ON DESCRIPTIONS OF PRODUCTS OF SIMPLICES

LI YU AND MIKIYA MASUDA

Abstract. We give several new criteria to judge whether a simple convex polytope in a Euclidean space is combinatorially equivalent to a product of simplices. These criteria are mixtures of combinatorial, geometrical and topological conditions that are inspired by the ideas from toric topology.

1. Background

An n-dimensional convex polytope P^n is the convex hull of a finite set of points in a Euclidean space $\mathbb{R}^d$. Any codimension-one face of P^n is called a *facet* of P^n. We call P^n *simple* if each vertex of P^n is the intersection of exactly n facets of P^n. Two convex polytopes P^n and Q^n are *combinatorially equivalent* if their face lattices are isomorphic. Topologically, combinatorial equivalence corresponds to the existence of a (piecewise linear) homeomorphism between the two polytopes that restricts to homeomorphisms between their facets, and hence all their faces (see [14, Chapter 2.2]).

In this paper, we will give several new criteria to judge whether a simple convex polytope is combinatorially equivalent to product of simplices (Theorem 2.2 and Theorem 2.7). Some of these criteria are purely combinatorial, while others are phrased in geometrical or topological terms. These criteria are mainly inspired by the ideas from toric topology. So in the following we first explain some basic constructions and facts in toric topology that are relevant to our discussion.

An *abstract simplicial complex* on a set $[m] = \{v_1, \cdots, v_m\}$ is a collection K of subsets $\sigma \subseteq [m]$ such that if $\sigma \in K$, then any subset of σ also belongs to K. We always assume that the empty set belongs to K and refer to $\sigma \in K$ as a *simplex* of K. In particular, any element of $[m]$ is called a *vertex* of K. To avoid ambiguity in our argument, we also use $V(K)$ and $V(\sigma)$ to refer to the vertex sets of K and any simplex σ in K.

Key words and phrases. Convex polytope, product of simplices, moment-angle complex.

2010 *Mathematics Subject Classification.* 52B11, 52B70, 53C21, 53C23, 57S17.

This work is partially supported by the NSFC (Grant No.11371188) and the PAPD (priority academic program development) of Jiangsu higher education institutions.

Any finite abstract simplicial complex K admits a *geometric realization* in some Euclidean space. But sometimes we also use K to denote its geometric realization when the meaning is clear in the context.

Given a finite abstract simplicial complex K on a set $[m]$ and a pair of spaces (X, A) with $A \subset X$, we can construct of a topological space $(X, A)^K$ by:

$$(X,A)^K = \bigcup_{\sigma \in K} (X,A)^\sigma, \text{ where } (X,A)^\sigma = \prod_{v_j \in \sigma} X \times \prod_{v_j \notin \sigma} A. \tag{1}$$

Here $\prod$ means Cartesian product. So $(X, A)^K$ is a subspace of the Cartesian product of m copies of X. It is called the *polyhedral product* or the *generalized moment-angle complex* of K and (X, A). In particular, $\mathcal{Z}_K = (D^2, S^1)^K$ and $\mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K$ are called the *moment-angle complex* and *real moment-angle complex* of K, respectively (see [3, Section 4.1]). The natural actions of $(\mathbb{Z}_2)^m$ on $(D^1)^m$ and $(S^1)^m$ on $(D^2)^m$ induce *canonical actions* of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_K$ and $(S^1)^m$ on $\mathcal{Z}_K$, respectively.

When K is the boundary of the dual of a simple convex polytope P, the $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ are closed manifolds, also denoted by $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ respectively. In this case, $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ are called *moment-angle manifold* and *real moment-angle manifold* of P, respectively. These manifolds can be constructed in another way as described below (see [6, Construction 4.1]).

Let P^n be an n-dimensional simple convex polytope. Let $\mathcal{F}(P^n) = \{F_1, \cdots, F_m\}$ be the set of facets of P^n. Let $\{e_1, \cdots, e_m\}$ be a basis of $(\mathbb{Z}_2)^m$ and define a map $\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^m$ by $\lambda(F_i) = e_i$. Then we can construct a space

$$M(P^n, \lambda) := P^n \times (\mathbb{Z}_2)^m / \sim \tag{2}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_p^\lambda$ where G_p^λ is the subgroup of $(\mathbb{Z}_2)^m$ generated by the set $\{\lambda(F_i) \,|\, p \in F_i\}$. Let $\pi_\lambda : M(P^n, \lambda) \to P^n$ be the quotient map. One can show that $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to $M(P^n, \lambda)$ and the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_{P^n}$ can be written on $M(P^n, \lambda)$ as:

$$g' \cdot [(p, g)] = [(p, g' + g)], \ p \in P^n, \ g, g' \in (\mathbb{Z}_2)^m. \tag{3}$$

The moment-angle manifold $\mathcal{Z}_{P^n}$ can be similarly constructed from P^n and a map $\Lambda : \mathcal{F}(P^n) \to \mathbb{Z}^m$ where $\{\Lambda(F_1), \cdots, \Lambda(F_m)\}$ is a unimodular basis of $\mathbb{Z}^m$.

In addition, $\mathbb{R}\mathcal{Z}_{P^n}$ and $\mathcal{Z}_{P^n}$ are smooth manifolds. In fact, there is a unique equivariant smooth structure on $\mathbb{R}\mathcal{Z}_{P^n}$ (or $\mathcal{Z}_{P^n}$) with respect to the canonical $(\mathbb{Z}_2)^m$-action (or $(S^1)^m$-action) and, the orbit space $\mathbb{R}\mathcal{Z}_{P^n}/(\mathbb{Z}_2)^m$ (or $\mathcal{Z}_{P^n}/(S^1)^m$) with the stratification determined by the canonical $(\mathbb{Z}_2)^m$-action (or $(S^1)^m$-action) is diffeomorphic to the convex polytope P^n as a smooth manifold with corners (see [9, Proposition 3.8] and [12, Theorem 5.6]). Moreover, for any proper face f of P^n, $\pi_\lambda^{-1}(f)$ is an embedded closed smooth submanifold of $\mathbb{R}\mathcal{Z}_{P^n}$ which is the fixed point set of the subgroup of $(\mathbb{Z}_2)^m$ generated by $\{\lambda(F_i) \mid f \in F_i\}$ under the canonical $(\mathbb{Z}_2)^m$-action.

2. Descriptions of products of simplices

For any $k \in \mathbb{N}$, let Δ^k denote the standard k-dimensional simplex, which is

$$\Delta^k = \{(x_1, \cdots, x_k, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1 + \cdots + x_{k+1} = 1, x_1, \cdots, x_{k+1} \geq 0\}.$$

For any $n_1, \cdots, n_r \in \mathbb{N}$, consider $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ as a product of $\Delta^{n_1}, \cdots, \Delta^{n_r}$ in the Cartesian product $\mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_r+1}$.

Next, we first list some descriptions of products of simplices that appeared in Wiemeler's paper [13].

Theorem 2.1 (Wiemeler [13])**.** *Let P^n be an n-dimensional simple convex polytope with m facets, $n \geq 3$. Then the following statements are equivalent.*

(**a**) *P^n is combinatorially equivalent to a product of simplices.*

(**b**) *Any 2-dimensional face of P^n is either a 3-gon or a 4-gon.*

(**c**) *There exists a quasitoric manifold M^{2n} over P^n which admits a nonnegatively curved Riemannian metric that is invariant under the canonical $(S^1)^n$-action on M^{2n}.*

A *quasitoric manifold* M^{2n} over P^n is the quotient space of $\mathcal{Z}_{P^n}$ under a free action of a rank $m-n$ toral subgroup of $(S^1)^m$ (see [6]). There is a canonical $(S^1)^n$-action on M^{2n} induced from the canonical action of $(S^1)^m$ on $\mathcal{Z}_{P^n}$, which makes M^n a *torus manifold* (see [7]).

Theorem 2.1(**b**) is a corollary of [13, Proposition 4.5] and Theorem 2.1(**c**) is a corollary of [13, Lemma 4.2]. Note that Theorem 2.1(**b**) is a particularly useful description of products of simplices. Indeed, the proofs of many other descriptions of products of simplices in this paper boil down to this one first. But the proof of [13, Proposition 4.5] is a little long and not particularly easy to follow. We will give a shorter proof of Theorem 2.1(**b**) in the appendix to make our paper more self-contained.

Next, we give more descriptions of products of simplices from combinatorial and topological viewpoints. For convenience let us introduce some notations first.

- For any topological space X and any field $\mathbf{k}$, let

$$\mathrm{hrk}(X; \mathbf{k}) = \sum_{i=0}^{\infty} \dim_{\mathbf{k}} H^i(X; \mathbf{k}).$$

- For a simplical complex K and a subset I of the vertex set $V(K)$ of K, let K_I denote the *full subcomplex* of K obtained by restricting to I.

In addition, for a simplicial complex K on $[m] = \{v_1, \cdots, v_m\}$, we can define a new simplicial complex $L(K)$ from K, called the *double of* K, where $L(K)$ is a simplicial complex on the vertex set $[2m] = \{v_1, v_1', \cdots, v_m, v_m'\}$ determined by the following condition: $\omega \subset [2m]$ is a minimal (by inclusion) missing simplex of $L(K)$ if and only if ω is of the form $\{v_{i_1}, v_{i_1}', \cdots, v_{i_k}, v_{i_k}'\}$ where $\{v_{i_1}, \cdots, v_{i_k}\}$ is a missing simplex of K. Note that any minimal missing simplex in $L(K)$ must have even number of vertices.

The following are some basic facts about $L(K)$ (see Ustinovsky [10, 11]).

- K_1 is simplicially isomorphic to K_2 if and only if $L(K_1)$ is simplicially isomorphic to $L(K_2)$.
- $\dim(L(K)) = m + \dim(K)$ ([11, Lemma 1.2]).
- $L(K_1 * K_2) = L(K_1) * L(K_2)$ (here $*$ is the join of two simplical complexes).
- If $K = \partial P^*$ where P^* is the simplicial polytope dual to a simple convex polytope P, then $L(K) = \partial L(P)^*$ where $L(P)$ is a simple convex polytope called the *double of* P (see [10] for the construction of $L(P)$).
- $L(\partial\Delta^k) = \partial\Delta^{2k+1}$.

Theorem 2.2. *Let P be an n-dimensional simple convex polytope with m facets and let K be the boundary of the simplicial polytope dual to P. Then the following statements are all equivalent.*

(**a**) *P is combinatorially equivalent to a product of simplices.*
(**b**) *K is simplicially isomorphic to $\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_r}$ for some $n_1 \cdots, n_r \in \mathbb{N}$.*
(**c**) *$L(K)$ is simplicially isomorphic to $\partial\Delta^{l_1} * \cdots * \partial\Delta^{l_r}$ for some $l_1 \cdots, l_r \in \mathbb{N}$.*
(**d**) *For any vertex x of P, the intersection of all the facets of P that are not incident to x is nonempty (which must be exactly one face of P).*
(**e**) *For any $(n-1)$-dimensional simplex σ in K, the full subcomplex $K_{V(K)-V(\sigma)}$ of K is a nonempty simplex.*
(**f**) *For some field $\mathbf{k}$, $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_K; \mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_P; \mathbf{k}) = 2^{m-n}$.*
(**g**) *For some field $\mathbf{k}$, $\mathrm{hrk}(\mathcal{Z}_K; \mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\mathrm{hrk}(\mathcal{Z}_P; \mathbf{k}) = 2^{m-n}$.*

Proof. The equivalence of (**a**) and (**b**) is trivial.

2.1. (**b**) $\Leftrightarrow$ (**c**). If $K = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_r}$, then

$$L(K) = L(\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_r}) = L(\partial\Delta^{n_1}) * \cdots * L(\partial\Delta^{n_r}) = \partial\Delta^{2n_1+1} * \cdots * \partial\Delta^{2n_r+1}.$$

Conversely, suppose $L(K) = \partial\Delta^{l_1} * \cdots * \partial\Delta^{l_r}$. Notice that for each $1 \le j \le r$, Δ^{l_j} is a minimal missing simplex of $L(K)$. So Δ^{l_j} must have even number of vertices, which implies that l_j is an odd integer. Let $l_j = 2n_j + 1$, $1 \le j \le r$.

Then we have $L(K) = \partial\Delta^{2n_1+1} * \cdots * \partial\Delta^{2n_r+1} = L(\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_r})$. This implies $K = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_r}$. $\square$

2.2. (**a**) $\Rightarrow$ (**d**). Let $P = \Delta^{n_1} \times \cdots \times \Delta^{n_r}$ where $n_1 + \cdots + n_r = n$. For each $1 \le i \le r$, let $\{v_0^i, \cdots, v_{n_i}^i\}$ be the set of all vertices of Δ^{n_i}. Then the set of all the vertices of P can be written as

$$\{\widetilde{v}_{j_1\cdots j_r} = v_{j_1}^1 \times \cdots \times v_{j_r}^r \mid 0 \le j_i \le n_i,\ i = 1, \cdots, r\}.$$

Note that any facet of P is the product of a codimension-one face of some Δ^{n_i} and the remaining simplices. So the set of all the facets of P is (see [5])

$$\mathcal{F}(P) = \{F_{k_i}^i \mid 0 \le k_i \le n_i,\ i = 1, \cdots, r\}, \tag{4}$$

where $F_{k_i}^i = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f_{k_i}^i \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_r}$ and $f_{k_i}^i$ is the codimension-one face of the simplex Δ^{n_i} which is opposite to the vertex $v_{k_i}^i$. So there are total of $n + r$ facets in P. The n facets of P that are incident to $\widetilde{v}_{j_1\cdots j_r}$ are:

$$\mathcal{F}(\Delta^{n_1} \times \cdots \times \Delta^{n_r}) - \{F_{j_i}^i \mid i = 1, \cdots, r\}.$$

Those facets that are not incident to $\widetilde{v}_{j_1\cdots j_r}$ are $F_{j_1}^1, \cdots, F_{j_r}^r$ whose intersection $F_{j_1}^1 \cap \cdots \cap F_{j_r}^r$ is exactly one face $f_{j_1}^1 \times \cdots \times f_{j_r}^r$ (of dimension $n - r$). $\square$

2.3. (**d**) $\Leftrightarrow$ (**e**). Note that K is a $(n-1)$-dimensional simplicial sphere with m vertices where each vertex x of P corresponds to a unique $(n-1)$-simplex K, denoted by σ_x. Let W_x be the union of those facets of P that are not incident to x. The face poset of $K_{V(K)-V(\sigma_x)}$ is the reverse poset of the face poset of W_x. Then since the polytope P is simple, if the intersection of all the facets in W_x is nonempty, it must be exactly one face of P (which is equivalent to saying that $K_{V(K)-V(\sigma_x)}$ is a simplex). So (**d**) is equivalent to (**e**). $\square$

2.4. (**d**) $\Rightarrow$ (**a**). We first prove the following claim.

Claim: If the condition (**d**) holds for P, then (**d**) also holds for any facet of P.

Indeed, each facet F of P corresponds to a unique vertex v_F of K. Let $\mathrm{Star}_K(v_F)$ and $\mathrm{link}_K(v_F)$ denote the star and the link of v_F in K. Then $\mathrm{link}_K(v_F)$ is an $(n-2)$-dimensional simplicial sphere which is the boundary of the simplicial polytope dual to F. For any $(n-2)$-simplex $\tau \in \mathrm{link}_K(v_F)$, there are exactly two $(n-1)$-simplices σ_1, σ_2 in K that contains τ. We let $V(\sigma_1) = V(\tau) \cup \{v_F\}$ and $V(\sigma_2) = V(\tau) \cup \{u\}$ for some vertex u of K. Since the condition (**d**) and (**e**) are equivalent, by applying (**e**) to K we deduce that $K_{V(K)-V(\sigma_2)}$ is a simplex in K, denoted by ξ. So

$$V(K) = V(\xi) \cup V(\sigma_2) = V(\xi) \cup V(\tau) \cup \{u\}. \tag{5}$$

Notice that $v_F \notin \sigma_2$, so ξ contains v_F and hence $\xi \subset \mathrm{Star}_K(v_F)$. Let L_τ denote the full subcomplex of $\mathrm{link}_K(v_F)$ restricting to $V(\mathrm{link}_K(v_F)) - V(\tau)$.

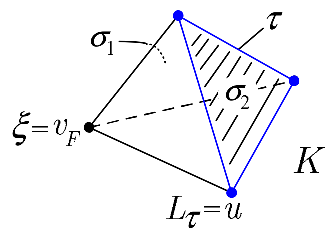

FIGURE 1. The boundary of a simplex

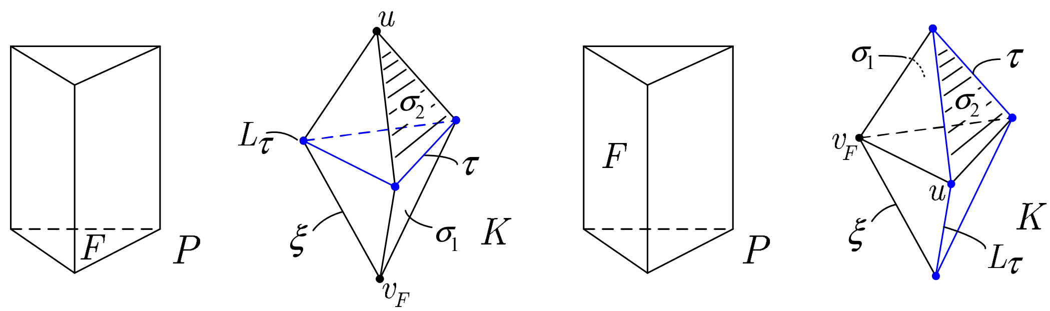

FIGURE 2

- If $\dim(\xi) = 0$, then $\xi = \{v_F\}$. This implies that the number of vertices of K is $n+1$. So P must be an n-simplex since P is a simple convex polytope of dimension n. In this case $L_\tau = \{u\}$ (See Figure 1).
- If $\dim(\xi) \geq 1$, then $\xi \cap \mathrm{link}_K(v_F)$ is a nonempty simplex in $\mathrm{link}_K(v_F)$ with dimension $\dim(\xi) - 1$. We have two possible cases as follows.
  - If $u \notin \mathrm{link}_K(v_F)$, the vertex set of L_τ is
  $$V(L_\tau) = V(\xi) - \{v_F\} = V(\xi \cap \mathrm{link}_K(v_F)) \quad \text{(by (5))}.$$
  Then since L_τ is a full subcomplex of $\mathrm{link}_K(v_F)$ on $V(L_\tau)$, we have $L_\tau = \xi \cap \mathrm{link}_K(v_F)$ is a nonempty simplex (see the left two pictures in Figure 2).
  - If $u \in \mathrm{link}_K(v_F)$, the vertex set of L_τ is
  $$V(L_\tau) = (V(\xi) - \{v_F\}) \cup \{u\} = V(\xi \cap \mathrm{link}_K(v_F)) \cup \{u\} \quad \text{(by (5))}.$$
  Note that both the simplex $\xi \cap \mathrm{link}_K(v_F)$ and u belong to $K_{V(K)-V(\sigma_1)}$. Then since $K_{V(K)-V(\sigma_1)}$ is a simplex by the condition (**e**), the simplex $\xi \cap \mathrm{link}_K(v_F)$ and u spans a unique simplex (of dimension $\dim(\xi)$) which must coincide with L_τ (see the right two pictures in Figure 2).

So we have shown that the condition (**e**) holds for $\mathrm{link}_K(v_F)$. Dually it means that the condition (**d**) holds for the facet F. So the above claim is proved.

By iterating the above argument, we deduce that the condition (**d**) holds for all two dimensional faces of P. This forces any 2-dimensional face of P to be either a 3-gon or a 4-gon. Then by Theorem 2.1(**b**), the polytope P is combinatorially equivalent to a product of simplices. $\square$

2.5. **(a) $\Rightarrow$ (f) and (g).** If $P = \Delta^{n_1} \times \cdots \times \Delta^{n_r}$, $n_1 + \cdots + n_r = n$, then

$$\mathcal{Z}_P = S^{2n_1+1} \times \cdots \times S^{2n_r+1}, \quad \mathbb{R}\mathcal{Z}_P = S^{n_1} \times \cdots \times S^{n_r}.$$

The number of facets of P is $m = n + r$. It is clear that for any field $\mathbf{k}$,

$$\mathrm{hrk}(\mathcal{Z}_P; \mathbf{k}) = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_P; \mathbf{k}) = 2^r = 2^{m-n}.$$

2.6. **(f) $\Rightarrow$ (a).** For any vertex v of K, let m_v be the number of vertices in $\mathrm{link}_K(v)$. According to the proof of [11, Theorem 3.2] (note that the argument there works for any coefficient), there is a subspace X of $\mathbb{R}\mathcal{Z}_K$ so that

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_P; \mathbf{k}) = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_K; \mathbf{k}) \geq \mathrm{hrk}(X; \mathbf{k}),$$

and X is the disjoint union of the 2^{m-m_v-1} copies of $\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)}$. So we have

$$2^{m-n} = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_K; \mathbf{k}) \geq 2^{m-m_v-1}\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)}; \mathbf{k}).$$

Then $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)}; \mathbf{k}) \leq 2^{m_v-n+1}$. On the other hand, [11, Theorem 3.2] tells us that $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)}; \mathbf{k}) \geq 2^{m_v-n+1}$ (since $\dim(\mathrm{link}_K(v)) = n-2$). So we obtain

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)}; \mathbf{k}) = 2^{m_v-n+1}.$$

Note if v is the vertex corresponding to a facet F of P, then $\mathbb{R}\mathcal{Z}_{\mathrm{link}_K(v)} = \mathbb{R}\mathcal{Z}_F$. So we have shown that if the condition (**f**) holds for P, it should hold for any facet of P as well.

By iterating the above argument, we deduce that the condition (**f**) holds for all the two dimensional faces of P. It is not hard to see that the real moment-angle manifold of a k-gon is a closed connected orientable surface with genus $1+(k-4)2^{k-3}$ (see [3, Proposition 4.1.8]). So any 2-dimensional face of P is either a 3-gon or a 4-gon. Then by Theorem 2.1(**b**), the polytope P is combinatorially equivalent to a product of simplices. $\square$

2.7. **(g) $\Rightarrow$ (a).** First of all, [11, Lemma 2.2] says that there is a homeomorphism $\mathcal{Z}_K \cong \mathbb{R}\mathcal{Z}_{L(K)}$. Since K has m vertices, $\dim(L(K)) = m + \dim(K) = m + n - 1$. So if $\mathrm{hrk}(\mathcal{Z}_K; \mathbf{k}) = 2^{m-n}$, we have

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{L(K)}; \mathbf{k}) = 2^{m-n} = 2^{2m-(m+n-1)-1} = 2^{2m-\dim(L(K))-1}.$$

So (**f**) holds for the simplicial sphere $L(K)$. Since we have already shown (**f**) $\Rightarrow$ (**a**) and (**a**) $\Rightarrow$ (**b**), $L(K)$ satisfies (**b**). Then K itself satisfies (**b**) by the equivalence of (**b**) and (**c**). So P is combinatorially equivalent to a product of simplices. $\square$

Remark 2.3. The equivalences of (**b**), (**f**) and (**g**) in Theorem 2.2 are stated in [3, Section 4.8] as an exercise.

Remark 2.4. Generally speaking, the number of 2-dimensional faces in a simple convex polytope P is a lot more than the number of vertices and facets of P. So in practice if we write an algorithm to judge whether P is combinatorially equivalent to a product of simplices, using Theorem 2.2(**d**) or (**e**) should be more efficient than using Theorem 2.1(**b**).

Next, we give some descriptions of products of simplices in terms of geometric conditions on real moment-angle manifolds of simple convex polytopes. We first recall a concept in metric geometry (see [4, Definition 3.1.12]).

Definition 2.5 (Quotient Metric Space)**.** Let (X, d) be a metric space and let $\mathcal{R}$ be an equivalence relation on X. The quotient semi-metric $d_{\mathcal{R}}$ is defined as

$$d_{\mathcal{R}}(x,y) = \inf\Big\{ \sum_{i=1}^{k} d(p_i, q_i) \,:\, p_1 = x,\, q_k = y,\, k \in \mathbb{N} \Big\},$$

where the infimum is taken over all choices of $\{p_i\}$ and $\{q_i\}$ such that the point q_i is $\mathcal{R}$-equivalent to p_{i+1} for all $i = 1, \cdots, k-1$. Moreover, by identifying points with zero $d_{\mathcal{R}}$-distance, we obtain a metric space $(X/\mathcal{R}, d)$ called the *quotient metric space* of (X, d).

Suppose P is a simple convex polytope in a Euclidean space $\mathbb{R}^d$. Consider P to be equipped with the *intrinsic metric*. More precisely, the intrinsic metric on P defines the distance between any two points x and y in P to be the infimum of lengths of piecewise smooth paths in P that connect x and y. Note that the intrinsic metric on P coincides with the subspace metric on P since P is convex.

By the construction in (2), $\mathbb{R}\mathcal{Z}_P = M(P, \lambda)$ is a closed manifold obtained by gluing 2^m copies of P along their facets. We can assume that the 2^m copies of P are congruent convex polytopes inside the same Euclidean space and the gluings of their facets are all isometries. Then by Definition 2.5 we obtain a quotient metric on $\mathbb{R}\mathcal{Z}_P$, denoted by d_P. It is clear that the metric d_P is invariant with respect to the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_P$ (see (3)).

Remark 2.6. We can also call $(\mathbb{R}\mathcal{Z}_P, d_P)$ a *Euclidean polyhedral space*, which just means that it is built from Euclidean polyhedra (see [4, Definition 3.2.4]).

Note that if P' is another simple convex polytope combinatorially equivalent to P but not congruent to P, the two metric spaces $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ and $(\mathbb{R}\mathcal{Z}_P, d_P)$ are not isometric in general (though $\mathbb{R}\mathcal{Z}_{P'}$ is homeomorphic to $\mathbb{R}\mathcal{Z}_P$).

Theorem 2.7. *Let P be an n-dimensional simple convex polytope, $n \geq 2$, with m facets. Then the following statements are all equivalent.*

(**a**) *P is combinatorially equivalent to a product of simplices.*
(**b**) *There exists a non-negatively curved Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ that is invariant under the canonical $(\mathbb{Z}_2)^m$-action on $\mathbb{R}\mathcal{Z}_P$.*
(**c**) *There exists a simple convex polytope P' combinatorially equivalent to P so that the metric space $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is non-negatively curved.*
(**d**) *There exists a simple convex polytope P' combinatorially equivalent to P so that all the dihedral angles of P' are non-obtuse.*

Note that a Riemannian metric on a manifold is non-negatively curved means that its sectional curvature is everywhere non-negative, while a metric space being non-negatively curved is defined via comparison of triangles (see [4, Section 4]).

Proof. (**a**) $\Rightarrow$ (**b**) The real moment-angle manifold of a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ is diffeomorphic to a product of standard spheres $S^{n_1} \times \cdots \times S^{n_r}$ where $S^k = \{(x_1, \cdots, x_{k+1}) \in \mathbb{R}^{k+1} \,|\, x_1^2 + \cdots + x_{k+1}^2 = 1\}$ for any $k \in \mathbb{N}$. Let S^k be equipped with the induced Riemannian metric from $\mathbb{R}^{k+1}$. Then it is easy to check that $S^{n_1} \times \cdots \times S^{n_r}$ is a nonnegatively curved Riemannian manifold with respect to the product of the Riemannian metrics on $S^{n_1}, \cdots, S^{n_r}$.

(**b**) $\Rightarrow$ (**a**) Recall the definition of $\pi_\lambda : M(P, \lambda) = \mathbb{R}\mathcal{Z}_P \to P$ in (2). For any proper face f of P, let $M_f = \pi_\lambda^{-1}(f)$. It is easy to see the following.

- M_f is an embedded closed submanifold of $\mathbb{R}\mathcal{Z}_P$ which has $2^{m+\dim(f)-n-m_f}$ connected components, where m_f is the number of facets of f.
- Each connected component of M_f is diffeomorphic to $\mathbb{R}\mathcal{Z}_f$.

Note that M_f is the fixed point set of a rank $n - \dim(f)$ subgroup of $(\mathbb{Z}_2)^m$ under the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_P$. Then since the Riemannian metric is $(\mathbb{Z}_2)^m$-invariant, each component of M_f is a totally geodesic submanifold of $\mathbb{R}\mathcal{Z}_P$ (see [8, Theorem 5.1]), and so is non-negatively curved with respect to the induced Riemannian metric from $\mathbb{R}\mathcal{Z}_P$. This implies that the condition (**b**) holds for $\mathbb{R}\mathcal{Z}_f$ as well.

In particular when $\dim(f) = 2$, the $\mathbb{R}\mathcal{Z}_f$ is a closed connected surface with non-negatively curved Riemannian metric. Then by Gauss-Bonnet Theorem, the Eular characteristic $\chi(\mathbb{R}\mathcal{Z}_f) \geq 0$, which implies that f has to be a 3-gon or a 4-gon. Then by Theorem 2.1(**b**), the polytope P is combinatorially equivalent to a product of simplices.

(**a**) $\Rightarrow$ (**c**) Suppose P is combinatorially equivalent to $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ where $n_1 + \cdots + n_r = n$. Then the number of facets of P is $m = n + r$. Consider the standard simplex Δ^k as a metric subspace of $\mathbb{R}^{k+1}$ with the intrinsic metric. Let $P' = \Delta^{n_1} \times \cdots \times \Delta^{n_r}$ be the product of the r metric spaces $\Delta^{n_1}, \cdots, \Delta^{n_r}$.

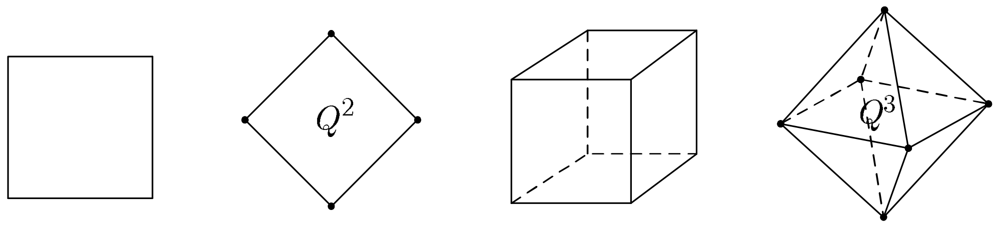

FIGURE 3. Cross-polytopes of dimension 2 and 3

Claim: As a metric space $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is isometric to the product of the r metric spaces $(\mathbb{R}\mathcal{Z}_{\Delta^{n_1}}, d_{\Delta^{n_1}}), \cdots, (\mathbb{R}\mathcal{Z}_{\Delta^{n_r}}, d_{\Delta^{n_r}})$.

Recall how we enumerate the facets of $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$ in (4). If we glue two copies of P' along the facet $F^i_{k_i} = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_{k_i} \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_r}$, we obtain $\Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times \big(\Delta^{n_i} \cup_{f^i_{k_i}} \Delta^{n_i}\big) \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_r}$. Then we decompose the gluing procedure in the construction (2) for $\mathbb{R}\mathcal{Z}_{P'}$ into r steps. The i-th step only glues those facets of the form $\{F^i_{k_i}, 0 \leq k_i \leq n_i\}$ in the 2^m copies of P', which gives us the factor $(\mathbb{R}\mathcal{Z}_{\Delta^{n_i}}, d_{\Delta^{n_i}})$, while fixing all other factors in the product. So after the first step we obtain 2^{m-n_1-1} copies of $\mathbb{R}\mathcal{Z}_{\Delta^{n_1}} \times \Delta^{n_2} \times \cdots \times \Delta^{n_r}$. After the second step we obtain $2^{m-n_1-n_2-2}$ copies of $\mathbb{R}\mathcal{Z}_{\Delta^{n_1}} \times \mathbb{R}\mathcal{Z}_{\Delta^{n_2}} \times \Delta^{n_3} \times \cdots \times \Delta^{n_r}$ and so on. Then our claim follows.

Moreover, observe that for any $k \in \mathbb{N}$, $(\mathbb{R}\mathcal{Z}_{\Delta^k}, d_{\Delta^k})$ is isometric to the boundary of the $(k+1)$-dimensional cross-polytope Q^{k+1} whose vertices are

$$\{(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0),\ (0, \cdots, 0, \overset{i}{-1}, 0, \cdots, 0)\,;\ i = 1, \cdots, k+1\}.$$

Recall that the n-dimensional *cross-polytope* is the simplicial polytope dual to the n-dimensional cube (see Figure 3 for the case $n = 2, 3$).

It is well known that the intrinsic metric on any *convex hypersurface* (i.e. the boundary of a compact convex set with nonempty interior) in a Euclidean space $\mathbb{R}^n$ $(n \geq 3)$ is non-negatively curved (see [4, p.359]). Then since Q^{k+1} is a convex polytope in $\mathbb{R}^{k+1}$, $(\mathbb{R}\mathcal{Z}_{\Delta^k}, d_{\Delta^k})$ is non-negatively curved for any $k \geq 2$. When $k = 1$, the boundary of Q^2 is a piecewise smooth simple curve in $\mathbb{R}^2$. But by definition (see [4, Definition 4.1.9]), the intrinsic metric on any piecewise smooth simple curve is non-negatively curved because any geodesic triangle on the curve is degenerate. So we can conclude that $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is non-negatively curved because the product of non-negatively curved Alexandrov spaces is still non-negatively curved (see [4, Chapter 10]).

(c) $\Rightarrow$ **(d)** If the metric $d_{P'}$ on $\mathbb{R}\mathcal{Z}_{P'}$ is non-negatively curved, we want to show that the dihedral angle between any two adjacent facets F_1 and F_2 of P' is non-obtuse. Otherwise, assume that the dihedral angle θ between F_1 and F_2 is

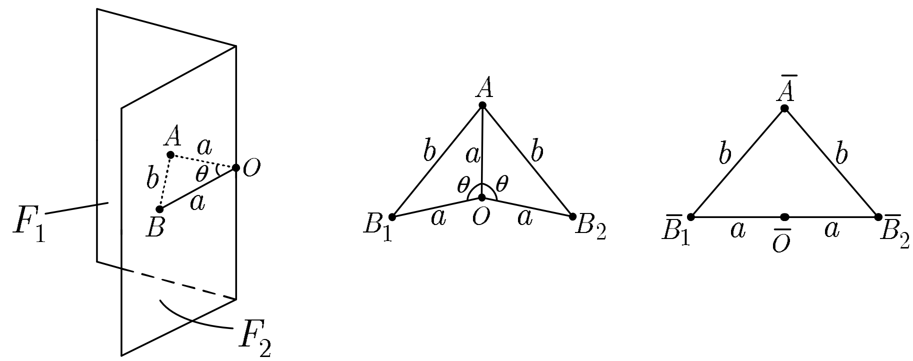

FIGURE 4. Comparison of triangles

obtuse. Choose a point O in the relative interior of $F_1 \cap F_2$, a point $A \in F_1$ and $B \in F_2$ so that the line segments $\overline{OA}$ and $\overline{OB}$ are perpendicular to $F_1 \cap F_2$. Then $\angle AOB = \theta$. Suppose the lengths of the line segaments $\overline{OA}$, $\overline{OB}$ and $\overline{AB}$ are

$$|\overline{OA}| = |\overline{OB}| = a, \quad |\overline{AB}| = b.$$

In the gluing construction (2) for $\mathbb{R}\mathcal{Z}_{P'}$, consider two copies of P' glued along the facet F_1. We then have an isosceles triangle $\triangle AB_1B_2$ in $\mathbb{R}\mathcal{Z}_{P'}$ (see Figure 4). When a is small enough, the distance between B_1 and B_2 in $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is $2a$ by the definition of the quotient metric because $\overline{B_1O} \cup \overline{OB_2}$ is the shortest path between B_1 and B_2 in $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$. Moreover, let $\triangle \bar{A}\bar{B}_1\bar{B}_2$ be a triangle in the Euclidean plane $\mathbb{R}^2$ which have the same lengths of sides as $\triangle AB_1B_2$. Then since θ is obtuse, it is clear that $\triangle AB_1B_2$ is strictly *thinner* than $\triangle \bar{A}\bar{B}_1\bar{B}_2$, i.e.

$$\angle AB_1B_2 < \angle \bar{A}\bar{B}_1\bar{B}_2, \quad \angle AB_2B_1 < \angle \bar{A}\bar{B}_2\bar{B}_1, \quad \angle B_1AB_2 < \angle \bar{B}_1\bar{A}\bar{B}_2.$$

But this contradicts our assumption that the metric $d_{P'}$ on $\mathbb{R}\mathcal{Z}_{P'}$ is non-negatively curved (see [4, Section 4.1.5]). Therefore, θ has to be non-obtuse.

(**d**) $\Rightarrow$ (**a**) Suppose F_1, F_2 and F_3 are three facets of P' with $F_1 \cap F_2 \cap F_3 \neq \varnothing$. Then $F_1 \cap F_2$ and $F_1 \cap F_3$ are codimension-one faces of F_1. By our assumption, the dihedral angles of (F_1, F_2), (F_1, F_3) and (F_2, F_3) are all non-obtuse. We claim that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in F_1 is non-obtuse as well.

Indeed, we can assume that P' sits inside $\mathbb{R}^n$ and let $\eta_i \in \mathbb{R}^n$ $(i = 1, 2, 3)$ be a normal vector of F_i pointing to the interior of P (see Figure 5). By choosing a proper coordinate system of $\mathbb{R}^n$, we can assume that $\eta_1 = (0, \cdots, 0, 1) \in \mathbb{R}^n$ and F_1 lies in the coordinate hyperplane $\{x_n = 0\} \subset \mathbb{R}^n$. Let $\eta_2 = (a_1, \cdots, a_{n-1}, a_n)$, $\eta_3 = (b_1, \cdots, b_{n-1}, b_n)$. Since the dihedral angles of (F_1, F_2), (F_1, F_3) and (F_2, F_3) are all non-obtuse, the inner products of η_1, η_2, η_3 satisfy

$$\eta_1 \cdot \eta_2 = a_n \leq 0, \quad \eta_1 \cdot \eta_3 = b_n \leq 0, \quad (\eta_2, \eta_3) = a_1b_1 + \cdots + a_{n-1}b_{n-1} + a_nb_n \leq 0.$$

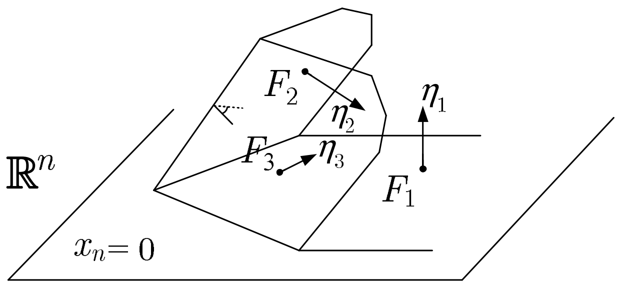

FIGURE 5. Dihedral angles of a simple convex polytope

$$\Longrightarrow \quad a_1b_1 + \cdots + a_{n-1}b_{n-1} \leq 0. \tag{6}$$

Note that $(a_1, \cdots, a_{n-1}, 0)$ and $(b_1, \cdots, b_{n-1}, 0)$ are normal vectors of $F_1 \cap F_2$ and $F_1 \cap F_3$ inside F_1 respectively. So (6) implies that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in F_1 is non-obtuse. Our claim is proved.

By iterating the above arguments, we can show that for any 2-dimensional face f of P', any interior angle of f is non-obtuse. Since f is a Euclidean polygon, it must be either a 3-gon or a 4-gon. So since P is combinatorially equivalent to P', any 2-face of P is either a 3-gon or a 4-gon, too. Then by Theorem 2.1(**b**), the polytope P is combinatorially equivalent to a product of simplices. □

Remark 2.8. In the statement of Theorem 2.7(**b**), if we do not require the Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ to be $(\mathbb{Z}_2)^m$-invariant, it is still likely that P has to be combinatorially equivalent to a product of simplices (see [9, Section 5.2]). But we do not know how to prove this so far.

3. Appendix

Here we give another proof of Theorem 2.1(**b**). For brevity, we say that a simplicial complex is a *sphere join* if it is isomorphic to $\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ for some $n_1 \cdots, n_q \in \mathbb{N}$. One dimensional sphere join is either $\partial\Delta^2$ (boundary of a triangle) or $\partial\Delta^1 * \partial\Delta^1$ (boundary of a square). Let us first prove the following theorem.

Theorem 3.1. *Let K be a simplicial complex of dimension n. Suppose that K satisfies the following two conditions:*

(a) *K is a pseudomanifold,*
(b) *the link of any vertex of K is a sphere join of dimension $n-1$,*

Then K is a sphere join.

Proof. First of all, assumption (b) implies that the link of any k-simplex in K is a sphere join of dimension $n-k-1$. We denote a simplex spanned by vertices

$v_0, v_1, \ldots, v_m$ by $[v_0, v_1, \ldots, v_m]$ and its boundary complex by $\partial[v_0, v_1, \ldots, v_m]$. Let w be a vertex of K. By assumption (b) the link $\mathrm{link}_K w$ of w in K is of the form $\mathrm{link}_K w = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ where $n_1 + \cdots + n_q = n$. Denote the vertices of $\partial\Delta^{n_k}$ by $v_0^k, v_1^k, \ldots, v_{n_k}^k$ for $k = 1, 2, \ldots, q$, so that

$$\mathrm{link}_K w = \partial[v_0^1, v_1^1, \ldots, v_{n_1}^1] * \cdots * \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q]. \tag{7}$$

Let I be the set of vertices $v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^q, \ldots, v_{n_q}^q$. Then $[I]$ is a maximal simplex in $\mathrm{link}_K w$ and the simplex $[I, w]$ spanned by I and w is of dimension n. Since K is a pseudomanifold by assumption (a), there is a unique vertex v in K such that $[I, v] \cap [I, w] = [I]$. We have two cases below.

Case 1. The case where $v \notin \mathrm{link}_K w$. In this case we claim $K = \partial[v, w] * \mathrm{link}_K w$. The proof is as follows. Choose an element from I arbitrarily, say v_j^i ($1 \le i \le q$, $1 \le j \le n_i$). Set $\bar{I} = (I \backslash \{v_j^i\}) \cup \{v_0^i\}$. Then $[\bar{I}]$ is an $(n-1)$-simplex of $\mathrm{link}_K w$ by (7), so there is a unique vertex $\bar{v}$ of K such $[\bar{I}, \bar{v}] \cap [\bar{I}, w] = [\bar{I}]$ as before since K is a pseudomanifold. Now we shall observe the link of an $(n-2)$-simplex $[I \cap \bar{I}] = [I \backslash \{v_j^i\}]$ in K. By our construction, the following are four n-simplices in K containing $[I \cap \bar{I}]$:

$$[I \cap \bar{I}, v_j^i, w],\ [I \cap \bar{I}, v_0^i, w],\ [I \cap \bar{I}, v_j^i, v],\ [I \cap \bar{I}, v_0^i, \bar{v}].$$

Therefore the vertices $v_j^i, w, v_0^i, v, \bar{v}$ are in the link of the $(n-2)$-simplex $[I \cap \bar{I}]$. But by assumption (b), this link is a sphere join of dimension one which can have at most four vertices. Note that v_j^i, w, v_0^i are mutually distinct and $v, \bar{v}$ are different from v_j^i, w, v_0^i. So we must have $\bar{v} = v$. Now let v_j^i run over all elements of I, then $\bar{I}$ runs over all $(n-1)$-simplices in $\mathrm{link}_K w$. This shows that $\partial[v, w] * \mathrm{link}_K w$ is a subcomplex of K. However, $\partial[v, w] * \mathrm{link}_K w$ and K are both pseudomanifolds and have the same dimension, so they must agree. This proves the claim.

Case 2. The case where $v \in \mathrm{link}_K w$, so v is one of $v_0^1, v_0^2, \ldots, v_0^q$. We may assume $v = v_0^1$ without loss of generality. Then

$$[v, I] = [v_0^1, v_1^1, \ldots, v_{n_1}^1, v_1^2, \ldots, v_{n_2}^2, \ldots, v_1^q, \ldots, v_{n_q}^q] \quad \text{is an } n\text{-simplex in } K. \tag{8}$$

We look at $\mathrm{link}_K v$. Since $v = v_0^1$, it follows from (7) that $\mathrm{link}_K v$ contains

$$\partial[v_1^1, \ldots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q] \tag{9}$$

as a subcomplex. This together with assumption (b) implies that there is a vertex w' different from vertices in (9) such that $\operatorname{link}_K v$ is one of the following:

$$\begin{aligned}
&\partial[w', v_1^1, \dots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \dots, v_{n_q}^q],\\
&\partial[v_1^1, \dots, v_{n_1}^1] * \partial[w', v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \dots, v_{n_q}^q],\\
&\qquad\vdots \qquad\qquad\qquad \vdots\\
&\partial[v_1^1, \dots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[w', v_0^q, v_1^q, \dots, v_{n_q}^q].
\end{aligned}$$

However, the fact (8) implies that none of the above occurs except the first one. So we have

$$\operatorname{link}_K v = \partial[w', v_1^1, \dots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \dots, v_{n_q}^q]. \tag{10}$$

The simplex $[I]$ is in $\operatorname{link}_K v$ by (10) and the n-simplices $[I, v]$ and $[I, w]$ share $[I]$. We know that w is different from $v_0^2, \dots, v_0^q$. Therefore, if $w' \neq w$, then we are in the same situation as Case 1 above (the role of v and w are interchanged). Therefore one concludes

$$K = \partial[w, v] * \operatorname{link}_K v.$$

In particular, $[w, v]$ is not a 1-simplex of K. But this contradicts the assumption that $v \in \operatorname{link}_K w$. Therefore $w' = w$ and by (10) we have

$$\operatorname{link}_K v = \partial[w, v_1^1, \dots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \dots, v_{n_q}^q]. \tag{11}$$

Remember that $v = v_0^1$. We claim that K contains

$$\partial[w, v_0^1, v_1^1, \dots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \dots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \dots, v_{n_q}^q] \tag{12}$$

as a subcomplex. Indeed, any n-simplex in (12) is spanned by $n+1$ vertices which consist of $n_1 + 1$ vertices from $\partial[w, v_0^1, v_1^1, \dots, v_{n_1}^1]$ and n_i vertices from $\partial[v_0^i, v_1^i, \dots, v_{n_i}^i]$ for $i = 2, 3, \dots, q$. Since $v_0^1 = v$, either w or v is in the $n_1 + 1$ vertices from $\partial[w, v_0^1, v_1^1, \dots, v_{n_1}^1]$. If w (resp. v) is in the $n_1 + 1$ vertices from $\partial[w, v_0^1, v_1^1, \dots, v_{n_1}^1]$, then any n-simplex formed this way is in K by (7) (resp. (11)). This proves the claim.

Finally, since K and the subcomplex (12) are both pseudomanifolds and have the same dimension, they must agree. So we finish the proof of the theorem. □

***Proof of Theorem 2.1(b)*:** Suppose any 2-dimensional face of P is either a 3-gon or a 4-gon. We want to show that P is combinatorially equivalent to a product of simplices, or equivalently ∂P^* is a sphere join. Let us do induction on the dimension of P. When $\dim P = 2$, the proof is trivial. If $\dim P \geq 3$, we will show that ∂P^* satisfies the two conditions in Theorem 3.1. Condition (a) is obvious. By induction assumption, all facets of P are product of simplices which means that ∂P^* satisfies condition (b). So we finish the induction by Theorem 3.1.

□

Acknowledgement

The authors want to thank Shicheng Xu and Jiaqiang Mei for some helpful discussions on the geometry of Alexandrov spaces.

References

[1] I. V. Baskakov, *Cohomology of K-powers of spaces and the combinatorics of simplicial divisions*, (Russian) Uspekhi Mat. Nauk 57 (2002), no. **5** (347), 147–148; translation in Russian Math. Surveys 57 (2002), no. **5**, 989–990.

[2] V. M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, 24. American Mathematical Society, Providence, RI, 2002.

[3] V. M. Buchstaber and T.E. Panov, *Toric Topology*, Mathematical Surveys and Monographs, Vol. 204, American Mathematical Society, Providence, RI, 2015,

[4] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, **33**. American Mathematical Society, Providence, RI, 2001.

[5] S. Choi, M. Masuda and D.Y. Suh, *Quasitoric manifolds over a product of simplices*, Osaka J. Math. vol. **47** (2010), 109-129.

[6] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no.**2**, 417–451.

[7] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. 40 (2003), No. 1, 1–68.

[8] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin (1972).

[9] S. Kuroki, M. Masuda and L. Yu, *Small covers, infra-solvmanifolds and curvature*, Forum Mathematicum 27 (2015), No. 5, 2981-3004.

[10] Y. M. Ustinovsky, *Doubling operation for polytopes and torus actions*, Russian Math. Surveys 64 (2009) No. 5, 181–182.

[11] Y. M. Ustinovsky, *Toral rank conjecture for moment-angle complexes*, Mathematical Notes Vol. 90 (2011) No. 2, 279–283.

[12] M. Wiemeler, *Exotic torus manifolds and equivariant smooth structures on quasitoric manifolds*, Math. Z. 273 (2013), no. **3-4**, 1063–1084.

[13] M. Wiemeler, *Torus manifolds and non-negative curvature*, J. Lond. Math. Soc., II. Ser. 91 (2015), No. 3, 667-692

[14] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, 152 (revised first edition), Springer-Verlag, New York, 1998.

Department of Mathematics and IMS, Nanjing University, Nanjing, 210093, P.R.China

E-mail address: yuli@nju.edu.cn

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

E-mail address: masuda@sci.osaka-cu.ac.jp