1+o(1))\int_{B_{\delta/\varepsilon_n}}\frac{U^{2^*(s)-2}\psi_n^2}{|\frac{\Phi(\varepsilon_n z)}{\varepsilon_n}|^s}|D\Phi(\varepsilon_n z)|dz \tag{30}$$

On the other hand we have

$$
\begin{aligned}
1 &= \|\omega_n\|^2_{\lambda_n} \\
&\geq \sum_{i,j} \int_{B_\delta} a_{ij}(y) \left(\frac{\partial \omega_n}{\partial y_i}(y) \frac{\partial \omega_n}{\partial y_j}(y) \right) |D\Phi| dy + \lambda_n \int_{B_\delta} \omega_n^2 |D\Phi| dy \\
&\geq (1 + o(1)) \int_{B_{\delta/\varepsilon_n}} |\nabla \psi_n(z)|^2 dz \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
1 &= \|\omega_n\|^2_{\lambda_n} \\
&\geq C \left(\int_{B_\delta} \omega_n^{\frac{2N}{N-2}} |D\Phi| dy \right)^{\frac{N-2}{N}} \\
&= C(1 + o(1)) \left(\int_{B_{\delta/\varepsilon_n}} \psi_n^{\frac{2N}{N-2}} dz \right)^{\frac{N-2}{N}}.
\end{aligned}
$$

Therefore after passing to a subsequence we have

$$
\psi_n \to \psi_\infty \quad \text{weakly in } D^{1,2}_{loc}(\mathbb{R}^N), \quad \text{and} \quad \psi_n \to \psi_\infty \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N).
$$

We can see that

$$
\langle \nabla \psi_\infty, \nabla U \rangle_{L^2(\mathbb{R}^N)} = 0, \quad \left\langle \nabla \psi_\infty, \nabla \left(\frac{\partial}{\partial \lambda} \bigg|_{\lambda = 1} U_\lambda \right) \right\rangle_{L^2(\mathbb{R}^N)} = 0. \qquad (32)
$$

Moreover from (30) and (31) it follows that

$$
\int_{\mathbb{R}^N} |\nabla \psi_\infty|^2 dz \leq 1 \leq (2^*(s) - 1) \int_{\mathbb{R}^N} \frac{U^{2^*(s)-2} \psi_\infty^2}{|z|^s} dz,
$$

and hence

$$
\frac{\int_{\mathbb{R}^N} |\nabla \psi_\infty|^2 dz}{\int_{\mathbb{R}^N} \frac{U^{2^*(s)-2} \psi_\infty^2}{|z|^s} dz} \leq 2^*(s) - 1. \qquad (33)
$$

However, (32) and (33) contradict the following lemma.

Lemma 4.4 ([24])**.** *We consider the eigenvalue problem:*

$$\begin{cases} -\Delta\psi = \mu\frac{U^{2^*(s)-1}}{|z|^s}\psi & \text{in} \quad \mathbb{R}^N, \\ \psi \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{34}$$

Then the first two eigenvalues of (34) are $\mu_1 = 1$, $\mu_2 = 2^*(s) - 1$ and the corresponding eigenfunction ψ_1 and ψ_2 satisfy

$$\psi_1 \in \text{span}\{U_\varepsilon\} \quad \text{and} \quad \psi_2 \in \text{span}\left\{ \frac{d}{d\varepsilon}\Big|_{\varepsilon=1} U_\varepsilon \right\}$$

respectively.

□

Recall that $Lh_\lambda = 0$ and $h_\lambda = O(\varepsilon^{-\gamma\sqrt{\lambda}})$. Multiplying (29) by ω_λ and integrating on B_δ by parts, we have

$$\|\omega_\lambda\|_\lambda^2 + O(\varepsilon^{-\gamma\sqrt{\lambda}})\|\omega_\lambda\|_\lambda = \int_{B_\delta} \frac{(c_\lambda PU_\varepsilon + h_\lambda + w_\lambda)^{2^*(s)-1} w_\lambda}{|\Phi(y)|^s}|D\Phi|dy.$$

For the right hand side we have

$$\begin{aligned} &\int_{B_\delta} \frac{(c_\lambda PU_\varepsilon + h_\lambda + w_\lambda)^{2^*(s)-1} w_\lambda}{|\Phi(y)|^s}|D\Phi|dy \\ = \;& c_\lambda^{2^*(s)-1}\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy \\ &+(2^*(s)-1)c_\lambda^{2^*(s)-2}\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s}|D\Phi|dy + O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda) \end{aligned}$$

where $\sigma = \min\{3, 2^*(s)\}$. Thus we have

$$\begin{aligned} &\|\omega_\lambda\|_\lambda^2 - (2^*(s)-1)c_\lambda^{2^*(s)-2}\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s}|D\Phi|dy \\ = \;& c_\lambda^{2^*(s)-1}\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy + O(\|\omega_\lambda\|_\lambda^\sigma + \varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned} \tag{35}$$

Since $0 < PU_\varepsilon < U_\varepsilon$ and from Lemma 4.3 we have

$$\begin{aligned} \|\omega_\lambda\|_\lambda^2 \;=\; & \frac{2^*(s)-1+\sigma}{\sigma}(1+o(1))\int_{B_\delta} \frac{PU_\varepsilon^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy \\ &+O(\varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned} \tag{36}$$

Set

$$\tilde{Q}_\lambda(f) := \frac{\|f\|_\lambda^2}{\left(\int_{B_\delta} \frac{f^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy\right)^{2/2^*(s)}}.$$

Lemma 4.5.

$$\begin{aligned} Q_\lambda(v_\lambda) &= \frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(cPU) \\ &\quad -(1+o(1))\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s^{-\frac{N-2}{2-s}} \int_{B_\delta} \frac{PU^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy \\ &\quad +O(e^{-\sqrt{\lambda}}\|\omega_\lambda\|_\lambda). \end{aligned}$$

Proof. From Theorem 3.1 it follows

$$\begin{aligned} Q_\lambda(v_\lambda) &= \frac{\int_{\Omega\cap\mathcal{N}_0} |\nabla v_\lambda|^2 + \lambda v_\lambda^2 dx}{\left(\int_{\Omega\cap\mathcal{N}_0} \frac{v_\lambda^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}} + O(e^{-\gamma\sqrt{\lambda}}) \\ &= \frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(\tilde{v}_\lambda) + O(\varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda) \end{aligned} \tag{37}$$

Since $\tilde{v}_\lambda = c_\lambda PU_\varepsilon + \omega_\lambda + h_\lambda$ we have

$$\|\tilde{v}_\lambda\|_\lambda^2 = \|c_\lambda PU_\varepsilon\|_\lambda^2 + \|\omega_\lambda\|_\lambda^2 + O(e^{-\gamma\sqrt{\lambda}}).$$

On the other hand,

$$\begin{aligned} &\left(\int_{B_\delta} \frac{\tilde{v}_\lambda^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy\right)^{2/2^*(s)} \\ = &\left(\int_{B_\delta} \frac{(cPU)^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy\right)^{2/2^*(s)} + \frac{2}{2^*(s)}\left(\int_{B_\delta} \frac{(cPU)^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy\right)^{2/2^*(s)-1} \\ &\times \int_{B_\delta} \frac{2^*(s)\,(cPU)^{2^*(s)-1}\,\omega_\lambda + \frac{2^*(s)(2^*(s)-1)}{2}\,(cPU)^{2^*(s)-2}\,\omega_\lambda^2}{|\Phi(y)|^s}|D\Phi|dy \\ &+O(\|\omega_\lambda\|_\lambda^\sigma + e^{-\sqrt{\lambda}}). \end{aligned}$$

Hence we obtain

$$\begin{aligned}
& Q_\lambda(v_\lambda) \\
= \; & \frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(cPU_\varepsilon)\Bigg[1+(1+o(1)) \\
& \times\left\{\frac{\|\omega_\lambda\|_\lambda^2}{\|cPU_\varepsilon\|_\lambda^2}-2\frac{\int_{B_\delta}\frac{PU_\varepsilon^{2^*(s)-1}\omega}{|\Phi(y)|^s}|D\Phi|dy}{c\int_{B_\delta}\frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy}-(2^*(s)-1)\frac{\int_{B_\delta}\frac{PU_\varepsilon^{2^*(s)-2}\omega_\lambda^2}{|\Phi(y)|^s}|D\Phi|dy}{c^2\int_{B_\delta}\frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy}\right\}\Bigg] \\
& +O(\|\omega_\lambda\|_\lambda^\sigma+\varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda).
\end{aligned}$$

Using (35), (36), (37), $c_\lambda = 1+o(1)$, and

$$\lim_{\lambda\to\infty}\|PU_\varepsilon\|_\lambda^2=\lim_{\lambda\to\infty}\int_{B_\delta}\frac{PU_\varepsilon^{2^*(s)}}{|\Phi(y)|^s}|D\Phi|dy=\mu_s^{\frac{N-s}{2-s}},$$

we obtain

$$\begin{aligned}
Q_\lambda(v_\lambda) \; = \; & \frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(cPU) \\
& -(1+o(1))\left(\frac{1}{2}\right)^{\frac{2-s}{N-s}}\mu_s^{-\frac{N-2}{2-s}}\int_{B_\delta}\frac{PU^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy \\
& +O(\|\omega_\lambda\|_\lambda^\sigma+\varepsilon^{-\gamma\sqrt{\lambda}}\|\omega_\lambda\|_\lambda).
\end{aligned}$$

□

Lemma 4.6.

$$\|\omega_\lambda\|_\lambda = O(e^{-\sqrt{\lambda}})+\begin{cases}O\left(\varepsilon+\lambda\varepsilon^2\right) & (N\geq 7)\\ o(\lambda\varepsilon) & (N=5,6).\end{cases}$$

and

$$\int_{B_\delta}\frac{PU^{2^*(s)-1}\omega_\lambda}{|\Phi(y)|^s}|D\Phi|dy=\begin{cases}O(\varepsilon^2+\lambda^2\varepsilon^4) & (N\geq 7)\\ o(\lambda\varepsilon^2) & (N=5,6),\end{cases}$$

Hence

$$Q_\lambda(v_\lambda)=\frac{1}{2^{\frac{2-s}{N-s}}}\tilde{Q}_\lambda(cPU)+O(e^{-\sqrt{\lambda}})+\begin{cases}O\left(\varepsilon^2+\lambda^2\varepsilon^4\right) & (N\geq 7)\\ o(\lambda\varepsilon^2) & (N=5,6).\end{cases}$$

To end the proof of Theorem 4.1 we calculate $\tilde{Q}_\lambda(cPU)$. Note that v_λ exists and

$$Q_\lambda(v_\lambda) < \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s \tag{38}$$

when Ω satisfies $H(0) > 0$. We replacing $c_\lambda PU_\varepsilon$ by ϕ in (6), (7) and (8). Consequently by using (38) we have

$$Q_\lambda(v_\lambda) = \left(\frac{1}{2}\right)^{\frac{2-s}{N-s}} \mu_s - C_1 H(0)\varepsilon + C_2\lambda\varepsilon^2 + o(\lambda\varepsilon^2), \quad \varepsilon = O\left(\frac{1}{\lambda}\right).$$

□

5. Minimization problem

Theorem 5.1. *Assume that $N \geq 5$ and Ω satisfies $H(0) \leq 0$. Then there exist $\lambda_* = \lambda_*(\Omega)$ such that*

(i) If $0 < \lambda < \lambda_*$ then $\mu_{s,\lambda}^N(\Omega)$ is attained.
(ii) If $\lambda > \lambda_*$ then $\mu_{s,\lambda}^N(\Omega)$ is not attained.

Proof. By Theorem 4.1 the minimizer of $\mu_s^N(\Omega)$ does not exist for λ sufficiently large (if the minimizer exists, $\mu_{s,\lambda}^N(\Omega) > \mu_s/2^{(2-s)/(N-s)}$ and this contradicts (ii) in Lemma 2.1). Thus there exists $\lambda_* = \lambda_*(\Omega)$ such that part (i) of Lemma 2.2 holds true as $\tilde{\lambda} = \lambda_*$. Consequently from Lemma 2.4 we can prove (i) and (ii) immediately. □

The following theorem holds for all domains (we don't require the condition of the mean curvature at 0).

Theorem 5.2. *There exist $\lambda_0 > 0$ such that if $\lambda < \lambda_0$ then the minimizer of $\mu_{s,\lambda}^N(\Omega)$ is unique.*

Proof. In order to prove this theorem we argue in the same way as [27]. Assume that v_λ is a least-energy solution of (1). Then

$$\int_\Omega \frac{v_\lambda^{2^*(s)}}{|x|^s} dx = \mu_{s,\lambda}^N(\Omega)^{\frac{N-s}{2-s}} \to 0 \quad \text{as} \quad \lambda \to 0.$$

From Lemma 3.4 we have $\|v_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.

Set $\lambda_i \to 0$ as $i \to \infty$. Let u_i, v_i be the least-energy solutions of (1) when $\lambda = \lambda_i$ such that $\|u_i - v_i\|_{L^\infty(\Omega)} \neq 0$. Define $A_i = \|u_i - v_i\|_{L^\infty(\Omega)}$ and $z_i = A_i^{-1}(u_i - v_i)$. Then z_i satisfies $0 \leq z_i \leq 1$ in Ω, $\|z_i\|_{L^\infty(\Omega)} = 1$, and

$$\begin{cases} -\Delta z_i + \lambda z_i = \frac{u_i^{2^*(s)-1} - v_i^{2^*(s)-1}}{(u_i - v_i)|x|^s} z_i & \text{in} \quad \Omega, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Note that by the mean value theorem, we can see that

$$\frac{u_i^{2^*(s)-1} - v_i^{2^*(s)-1}}{(u_i - v_i)|x|^s} \to 0 \quad \text{as} \quad i \to \infty.$$

Thus by the elliptic regularity theory there exists $z_0 \in C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$ such that $z_i \to z_0$ in $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$ and

$$\begin{cases} -\Delta z_0 = 0 & \text{in} \quad \Omega, \\ \frac{\partial z_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Hence $z_0 \equiv 1$ since $\|z_i\|_{L^\infty(\Omega)} = 1$ for all i.

On the other hand, since u_i and v_i are solutions of (1) we have

$$\int_\Omega \frac{u_i^{2^*(s)-2} - v_i^{2^*(s)-2}}{|x|^s} u_i v_i dx = 0.$$

Since $u_i > 0$ and $v_i > 0$ we see $u_i - v_i$ changes the sign for all i. This is a contradiction. □

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