COHOMOLOGICAL RIGIDITY OF MANIFOLDS DEFINED BY RIGHT-ANGLED 3-DIMENSIONAL POLYTOPES

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Abstract. We consider the class $\mathcal{P}$ of 3-dimensional simple polytopes P which are flag and do not have 4-belts of facets. It includes fullerenes, i. e. simple 3-polytopes with only 5-gonal and 6-gonal facets. According to a theorem of Pogorelov, any polytope from $\mathcal{P}$ admits a right-angled realisation in Lobachevsky 3-space, and such a realisation is unique up to isometry.

We study two families of smooth manifolds associated with polytopes from the class $\mathcal{P}$. The first family consists of 6-dimensional quasitoric manifolds over polytopes from $\mathcal{P}$. The second family consists of 3-dimensional small covers of polytopes from $\mathcal{P}$ or, equivalently, hyperbolic 3-manifolds of Löbell type. Our main result is that both families are cohomologically rigid, i. e. two manifolds M and M' from either of the families are diffeomorphic if and only if their cohomology rings are isomorphic. We also prove that if M and M' are diffeomorphic, then their corresponding polytopes P and P' are combinatorially equivalent. These results are intertwined with the classical subjects of geometry and topology, such as combinatorics of 3-polytopes, the Four Colour Theorem, diffeomorphism classification of 6-manifolds and invariance of Pontryagin classes. The proofs use techniques of toric topology.

Contents

The research of the first, second and fourth author was supported by the Russian Foundation for Basic Research (grants RFBR 14-01-00537 and 16-51-55017-GFEN). The second author was supported by Young Russian Mathematics award. The third author was supported by JSPS Grant-in-Aid for Scientific Research (C) 16K05152. The fourth author was supported by a grant from Dmitri Zimin's 'Dynasty' foundation.

1. Introduction

There is the following naive question from the early days of differential topology: given two closed smooth manifolds M and M', when does an isomorphism $H^*(M) \cong H^*(M')$ of integral cohomology rings imply that M and M' are diffeomorphic? This is generally regarded as an unlikely case, as in the 20th century topologists discovered many important series of manifolds for which the cohomology ring, or even the homotopy type, does not determine the diffeomorphism class. Three-dimensional lens spaces, Milnor's exotic spheres and Donaldson's four-dimensional manifolds are prominent examples of different level of complexity.

We say that a family of closed smooth manifolds is *cohomologically rigid* if a cohomology ring isomorphism $H^*(M) \cong H^*(M')$ implies a diffeomorphism $M \cong M'$ for any two manifolds in the family.

In this paper we establish cohomological rigidity for two particular families of manifolds of dimension 3 and 6, respectively. Each of these families arises from a particularly important class of combinatorial polytopes, which we refer to as the *Pogorelov class* $\mathcal{P}$. It consists of simple 3-dimensional polytopes which are flag and do not have 4-belts. In particular, polytopes in $\mathcal{P}$ do not have triangular and quadrangular facets. The class $\mathcal{P}$ includes combinatorial fullerenes, i. e. simple 3-polytopes with only pentagonal and hexagonal facets. By the results of Pogorelov [40] and Andreev [1], the class $\mathcal{P}$ consists precisely of combinatorial 3-polytopes which can be realised in Lobachevsky space $\mathbb{L}^3$ with right angles between adjacent facets (*right-angled* 3*-polytopes* for short). The conditions specifying the Pogorelov class also feature as the "no $\triangle$-condition" and "no $\square$-condition" in Gromov's theory of hyperbolic groups [29].

Our first family consists of *hyperbolic* 3*-manifolds of Löbell type*. These manifolds were introduced and studied by Vesnin [43]. They arise from right-angled realisations of polytopes from the Pogorelov class $\mathcal{P}$ (see the details in Subsection 2.4). Each hyperbolic 3-manifold N of Löbell type is composed of 8 copies of a polytope $P \in \mathcal{P}$ and is a *small cover* of P in the sense of Davis and Januszkiewicz [22]. We prove (in Theorem 5.4) that two such manifolds N and N' are diffeomorphic if and only if their $\mathbb{Z}_2$-cohomology rings are isomorphic.

Our second family arises from toric topology: it consists of quasitoric (or topological toric) manifolds whose quotient polytopes are in the class $\mathcal{P}$. These are 6-dimensional smooth manifolds acted on by a 3-torus T^3 with quotient $P \in \mathcal{P}$. We show (in Theorem 5.2 and Corollary 5.3) that this family is cohomologically rigid, i. e. two manifolds M and M' in the family are diffeomorphic if and only if their cohomology rings are isomorphic. In general a diffeomorphism between quasitoric manifolds M and M' does not imply that the corresponding polytopes P and P' are combinatorially equivalent, but this is the case when the quotient polytopes are in the class $\mathcal{P}$ (see Theorem 5.2).

Our proofs use both combinatorial and cohomological techniques of toric topology, and build upon recent important results of Fan, Ma and Wang on cohomological rigidity of moment-angle manifolds [27], [28]. These authors come to the class $\mathcal{P}$ independently, by considering the property of B-rigidity for simple polytopes (see Section 3). The class $\mathcal{P}$ also featured in the work of Buchstaber and Erokhovets [8] on combinatorial classification of fullerenes.

We note that cohomological rigidity is open for the whole family of toric or topological toric manifolds. In fact, we find it quite surprising that no counterexamples to the "toric cohomological rigidity problem" were found up to date.

In real dimension 6 the families of quasitoric and topological toric manifolds coincide and contain strictly the family of toric manifolds (smooth complete toric varieties). The family of quasitoric (or topological toric) manifolds whose quotient polytopes are in the class $\mathcal{P}$ is large enough, as there is at least one quasitoric manifold over any simple 3-polytope. Indeed, the Four Colour Theorem implies that any simple 3-polytope admits a "characteristic function" (see Proposition 2.8); this remarkable observation was made by Davis and Januszkiewicz in [22]. Toric manifolds whose associated polytopes are in $\mathcal{P}$ are fewer, but still abundant; many concrete examples were produced recently by Suyama [42]. On the other hand, there are no *projective* toric manifolds whose associated polytopes are in $\mathcal{P}$. This follows from a result of C. Delaunay [23] that a Delzant 3-polytope must have at least one triangular or quadrangular face.

Our results on cohomological rigidity of toric manifolds chime with the problem of diffeomorphism classification for simply connected oriented manifolds, which is a classical subject of algebraic and differential topology. The foundations of this classification in dimensions $\geqslant 5$ were laid in the works of Browder and Novikov (see [5], [39]). Novikov [38] showed that for a given simply connected oriented manifold M of dimension $\geqslant 5$ there are only finitely many manifolds M' for which there exists a homotopy equivalence $M \xrightarrow{\simeq} M'$ preserving the Pontryagin classes. Important classification results in dimension 6 were obtained in the works of Wall [44], Jupp [32] and Zhubr [45].

Toric, quasitoric or topological toric manifolds M are simply connected, and their cohomology rings $H^*(M)$ are generated by 2-dimensional classes. According to the general results mentioned above, two such manifolds in dimension 6 are diffeomorphic if there is an isomorphism of their cohomology rings preserving the first Pontryagin class p_1. Therefore, the toric cohomological rigidity problem in dimension 6 reduces to establishing the invariance of p_1 under cohomology isomorphisms. This turns out to be a purely combinatorial and linear algebra problem, see details in Section 6. However, we were not able to prove directly the invariance of p_1 under cohomology isomorphisms for toric manifolds over simple 3-polytopes from the class $\mathcal{P}$. One of our main results (Theorem 5.2) can be interpreted as a classification result for a particular family of simply connected 6-dimensional manifolds, and its proof is independent of the general classification results of Wall, Jupp and Zhubr.

Acknowledgements. The authors are grateful to

– Feifei Fan for introducing us to the results of the works [27] and [28] on the cohomological rigidity of moment-angle complexes;

– Alexandr Gaifullin for drawing our attention to Siebennman's no $\square$-condition in Gromov's theory of hyperbolic groups, which is equivalent to the absence of 4-belts in the case of polytopes;

– Sergei Novikov for stimulating discussions on the diffeomorphism classification of simply connected manifolds and invariance of Pontryagin classes;

– Andrei Vesnin for his help in linking our results to hyperbolic geometry; in particular, to hyperbolic 3-manifolds of Löbell type and the theorem of Pogorelov on right-angled 3-polytopes.

We thank the Institute for Mathematical Science, National University of Singapore, and the organisers of the Program on Combinatorial and Toric Homotopy, where we were able to exchange ideas which laid the foundation of this work. We also thank the JSPS program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers, based on Osaka City University Advanced Mathematical Institute (OCAMI), which gave us an opportunity to meet and work together on this project.

2. Preliminaries

Here we collect the necessary information about toric varieties, quasitoric manifolds and moment-angle manifolds; the details can be found in [11].

2.1. **Simple polytopes.** Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the scalar product $\langle\ ,\ \rangle$. A *convex polytope* P is a nonempty bounded intersection of finitely many half-spaces in some $\mathbb{R}^n$:

$$P=\bigl\{\boldsymbol{x}\in\mathbb{R}^n\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i\geqslant 0\quad\text{for } i=1,\dots,m\bigr\}, \tag{2.1}$$

where $\boldsymbol{a}_i\in\mathbb{R}^n$ and $b_i\in\mathbb{R}$. We often fix a presentation by inequalities (2.1) alongside with the polytope P. We assume that P is n-dimensional, that is, the dimension of the affine hull of P is n. We also assume that each inequality $\langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i\geqslant 0$ in (2.1) is not redundant, that is, cannot be removed without changing P. Then P has m *facets* $F_1,\dots,F_m$, where

$$F_i=\{\boldsymbol{x}\in P\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i=0\}.$$

Each facet is a polytope of dimension $n-1$. A *face* of P is a nonempty intersection of facets. Zero-dimensional faces are *vertices*, and one-dimensional faces are *edges*.

We refer to n-dimensional polytopes simply as *n-polytopes*.

Two polytopes P and Q are *combinatorially equivalent* ($P\simeq Q$) if there is a bijection between their faces preserving the inclusion relation. A *combinatorial polytope* is a class of combinatorially equivalent polytopes.

A polytope P is *simple* if exactly n facets meet at each vertex of P. A simple polytope P is called a *flag polytope* if every collection of its pairwise intersecting facets has a nonempty intersection. An *n-simplex* Δ^n is not flag for $n\geqslant 2$. An *n-cube* I^n is flag for any n.

We denote by G_P the vertex-edge graph of a polytope P, and refer to it simply as the *graph of* P. A graph is *simple* if it has no loops and multiple edges. A connected graph G is 3-*connected* if it has at least 6 edges and deletion of any one or two vertices with all incident edges leaves G connected. The following classical result describes the graphs of 3-polytopes.

Theorem 2.1 (Steinitz, see [46, Theorem 4.1])**.** *A graph G is the graph of a* 3-*polytope if and only it is simple, planar and* 3-*connected.*

For $k\geqslant 4$, a *k-belt* in a simple polytope P is a cyclic sequence $\mathcal{B}_k=(F_{i_1},\dots,F_{i_k})$ of facets in which only two consecutive facets (including F_{i_k},F_{i_1}) have nonempty intersection. For a 3-*belt* we assume additionally that $F_{i_1}\cap F_{i_2}\cap F_{i_3}=\varnothing$.

A 3-polytope P with a triangular facet has a 3-belt around it, unless $P=\Delta^3$. A simple 3-polytope $P\neq\Delta^3$ is flag if and only if it does not contain 3-belts.

A *fullerene* is a simple 3-polytope with only pentagonal and hexagonal facets. A simple calculation with Euler characteristic shows that the number of pentagonal facets in a fullerene is 12. The number of hexagonal facets can be arbitrary except for 1 (see [25, Proposition 2]). Also, any fullerene is a flag polytope without 4-belts (see [26] and [8, Corollary 3.16]).

2.2. **Toric varieties and manifolds.** A *toric variety* is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V. We only consider nonsingular complete (compact in the usual topology) toric varieties, also known as *toric manifolds*.

There is a bijective correspondence between the isomorphism classes of complex n-dimensional toric manifolds and complete nonsingular fans in $\mathbb{R}^n$. A *fan* is a finite collection $\Sigma=\{\sigma_1,\dots,\sigma_s\}$ of strongly convex cones σ_i in $\mathbb{R}^n$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *nonsingular* (or *regular*) if each of its cones σ_j is generated by part of a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (we choose the standard lattice for simplicity). In particular, each one-dimensional cone of Σ is generated by a primitive vector $\boldsymbol{a}_i \in \mathbb{Z}^n$. A fan Σ is *complete* if the union of its cones is the whole $\mathbb{R}^n$.

Projective toric varieties are particularly important. A projective toric manifold V is defined by a *lattice Delzant polytope* P. Given a simple n-polytope P with vertices in the lattice $\mathbb{Z}^n$, the *normal fan* Σ_P has n-dimensional cones σ_v corresponding to the vertices v of P, where σ_v is generated by the primitive inside-pointing normals to the facets of P meeting at v. The polytope P is *Delzant* whenever its normal fan Σ_P is nonsingular. The fan Σ_P defines a projective toric manifold V_P. Different lattice Delzant polytopes with the same normal fan produce different projective embeddings of the same toric manifold.

Irreducible torus-invariant subvarieties of complex codimension one in V correspond to one-dimensional cones of Σ. When V is projective, they also correspond to the facets of P. We assume that there are m one-dimensional cones (or facets), denote the corresponding primitive vectors by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and denote the corresponding codimension-1 subvarieties by $V_1, \ldots, V_m$.

Theorem 2.2 (Danilov–Jurkiewicz, see [11, Theorem 5.3.1])**.** *Let V be a toric manifold of complex dimension n, with the corresponding complete nonsingular fan Σ. The cohomology ring $H^*(V;\mathbb{Z})$ is generated by the degree-two classes $[v_i]$ dual to the invariant submanifolds V_i, and is given by*

$$H^*(V;\mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ *such that $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_k}$ do not span a cone of Σ;*

(b) $\displaystyle\sum_{i=1}^{m} \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle v_i$*, for any vector $\boldsymbol{x} \in \mathbb{Z}^n$.*

It is convenient to consider the integer $n \times m$-matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \tag{2.2}$$

whose columns are the vectors $\boldsymbol{a}_i$ written in the standard basis of $\mathbb{Z}^n$. Then the ideal (b) of Theorem 2.2 is generated by the n linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of Λ.

The quotient of a projective toric manifold V_P by the action of the compact torus $T^n \subset (\mathbb{C}^\times)^n$ is the polytope P. When a toric manifold V is not projective, the quotient V/T^n has a face structure of a *manifold with corners*. This face structure locally looks like that of a simple convex polytope, but globally may fail to be so even combinatorially. In the case $n = 3$, however, the quotient V/T^3 is combinatorially equivalent to a simple 3-polytope, by Steinitz's theorem (Theorem 2.1).

2.3. **Quasitoric manifolds.** In their 1991 work [22] Davis and Januszkiewicz suggested a topological generalisation of projective toric manifolds, which became known as quasitoric manifolds.

A *quasitoric manifold* over a combinatorial simple n-polytope P is a topological manifold M of dimension $2n$ with a locally standard action of T^n and a projection $\pi\colon M \to P$ whose fibres are the orbits of the T^n-action. (An action of T^n on M is *locally standard* if every point $x \in M$ is contained in a T^n-invariant neighbourhood equivariantly homeomorphic to an open subset in $\mathbb{C}^n$ with the standard coordinatewise action of T^n twisted by an automorphism of the torus. The orbit space of a locally standard action is a manifold with corners. For a quasitoric manifold M, the orbit space M/T^n is homeomorphic to P.)

Not every simple polytope can be the quotient of a quasitoric manifold. Nevertheless, quasitoric manifolds constitute a much larger family than projective toric manifolds, and enjoy more flexibility for topological applications.

Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be the set of facets of P. Each $M_i = \pi^{-1}(F_i)$ is a quasitoric submanifold of M of codimension 2, called a *characteristic submanifold*. The characteristic submanifolds $M_i \subset M$ are analogues of the invariant divisors V_i on a toric manifold V. Each M_i is fixed pointwise by a closed one-dimensional subgroup (a subcircle) $T_i \subset T^n$ and therefore corresponds to a primitive vector $\lambda_i \in \mathbb{Z}^n$ defined up to a sign. Choosing a direction of λ_i is equivalent to choosing an orientation for the normal bundle $\nu(M_i \subset M)$ or, equivalently, choosing an orientation for M_i, provided that M itself is oriented. An *omniorientation* of a quasitoric manifold M consists of a choice of orientation for M and each characteristic submanifold M_i.

The vectors λ_i are analogues of the generators $\boldsymbol{a}_i$ of the one-dimensional cones in the fan corresponding to a toric manifold V, or analogues of the normal vectors to the facets of P when V is projective. However, the vectors λ_i need not be the normal vectors to the facets of P in general.

There is an analogue of Theorem 2.2 for quasitoric manifolds:

Theorem 2.3 ([22])**.** *Let M be an omnioriented quasitoric manifold of dimension $2n$ over a simple n-polytope P. The cohomology ring $H^*(M;\mathbb{Z})$ is generated by the degree-two classes $[v_i]$ dual to the oriented characteristic submanifolds M_i, and is given by*

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ *such that* $F_{i_1} \cap \dots \cap F_{i_k} = \varnothing$ *in* P*;*
(b) $\sum_{i=1}^{m} \langle \lambda_i, \boldsymbol{x} \rangle v_i$, *for any vector* $\boldsymbol{x} \in \mathbb{Z}^n$.

We record a simple corollary for the latter use.

Corollary 2.4. *In the notation of Theorem 2.3,*

(a) *the product $[v_{i_1}] \cdots [v_{i_n}]$ of n different classes is a generator of $H^{2n}(M) \cong \mathbb{Z}$ if $F_{i_1} \cap \dots \cap F_{i_n} \neq \varnothing$ and is zero otherwise;*
(b) *for $i \neq j$, we have $[v_i][v_j] = 0$ if and only if $F_i \cap F_j = \varnothing$.*

By analogy with (2.2), we consider the integer *characteristic matrix*

$$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nm} \end{pmatrix} \tag{2.3}$$

whose columns are the vectors λ_i written in the standard basis of $\mathbb{Z}^n$. The matrix Λ has the following property:

$$\det(\lambda_{i_1}, \dots, \lambda_{i_n}) = \pm 1 \quad \text{whenever } F_{i_1} \cap \dots \cap F_{i_n} \neq \varnothing \text{ in } P. \tag{2.4}$$

Note that the ideal (b) of Theorem 2.3 is generated by the n linear forms $\lambda_{j1} v_1 + \dots + \lambda_{jm} v_m$ corresponding to the rows of Λ.

A map $\lambda \colon \mathcal{F} \to \mathbb{Z}^n$, $F_i \mapsto \lambda_i$, satisfying (2.4) is called a *characteristic function* for a simple n-polytope P. One can produce a characteristic matrix Λ from a characteristic function λ by fixing an order of facets. A *characteristic pair* (P, Λ) consists of a simple polytope P and its characteristic matrix Λ.

A quasitoric manifold M defines a characteristic pair $(P, \varLambda)$. On the other hand, each characteristic pair gives rise to a quasitoric manifold as follows.

Construction 2.5 ([22]). Let $(P, \varLambda)$ be a characteristic pair. For each facet F_i of P we denote by T_i the circle subgroup of $T^n = \mathbb{R}^n/\mathbb{Z}^n$ corresponding to the ith column $\lambda_i \in \mathbb{Z}^n$ of the characteristic matrix $\varLambda$. For each point $x \in P$, define a torus

$$T(x) = \prod_{i\colon\, x \in F_i} T_i,$$

assuming that $T(x) = \{1\}$ if there are no facets containing x. Property (2.4) implies that $T(x)$ embeds as a subgroup in T^n. Then define

$$M(P, \varLambda) = P \times T^n/\!\sim,$$

where the equivalence relation $\sim$ is given by $(x, t) \sim (x', t')$ whenever $x = x'$ and $t' - t \in T(x)$. One can see that $M(P, \varLambda)$ is a quasitoric manifold over P.

Change of basis in the lattice results in multiplying $\varLambda$ from the left by a matrix from $GL(n, \mathbb{Z})$. Changing the orientation of the ith characteristic submanifold M_i in the omniorientation data results in changing the sign of the ith column of $\varLambda$. A combinatorial equivalence between polytopes P and P' allows us to identify their sets of facets $\mathcal{F}$ and $\mathcal{F}'$ and therefore identify their characteristic functions. These observations lead us to the following definition.

Definition 2.6. Two characteristic pairs $(P, \varLambda)$ and $(P', \varLambda')$ are *equivalent* if

(a) P and P' are combinatorially equivalent, and
(b) $\varLambda' = A\varLambda B$, where $A \in GL(n, \mathbb{Z})$ and B is a diagonal $(m \times m)$-matrix with ± 1 on the diagonal.

Quasitoric manifolds $M(P, \varLambda)$ and $M(P', \varLambda')$ corresponding to equivalent pairs are equivariantly homeomorphic (in the weak sense). The latter means that there is a homeomorphism $f\colon M(P, \varLambda) \xrightarrow{\cong} M(P', \varLambda')$ satisfying $f(t \cdot x) = \psi(t) \cdot f(x)$ for any $t \in T^n$ and $x \in M(P, \varLambda)$, where $\psi\colon T^n \to T^n$ is the automorphism of the torus given by the matrix A. Furthermore, we have

Proposition 2.7 ([22, Proposition 1.8] and [11, Proposition 7.3.8]). *There is a one-to-one correspondence between equivariant homeomorphism classes of quasitoric manifolds and equivalence classes of characteristic pairs. In particular, for any quasitoric manifold M over P with characteristic matrix $\varLambda$, there is an equivariant homeomorphism $M \cong M(P, \varLambda)$.*

Remark. Both M and $M(P, \varLambda)$ were defined as topological manifolds in [22]. The manifold $M(P, \varLambda)$ can be endowed with a canonical smooth structure by defining it as the quotient of the moment-angle manifold $\mathcal{Z}_P$ by a smooth torus action, see [12] and Subsection 2.9. Nevertheless, for a smooth quasitoric manifold M, the existence of a *diffeomorphism* $M \cong M(P, \varLambda)$ is a delicate issue, see the discussion in [11, §7.3]. On the other hand, in the case of 6-dimensional quasitoric manifolds (which is our main concern in this paper), such a diffeomorphism follows from the classification results of Wall and Jupp discussed in Section 6.

In dimensions $n \geqslant 4$, there are simple n-polytopes P which do not admit any characteristic matrix $\varLambda$, see [22, 1.22]. Such a polytope cannot be the quotient of a quasitoric manifold. On the other hand, we have the following observation by Davis and Januszkiewicz, whose proof remarkably uses the Four Colour Theorem:

Proposition 2.8 ([22]). *Any simple 3-polytope admits a characteristic matrix $\varLambda$.*

Proof. Given a 4-colouring of the facets of P, we assign to a facet of ith colour the ith basis vector $\boldsymbol{e}_i \in \mathbb{Z}^3$ for $i = 1, 2, 3$ and the vector $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ for $i = 4$. The resulting $3 \times m$-matrix Λ satisfies (2.4), as any three of the four vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ form a basis of $\mathbb{Z}^3$. □

A projective toric manifold is a quasitoric manifold. A non-projective toric manifold V may fail to be quasitoric, as the quotient manifold with corners V/T^n is not necessarily a simple polytope, even combinatorially. First examples of this sort appear in dimension $n = 4$, see [41]. All complex 3-dimensional toric manifolds, even non-projective ones, are quasitoric by the Steinitz theorem (Theorem 2.1).

2.4. **Small covers and hyperbolic 3-manifolds.** Replacing the torus T^n in the definition of a quasitoric manifold by the subgroup $\mathbb{Z}_2^n \subset T^n$ (n commuting involutions), one obtains the definition of a small cover [22]. A *small cover* of a simple n-polytope P is a manifold N of dimension n with a locally standard action of $\mathbb{Z}_2^n$ and a projection $\pi\colon N \to P$ whose fibres are the orbits of the $\mathbb{Z}_2^n$-action.

The set of real points of a projective toric manifold V_P (i. e. the set of points fixed under complex conjugation) is a small cover of P; it is sometimes called a *real toric manifold*.

The theory of small covers parallels that of quasitoric manifolds, and we just outline the most crucial points.

Theorem 2.9 ([22])**.** *Let N be a small cover of a simple n-polytope P. The cohomology ring $H^*(N;\mathbb{Z}_2)$ is generated by the degree-one classes $[v_i]$ dual to the characteristic submanifolds N_i, and is given by*

$$H^*(N;\mathbb{Z}_2) \cong \mathbb{Z}_2[v_1, \ldots, v_m]/\mathcal{I}, \qquad \deg v_i = 1,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ *such that* $F_{i_1} \cap \cdots \cap F_{i_k} = \varnothing$ *in* P*;*

(b) $\sum_{i=1}^{m} \langle \lambda_i, \boldsymbol{x} \rangle v_i$, *for any vector* $\boldsymbol{x} \in (\mathbb{Z}_2)^n$.

The characteristic matrix Λ corresponding to a small cover N has entries in $\mathbb{Z}_2$ and satisfies the same condition (2.4). The equivalence of $\mathbb{Z}_2$-characteristic pairs is defined in the same way as in the quasitoric case, with $GL(n,\mathbb{Z})$ replaced by $GL(n,\mathbb{Z}_2)$. A small cover N of P is equivariantly homeomorphic to the "canonical model"

$$N(P,\Lambda) = P \times \mathbb{Z}_2^n/\sim$$

with the equivalence relation $\sim$ defined as in the quasitoric case. Note that $N(P,\Lambda)$ is composed of 2^n copies of the polytope P, patched together along their facets.

Reducing a $\mathbb{Z}$-characteristic matrix $\bmod 2$ we obtain a $\mathbb{Z}_2$-characteristic matrix. The following question is open.

Problem 1. *Assume given a $\mathbb{Z}_2$-characteristic pair (P,Λ) consisting of a simple n-polytope P and an $(n \times m)$-matrix Λ with entries in $\mathbb{Z}_2$ satisfying* (2.4)*. Can Λ be obtained by reduction $\bmod 2$ from an integer matrix satisfying the same condition* (2.4)*?*

The answer to the above problem is positive for 3-polytopes:

Proposition 2.10. *Every $\mathbb{Z}_2$-characteristic pair (P,Λ) with 3-dimensional P is the $\bmod 2$ reduction of a $\mathbb{Z}$-characteristic pair.*

Proof. One can check that any (3×3)-matrix with entries 0 or 1 and determinant 1 mod 2 has determinant ±1 when viewed as an integer matrix. Ineed, such a matrix either has a column with 2 zeros, or is $\begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix}$ up to permutation of rows and columns. The required property is then verified directly. □

A particularly important class of 3-dimensional small covers are *hyperbolic* 3-*manifolds of Löbell type*, introduced and studied by Vesnin in [43].

Construction 2.11. Let P be a polytope in 3-dimensional Lobachevsky space $\mathbb{L}^3$ with right angles between adjacent facets (a *right-angled* 3-*polytope* for short). Denote by $G(P)$ the group generated by reflections in the facets of P. It is a *right-angled Coxeter group* given by the presentation

$$G(P) = \langle g_1, \dots, g_m \mid g_i^2 = 1,\ g_i g_j = g_j g_i \text{ if } F_i \cap F_j \neq \varnothing\rangle,$$

where g_i denotes the reflection in the facet F_i. The reflections in adjacent facets commute because of the right-angledness. There are no relations between the reflections in non-adjacent faces, as the corresponding reflection hyperplanes do not intersect in $\mathbb{L}^3$. The group $G(P)$ acts on $\mathbb{L}^3$ discretely with finite isotropy subgroups and with fundamental domain P.

Now consider an epimorphism $\varphi\colon G(P) \to \mathbb{Z}_2^3$. As was observed in [43, Lemma 1], the subgroup $\operatorname{Ker}\varphi \subset G(P)$ does not contain elements of finite order if and only if the images of the reflections in any three facets of P that have a common vertex are linearly independent in $\mathbb{Z}_2^3$. In this case the group $\operatorname{Ker}\varphi$ acts freely on $\mathbb{L}^3$. The quotient $N = \mathbb{L}^3/\operatorname{Ker}\varphi$ is called a *hyperbolic* 3-*manifold of Löbell type.* It is composed of $|\mathbb{Z}_2^3| = 8$ copies of P and has a Riemannian metric of constant negative curvature. Furthermore, such a manifold N is aspherical (has the homotopy type of Eilenberg–Mac Lane space $K(G(P), 1)$), as its universal cover $\mathbb{L}^3$ is contractible.

Since the abelianisation of $G(P)$ is $\mathbb{Z}_2^m$, the epimorphism φ factors as $G(P) \xrightarrow{\text{ab}} \mathbb{Z}_2^m \xrightarrow{\varLambda} \mathbb{Z}_2^3$, where $\varLambda$ is a linear map. The above condition for the freeness of the action of $\operatorname{Ker}\varphi$ on $\mathbb{L}^3$ is equivalent to that $\varLambda$ satisfies (2.4) (i. e. $\varLambda$ is given by a $\mathbb{Z}_2$-characteristic matrix). We therefore can identify $N = \mathbb{L}^3/\operatorname{Ker}\varphi$ with the small cover $N(P, \varLambda)$.

Which combinatorial 3-polytopes have right-angled realisations in $\mathbb{L}^3$? Pogorelov stated necessary conditions in 1967, which were later shown by Andreev to be sufficient:

Theorem 2.12 ([40], [1])**.** *A combinatorial* 3*-polytope can be realised as a right-angled polytope in Lobachevsky space* $\mathbb{L}^3$ *if and only if it is simple, flag and does not have* 4*-belts. Furthermore, such a realisation is unique up to isometry.*

We refer to the class of simple flag 3-polytopes without 4-belts as the *Pogorelov class* $\mathcal{P}$. It will feature prominently throughout the rest of our paper.

A polytope from the class $\mathcal{P}$ does not have triangular or quadrangular facets. According to a result of Došlić [26] (see also [7, Corollary 3.16]), the Pogorelov class contains all fullerenes.

We summarise the constructions and results above as follows.

Theorem 2.13. *A small cover* $N(P, \varLambda)$ *of a* 3*-polytope* P *from the Pogorelov class* $\mathcal{P}$ *has the structure of a hyperbolic* 3*-manifold of Löbell type* $\mathbb{L}^3/\operatorname{Ker}\varphi$*, with the epimorphsim* φ *given by the composition* $G(P) \xrightarrow{\text{ab}} \mathbb{Z}_2^m \xrightarrow{\varLambda} \mathbb{Z}_2^3$*. Furthermore, such a* 3*-manifold* $N(P, \varLambda)$ *is aspherical.*

Remark. The conditions specifying the Pogorelov class $\mathcal{P}$ also feature in Gromov's theory of hyperbolic groups. Namely, the "no △-condition" from [29, §4.2.E] for a simplicial complex $\mathcal{K}$ is the absence of missing 2-faces, while the "no □-condition" is the absence of chordless 4-cycles. When $\mathcal{K}$ is the dual complex of a simple polytope, these two conditions translate to the absence of 3- and 4-belts, respectively.

The relationship between small covers and hyperbolic manifolds was also mentioned in the work of Davis and Januszkiewicz [22, p. 428], although the criterion for right-angled realisation of a polytope was stated there incorrectly.

2.5. **Topological toric manifolds.** A toric manifold is not necessarily a quasitoric manifold and a quasitoric manifold is also not necessarily a toric manifold. However, both toric and quasitoric manifolds are examples of *topological toric manifolds* introduced in [31]. Recall that a toric manifold admits an algebraic action of $(\mathbb{C}^\times)^n$ with an open dense orbit. It has local charts equivariantly isomorphic to a sum of complex one-dimensional *algebraic* representations of $(\mathbb{C}^\times)^n$. A topological toric manifold is a compact smooth $2n$-dimensional manifold with an effective smooth action of $(\mathbb{C}^\times)^n$ having an open dense orbit and covered by finitely many invariant open subsets each equivariantly diffeomorphic to a sum of complex one-dimensional *smooth* representation spaces of $(\mathbb{C}^\times)^n$. (The latter condition automatically follows from the existence of a dense orbit in the algebraic category, but not in the smooth category.)

The cohomology ring of a topological toric manifold is described similarly to the toric or quasitoric case; there is an analogue of Theorems 2.2 or 2.3, see [31, Proposition 8.3].

2.6. **Simplicial complexes and face rings.** Let $\mathcal{K}$ be an (abstract) *simplicial complex* on the set $[m] = \{1, \dots, m\}$, i. e. $\mathcal{K}$ is a collection of subsets $I \subset [m]$ such that for any $I \in \mathcal{K}$ all subsets of I also belong to $\mathcal{K}$. We always assume that the empty set $\varnothing$ and all one-element subsets $\{i\} \subset [m]$ belong to $\mathcal{K}$. We refer to $I \in \mathcal{K}$ as a *simplex* (or a *face*) of $\mathcal{K}$. Every abstract simplicial complex $\mathcal{K}$ has a *geometric realisation* $|\mathcal{K}|$, which is a polyhedron in a Euclidean space (a union of convex geometric simplices).

A *non-face* of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$. A *missing face* (or a *minimal non-face*) of $\mathcal{K}$ is an inclusion-minimal non-face of $\mathcal{K}$, that is, a subset $I \subset [m]$ such that I is not a simplex of $\mathcal{K}$, but every proper subset of I is a simplex of $\mathcal{K}$. A simplicial complex $\mathcal{K}$ is called a *flag complex* if each of its missing faces consists of two vertices. Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex. Every flag complex $\mathcal{K}$ is determined by its 1-skeleton $\mathcal{K}^1$, and is obtained from the graph $\mathcal{K}^1$ by filling in all complete subgraphs by simplices.

Let P be a simple n-polytope with m facets $F_1, \dots, F_m$. Then

$$\mathcal{K}_P = \bigl\{ I = \{i_1, \dots, i_k\} \in [m] \colon F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing \bigr\}$$

is a simplicial complex on $[m]$, called the *dual complex* of P. The vertices of $\mathcal{K}_P$ correspond to the facets of P, and the empty simplex $\varnothing$ corresponds to P itself. Geometrically, $|\mathcal{K}_P|$ is an $(n-1)$-dimensional sphere simplicially subdivided as the boundary of the *dual polytope* of P.

The definitions of flag polytopes and complexes agree: P is a flag polytope if and only if $\mathcal{K}_P$ is a flag complex. A k-belt in P with $k \geqslant 4$ corresponds to a *chordless k-cycle* in the graph $\mathcal{K}_P^1$.

We fix a commutative ring $\mathbf{k}$ with unit.

The *face ring* of $\mathcal{K}$ (also known as the *Stanley–Reisner ring*) is defined as the quotient of the polynomial algebra $\mathbf{k}[v_1, \dots, v_m]$ by the square-free monomial ideal generated by non-simplices of $\mathcal{K}$:

$$\mathbf{k}[\mathcal{K}] = \mathbf{k}[v_1, \dots, v_m] / \bigl(v_{i_1} \cdots v_{i_k} \, : \{i_1, \dots, i_k\} \notin \mathcal{K} \bigr).$$

As $\mathbf{k}[\mathcal{K}]$ is the quotient of the polynomial ring by a monomial ideal, it has a grading or even a multigrading (a $\mathbb{Z}^m$-grading). We use an even grading: $\deg v_i = 2$ and $\operatorname{mdeg} v_i = 2\boldsymbol{e}_i$, where $\boldsymbol{e}_i \in \mathbb{Z}^m$ is the ith standard basis vector.

Note that when $\mathcal{K} = \mathcal{K}_P$ for a simple polytope P, the ring $\mathbb{Z}[P]$ coincides with the quotient of $\mathbb{Z}[v_1, \dots, v_m]$ by the relations (a) in Theorem 2.2 or in Theorem 2.3.

A simplicial complex $\mathcal{K}$ is flag if and only if its face ring $\mathbf{k}[\mathcal{K}]$ is a *quadratic algebra*, i. e. the quotient of $\mathbf{k}[v_1, \dots, v_m]$ by an ideal generated by quadratic monomials (which have degree 4 in our grading).

2.7. **Moment-angle complexes and manifolds.** Let $\mathcal{K}$ be a simplicial complex on the set $[m]$, and let (D^2, S^1) denote the pair of a disc and its boundary circle. For each simplex $I = \{i_1, \dots, i_k\} \in \mathcal{K}$, set

$$(D^2, S^1)^I = \{(x_1, \dots, x_m) \in (D^2)^m \colon x_i \in S^1 \text{ when } i \notin I\}.$$

The *moment-angle complex* is defined as

$$\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (D^2, S^1)^I \subset (D^2)^m. \tag{2.5}$$

If $|\mathcal{K}|$ is homeomorphic to a sphere S^{n-1}, then $\mathcal{Z}_\mathcal{K}$ is a topological manifold. If $\mathcal{K}$ is the boundary of a convex simplicial polytope or is a starshaped sphere (the underlying complex of a complete simplicial fan), then $\mathcal{Z}_\mathcal{K}$ has a smooth structure.

In the polytopal case there is an alternative way to define $\mathcal{Z}_\mathcal{K}$ in terms of the dual simple polytope P. Namely, assume given a presentation of a convex n-dimensional polytope P by inequalities (2.1). Define the map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad \boldsymbol{x} \mapsto \big(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \dots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\big),$$

so $i_P(P) \subset \mathbb{R}^m_\geqslant = \{(y_1, \dots, y_m) \in \mathbb{R}^m \colon y_i \geqslant 0\}$. Also, define the map

$$\mu \colon \mathbb{C}^m \to \mathbb{R}^m_\geqslant, \quad (z_1, \dots, z_m) \mapsto (|z_1|^2, \dots, |z_m|^2).$$

Then define the space $\mathcal{Z}_P$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant \end{array} \tag{2.6}$$

The space $\mathcal{Z}_P$ can be written as an intersection of $(m-n)$ Hermitian quadrics in $\mathbb{C}^m$, and this intersection is nondegenerate precisely when the polytope P is simple. In the latter case, $\mathcal{Z}_P$ is a smooth $(m+n)$-dimensional manifold. Furthermore, the manifold $\mathcal{Z}_P$ is diffeomorphic to the moment-angle complex $\mathcal{Z}_{\mathcal{K}_P}$. In particular, the diffeomorphism type of $\mathcal{Z}_P$ depends only on the combinatorial type of P. We shall therefore not distinguish between $\mathcal{Z}_P$ and $\mathcal{Z}_{\mathcal{K}_P}$ and refer to it as the *moment-angle manifold* corresponding to a simple polytope P. The details of these constructions can be found in [11, Chapter 6].

The standard coordinatewise action of the m-torus T^m on $(D^2)^m$ or $\mathbb{C}^m$ induces the *canonical* T^m-action on $\mathcal{Z}_\mathcal{K}$ or $\mathcal{Z}_P$.

2.8. **Cohomology of moment-angle complexes.** We consider (co)homology with coefficients in $\mathbf{k}$. Denote by $\Lambda[u_1, \dots, u_m]$ the exterior algebra on m generators over $\mathbf{k}$ which satisfy the relations $u_i^2 = 0$ and $u_i u_j = -u_j u_i$.

The *Koszul complex* (or the *Koszul algebra*) of the face ring $\mathbf{k}[\mathcal{K}]$ is defined as the differential $\mathbb{Z} \oplus \mathbb{Z}^m$-graded algebra $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}], d)$, where

$$\operatorname{mdeg} u_i = (-1, 2\boldsymbol{e}_i), \quad \operatorname{mdeg} v_i = (0, 2\boldsymbol{e}_i), \qquad du_i = v_i, \quad dv_i = 0. \tag{2.7}$$

Cohomology of $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}], d)$ is the Tor-*algebra* $\operatorname{Tor}_{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$. It also inherits a $\mathbb{Z} \oplus \mathbb{Z}^m$-grading.

Theorem 2.14 ([4], [11, Theorem 4.5.5])**.** *There are isomorphisms of (multi)graded commutative algebras*

$$\begin{aligned} H^*(\mathcal{Z}_\mathcal{K}) &\cong \operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big)\\ &\cong H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[\mathcal{K}],d\big). \end{aligned}$$

The cohomology of $\mathcal{Z}_\mathcal{K}$ therefore acquires a multigrading, with the multigraded and ordinary graded components of $H^*(\mathcal{Z}_\mathcal{K})$ given by

$$H^{-i,2J}(\mathcal{Z}_\mathcal{K}) = \operatorname{Tor}^{-i,2J}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big),\quad H^\ell(\mathcal{Z}_\mathcal{K}) = \bigoplus_{-i+2|J|=\ell} H^{-i,2J}(\mathcal{Z}_\mathcal{K}),$$

where $J=(j_1,\ldots,j_m)\in\mathbb{Z}^m$ and $|J|=j_1+\cdots+j_m$.

The Koszul algebra $\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[\mathcal{K}],d\big)$ is infinitely generated as a $\mathbf{k}$-module. We define its quotient algebra

$$R^*(\mathcal{K}) = \Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[\mathcal{K}]\big/(v_i^2=u_iv_i=0,\ 1\leqslant i\leqslant m)$$

with the induced multigrading and differential (2.7). Note that $R^*(\mathcal{K})$ has a finite $\mathbf{k}$-basis. Passing to $R^*(\mathcal{K})$ does not change the cohomology. This can be proved either algebraically [11, Lemma 3.2.6] or using the following topological interpretation:

Lemma 2.15 ([11, Lemma 4.5.3])**.** *The algebra $R^*(\mathcal{K})$ coincides with the cellular cochains of $\mathcal{Z}_\mathcal{K}$ for the appropriate cell structure. In particular, there is an isomorphism of cohomology algebras*

$$H(R^*(\mathcal{K}))\cong H^*(\mathcal{Z}_\mathcal{K}).$$

The multigraded component $R^{-i,2J}(\mathcal{K})$ is zero unless all coordinates of the vector $J\in\mathbb{Z}^m$ are 0 or 1, and the same is true for the multigraded cohomology $H^{-i,2J}(\mathcal{Z}_\mathcal{K})$.

We can identify subsets $J\subset[m]$ with vectors $\sum_{j\in J}\boldsymbol{e}_j\in\mathbb{Z}^m$. Given $J=\{j_1,\ldots,j_k\}\subset[m]$, we denote by v_J the monomial $v_{j_1}\cdots v_{j_k}\in\mathbf{k}[v_1,\ldots,v_m]$, and similarly consider exterior monomials $u_J=u_{j_1}\cdots u_{j_k}\in\Lambda[u_1,\ldots,u_m]$. We also use the notation u_Jv_I for the monomial $u_J\otimes v_I$ in the Koszul algebra $\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[\mathcal{K}]$. Then $R^*(\mathcal{K})$ has a finite $\mathbf{k}$-basis consisting of monomials u_Jv_I where $J\subset[m]$, $I\in\mathcal{K}$ and $J\cap I=\varnothing$.

Given $J\subset[m]$, define the corresponding *full subcomplex* of $\mathcal{K}$ as

$$\mathcal{K}_J=\{I\in\mathcal{K}\colon I\subset J\}.$$

Consider simplicial cochains $C^*(\mathcal{K}_J)$ with coefficients in $\mathbf{k}$. Let $\alpha_L\in C^{p-1}(\mathcal{K}_J)$ be the basis cochain corresponding to an oriented simplex $L=(l_1,\ldots,l_p)\in\mathcal{K}_J$; it takes value 1 on L and vanishes on all other simplices. Define a $\mathbf{k}$-linear map

$$\begin{aligned} f\colon C^{p-1}(\mathcal{K}_J) &\longrightarrow R^{p-|J|,2J}(\mathcal{K}),\\ \alpha_L &\longmapsto \varepsilon(L,J)\,u_{J\setminus L}v_L, \end{aligned}\tag{2.8}$$

where $\varepsilon(L,J)$ is the sign given by $\varepsilon(L,J)=\prod_{j\in L}\varepsilon(j,J)$ and $\varepsilon(j,J)=(-1)^{r-1}$ if j is the rth element of the set $J\subset[m]$ written in increasing order.

Theorem 2.16 ([11, Theorem 3.2.9])**.** *The maps* (2.8) *combine to an isomorphism of cochain complexes $C^*(\mathcal{K}_J)\to R^{*,2J}(\mathcal{K})$ and induce an isomorphism*

$$\widetilde{H}^{|J|-i-1}(\mathcal{K}_J)\cong\operatorname{Tor}^{-i,2J}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big),$$

where $\widetilde{H}^k(\mathcal{K}_J)$ denotes the kth reduced simplicial cohomology group of $\mathcal{K}_J$.

Theorem 2.17 ([11, Theorem 4.5.8])**.** *There are isomorphisms of* $\mathbf{k}$*-modules*

$$H^{-i,2J}(\mathcal{Z}_\mathcal{K})\cong\widetilde{H}^{|J|-i-1}(\mathcal{K}_J),\qquad H^\ell(\mathcal{Z}_\mathcal{K})\cong\bigoplus_{J\subset[m]}\widetilde{H}^{\ell-|J|-1}(\mathcal{K}_J).$$

These isomorphisms combine to form a ring isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J),$$

where the ring structure on the right hand side is given by the canonical maps

$$H^{k-|I|-1}(\mathcal{K}_I) \otimes H^{\ell-|J|-1}(\mathcal{K}_J) \to H^{k+\ell-|I|-|J|-1}(\mathcal{K}_{I\cup J})$$

which are induced by the simplicial maps $\mathcal{K}_{I\cup J} \to \mathcal{K}_I * \mathcal{K}_J$ *for* $I \cap J = \varnothing$ *and zero otherwise.*

Proposition 2.18. *The* 3*-dimensional cohomology* $H^3(\mathcal{Z}_\mathcal{K})$ *is freely generated by the cohomology classes* $[u_iv_j] = [u_jv_i]$ *corresponding to pairs of vertices* i, j *such that* $\{i, j\} \notin \mathcal{K}$*. If* $\mathcal{K} = \mathcal{K}_P$ *for a simple polytope* P*, then these* 3*-dimensional cohomology classes correspond to pairs of non-adjacent facets* F_i, F_j*.*

Example 2.19. Let $\mathcal{K} =$ 1•—•2 3•—•4 be the union of two segments. Then nontrivial integral cohomology groups of $\mathcal{Z}_\mathcal{K}$ are given below together with the cocycles in the algebra $R^*(\mathcal{K})$ representing generators:

$H^0(\mathcal{Z}_\mathcal{K}) \cong \widetilde{H}^{-1}(\varnothing) \cong \mathbb{Z}$	1
$H^3(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{\lvert J\rvert=2} \widetilde{H}^0(\mathcal{K}_J) \cong \mathbb{Z}^4$	$u_1v_3,\ u_1v_4,\ u_2v_3,\ u_2v_4$
$H^4(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{\lvert J\rvert=3} \widetilde{H}^0(\mathcal{K}_J) \cong \mathbb{Z}^4$	$u_1u_2v_3,\ u_1u_2v_4,\ u_3u_4v_1,\ u_3u_4v_2$
$H^5(\mathcal{Z}_\mathcal{K}) \cong \widetilde{H}^0(\mathcal{K}) \cong \mathbb{Z}$	$u_1u_2u_4v_3 - u_1u_2u_3v_4$

Cochains in $C^0(\mathcal{K})$ are functions on the vertices of $\mathcal{K}$, and cocycles are functions which are constant on the connected components of $\mathcal{K}$. In our case, the cocycle $\alpha_{\{3\}} + \alpha_{\{4\}}$ represents a generator of $\widetilde{H}^0(\mathcal{K})$. It is mapped by (2.8) to the cocycle $u_1u_2u_4v_3 - u_1u_2u_3v_4$ representing a generator of $H^5(\mathcal{Z}_\mathcal{K})$.

Moment-angle complexes $\mathcal{Z}_\mathcal{K}$ may have nontrivial triple Massey products of 3-dimensional cohomology classes. First examples (found by Baskakov [3]) appear already for moment-angle manifolds corresponding to 3-polytopes (see also [11, §4.9]). A complete description of the triple Massey product $H^3(\mathcal{Z}_\mathcal{K}) \otimes H^3(\mathcal{Z}_\mathcal{K}) \otimes H^3(\mathcal{Z}_\mathcal{K}) \to H^8(\mathcal{Z}_\mathcal{K})$ is given by the following result of Denham and Suciu:

Theorem 2.20 ([24, Theorem 6.1.1])**.** *The following are equivalent:*

(a) *there exist cohomology classes* $\alpha, \beta, \gamma \in H^3(\mathcal{Z}_\mathcal{K})$ *for which the Massey product* $\langle\alpha, \beta, \gamma\rangle$ *is defined and non-trivial;*

(b) *the graph* $\mathcal{K}^1$ *contains an induced subgraph isomorphic to one of the five graphs in Figure 1.*

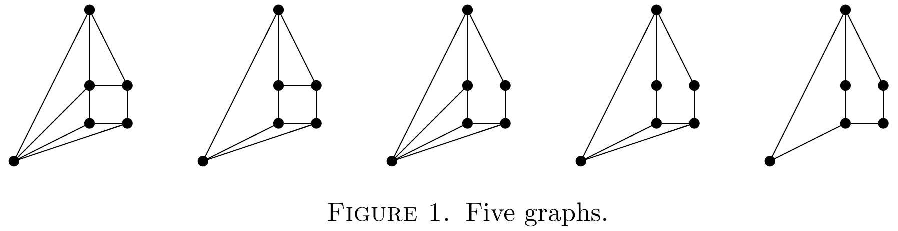


Figure 1. Five graphs.

2.9. **Moment-angle manifolds and quasitoric manifolds.** Let P be a simple n-polytope with the dual simplicial complex $\mathcal{K}_P$. The existence of a characteristic matrix (2.3) for P is equivalent to a choice of n linear forms

$$t_j = \lambda_{j1}v_1 + \cdots + \lambda_{jm}v_m, \quad j = 1, \dots, n \tag{2.9}$$

such that $\mathbb{Z}[\mathcal{K}_P]$ is a finitely generated free module over $\mathbb{Z}[t_1, \dots, t_n]$. Such $t_1, \dots, t_n$ form a linear *regular sequence* in $\mathbb{Z}[\mathcal{K}_P]$. This implies that $\mathbf{k}[\mathcal{K}_P]$ is a *Cohen–Macaulay ring* over any $\mathbf{k}$, but the condition is actually stronger, as it assumes the existence of a *linear* regular sequence over $\mathbb{Z}$ (and hence over any finite field).

Given a characteristic matrix (2.3) (or a linear regular sequence (2.9) in $\mathbb{Z}[\mathcal{K}_P]$), one can define the corresponding homomorphism of tori $\varLambda_T\colon T^m \to T^n$. Its kernel $\operatorname{Ker}\varLambda_T$ is an $(m-n)$-dimensional subtorus in T^m that acts *freely* on $\mathcal{Z}_P$. The quotient $\mathcal{Z}_P/\operatorname{Ker}\varLambda_T$ can be identified with the quasitoric manifold $M(P,\varLambda)$ from Construction 2.5. As $\mathcal{Z}_P$ is a smooth intersection of quadrics (2.6) and the action of torus is smooth, we obtain a canonical smooth structure on $M(P,\varLambda)$ as in [12].

We say that T^n-manifolds M and M' are *weakly equivariantly diffeomorphic* if there is a diffeomorphism $f\colon M \to M'$ and an automorphism $\theta\colon T^n \to T^n$ such that $f(t\cdot x) = \theta(t)\cdot f(x)$ for any $x \in M$ and $t \in T^n$. The following result is immediate.

Proposition 2.21. *If characteristic pairs $(P,\varLambda)$ and $(P',\varLambda')$ are equivalent, then the corresponding quasitoric manifolds $M(P,\varLambda)$ and $M(P',\varLambda')$ are weakly equivariantly diffeomorphic.*

The general homological properties of regular sequences imply yet another description of the cohomology of $\mathcal{Z}_P$:

Theorem 2.22 ([10, Theorem 4.2.11], [11, Lemma A.3.5])**.** *Let P be a simple n-polytope with m facets, and assume there exists a linear integral regular sequence (2.9). Denote by $\mathcal{J}$ the ideal in $\mathbb{Z}[v_1, \dots, v_m]$ generated by $t_1, \dots, t_n$. Then there is an isomorphism of cohomology rings*

$$H^*(\mathcal{Z}_P;\mathbb{Z}) \cong \operatorname{Tor}_{\mathbb{Z}[v_1,\dots,v_m]/\mathcal{J}}\bigl(\mathbb{Z}[\mathcal{K}_P]/\mathcal{J}, \mathbb{Z}\bigr).$$

Note that $\mathbb{Z}[\mathcal{K}_P]/\mathcal{J}$ is the cohomology ring of the quasitoric manifold $M(P,\varLambda)$, see Theorem 2.3. The theorem above implies that the spectral sequence of the principal T^{m-n}-fibration $\mathcal{Z}_P \to M(P,\varLambda)$ degenerates at the E_3 term.

3. Cohomological rigidity

We continue to consider cohomology with coefficients in a commutative ring with unit $\mathbf{k}$. When $\mathbf{k}$ is not specified explicitly, we assume $\mathbf{k} = \mathbb{Z}$.

Definition 3.1. We say that a family of closed manifolds is *cohomologically rigid* over $\mathbf{k}$ if manifolds in the family are distinguished up to homeomorphism by their cohomology rings with coefficients in $\mathbf{k}$. That is, a family is cohomologically rigid if a graded ring isomorphism $H^*(M_1;\mathbf{k}) \cong H^*(M_2;\mathbf{k})$ implies a homeomorphism $M_1 \cong M_2$ whenever M_1 and M_2 are in the family.

There is a homotopical and smooth versions of cohomological rigidity, with homeomorphisms replaced by homotopy equivalences and diffeomorphisms, respectively.

In toric topology, cohomological rigidity is studied for (quasi)toric manifolds and moment-angle manifolds. We refer to [37], [18] and [11, §7.8] for a more detailed survey of related results and problems. The main question here is as follows.

Problem 2. *Let M_1 and M_2 be two toric manifolds with isomorphic cohomology rings. Are they homeomorphic? In other words, is the family of toric manifolds cohomologically rigid? One can ask the same question for quasitoric and topological toric manifolds, and with homeomorphisms replaced by diffeomorphisms.*

The problem is solved positively for some particular families of toric and quasitoric manifolds, such as cohomologically trivial Bott towers (whose corresponding polytopes are combinatorial cubes) [36], $\mathbb{Q}$-cohomologically trivial Bott towers [14], $\mathbb{Z}_2$-cohomologically trivial Bott towers [15], Bott towers of real dimension up to 8 [13], quasitoric manifolds over a product of two simplices [21] and over some dual cyclic polytopes [30]. The problem is open for general Bott towers, and for (quasi)toric manifolds of real dimension 6 (over 3-dimensional polytopes). The latter case is the subject of this paper: we give a solution for a particular class of 3-polytopes.

There is also a cohomological rigidity problem for real toric objects, such as real toric manifolds, small covers, and real topological toric manifolds [31], with $\mathbb{Z}_2$-cohomology rings. This problem is solved positively for real Bott towers [16], [33], but negatively in some other cases [35].

Cohomological rigidity is also open for moment-angle manifolds, in both graded and bigraded versions:

Problem 3. *Let $\mathcal{Z}_{P_1}$ and $\mathcal{Z}_{P_2}$ be two moment-angle manifolds whose (bigraded) cohomology rings are isomorphic. Are they homeomorphic? In other words, is the family of moment-angle manifolds cohomologically rigid?*

A homeomorphism of two quasitoric manifolds over P_1 and P_2, or a homeomorphism of moment-angle manifolds $\mathcal{Z}_{P_1}$ and $\mathcal{Z}_{P_2}$ does not imply that the polytopes P_1 and P_2 are combinatorially equivalent, as shown by the next example.

Example 3.2. A *vertex truncation* operation [11, Construction 1.1.1] can be applied to a simple polytope P to produce a new simple polytope $\mathrm{vt}(P)$ with one more facet. If one applies this operation iteratively, then the combinatorial type of the resulting polytope depends, in general, on the choice and order of truncated vertices. For example, applying this operation three times to a 3-simplex one can produce three combinatorially different polytopes P_i, $i = 1, 2, 3$, with 7 facets each (their dual simplicial polytopes are known as *stacked*). The corresponding moment-angle manifolds $\mathcal{Z}_{P_i}$ are diffeomorphic, see [11, §4.6]. The polytopes P_i have Delzant realisations such that the correponding toric manifolds V_{P_i} are obtained from $\mathbb{C}P^3$ by blowing it up three times in three different ways. Each of V_{P_i} is diffeomorphic to a connected sum of 4 copies of $\mathbb{C}P^3$ (see the details in [37]).

One can look for classes of simple polytopes P whose combinatorial type is determined by the cohomology ring of any (quasi)toric manifold over P or by the cohomology ring of the moment-angle manifold $\mathcal{Z}_P$. This leads to the following two notions of rigidity for simple polytopes, considered in [37] and [6] respectively.

Definition 3.3. A simple polytope P is said to be *C-rigid* if any of the two conditions hold:

(a) there are no quasitoric manifolds M over P (equivalently, there are no linear regular sequences (2.9) in $\mathbb{Z}[\mathcal{K}_P]$), or
(b) whenever there exist a quasitoric manifold M over P and a quasitoric manifold M' over another polytope P' with a cohomology ring isomorphism $H^*(M) \cong H^*(M')$, there is a combinatorial equivalence $P \simeq P'$.

We say that a property of simple polytopes is *C-rigid* if for any ring isomorphism $H^*(M) \cong H^*(M')$, both P and P' either have or do not have the property.

Definition 3.4. A simple polytope P is said to be *B-rigid* if any cohomology ring isomorphism $H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$ of moment-angle manifolds implies a combinatorial equivalence $P \simeq P'$.

We say that a property of simple polytopes is *B-rigid* if for any ring isomorphism $H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$, both P and P' either have or do not have the property.

According to Example 3.2, a truncated simplex with at least 3 truncations (the dual to a stacked polytope with at least 3 stacks) is neither C-rigid nor B-rigid. Previously known examples of C-rigid polytopes include products of simplices and their single vertex truncations [19], as well as a product of a simplex and a polygon [20]. Also, C-rigidity was determined in [19] for all simple 3-polytopes with $\leqslant 9$ facets. The following relation between the two notions of rigidity can be extracted from the results of [19]:

Proposition 3.5. *If a simple polytope P is B-rigid, then it is C-rigid.*

Proof. Assume that we have a cohomology ring isomorphism $\varphi\colon H^*(M) \xrightarrow{\cong} H^*(M')$ for quasitoric manifolds M over P and M' over P'. We need to show that it implies a ring isomorphism $\psi\colon H^*(\mathcal{Z}_P) \xrightarrow{\cong} H^*(\mathcal{Z}_{P'})$, as the latter would give $P \simeq P'$ by B-rigidity. Let $\mathcal{J}$ and $\mathcal{J}'$ denote the corresponding ideals in $\mathbb{Z}[\mathcal{K}_P]$ and $\mathbb{Z}[\mathcal{K}_{P'}]$, respectively, generated by the linear regular sequences (2.9). Then we have a ring isomorphism $\varphi\colon \mathbb{Z}[\mathcal{K}_P]/\mathcal{J} \xrightarrow{\cong} \mathbb{Z}[\mathcal{K}_{P'}]/\mathcal{J}'$. We need to show that this isomorphism gives rise to a ring isomorphism

$$\operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]/\mathcal{J}}\bigl(\mathbb{Z}[\mathcal{K}_P]/\mathcal{J},\mathbb{Z}\bigr) \xrightarrow{\cong} \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_{m'}]/\mathcal{J}'}\bigl(\mathbb{Z}[\mathcal{K}_{P'}]/\mathcal{J}',\mathbb{Z}\bigr), \tag{3.1}$$

as the latter is nothing but an isomorphism $H^*(\mathcal{Z}_P) \xrightarrow{\cong} H^*(\mathcal{Z}_{P'})$ according to Theorem 2.22. This is is proved in [19, Lemma 3.7]. Namely, the isomorphism $\varphi\colon \mathbb{Z}[\mathcal{K}_P]/\mathcal{J} \xrightarrow{\cong} \mathbb{Z}[\mathcal{K}_{P'}]/\mathcal{J}'$ can be extended to a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}[v_1,\ldots,v_m]/\mathcal{J} & \xrightarrow{\cong} & \mathbb{Z}[v_1,\ldots,v_{m'}]/\mathcal{J}' \\
\downarrow & & \downarrow \\
\mathbb{Z}[\mathcal{K}_P]/\mathcal{J} & \xrightarrow[\cong]{\varphi} & \mathbb{Z}[\mathcal{K}_{P'}]/\mathcal{J}',
\end{array}$$

implying in particular that $m = m'$. The commutative diagram above gives rise to an isomorphism (3.1) by the standard properties of Tor. More specifically, the isomorphism φ gives an isomorphism of the Koszul algebras

$$\widetilde{\varphi}\colon \bigl(\Lambda[u_1,\ldots,u_m]/\mathcal{J}\otimes\mathbb{Z}[\mathcal{K}_P]/\mathcal{J},d\bigr) \xrightarrow{\cong} \bigl(\Lambda[u_1',\ldots,u_m']/\mathcal{J}'\otimes\mathbb{Z}[\mathcal{K}_{P'}]/\mathcal{J}',d\bigr), \tag{3.2}$$

where the ideals in the exterior algebras are defined by the same linear forms as in the face rings. Then (3.1) is obtained by passing to the cohomology. □

Remark. The argument above is essentially [19, Lemma 3.7]. The term “B-rigidity” was introduced in the last section of [19]. However, the implication of Proposition 3.5 was erroneously stated there in the opposite direction: “if P is C-rigid, then it is B-rigid”. This was a confusion. It is not known whether C-rigidity is equivalent to B-rigidity, and it is unlikely to be true in general.

4. The Pogorelov class: flag 3-polytopes without 4-belts

Recall that the Pogorelov class $\mathcal{P}$ consists of simple 3-polytopes P which are flag and do not have 4-belts (or, equivalently, of simple 3-polytopes $P \neq \varDelta^3$ without 3- and 4-belts). In this section we consider combinatorial properties of polytopes $P \in \mathcal{P}$ and cohomological properties of the corresponding moment-angle manifolds $\mathcal{Z}_P$. The key statements here are Theorem 4.8, Theorem 4.10 and Lemma 4.11; they will be used in the proof of the main results in the next section.

The first property is straightforward:

Proposition 4.1. *In a polytope $P \in \mathcal{P}$, there are no 3-gonal or 4-gonal facets.*

Lemma 4.2. *A simple* 3*-polytope* P *is flag if and only if each of its facets* F *is surrounded by a* k*-belt with* $k \geqslant 4$.

Proof. Note that k is the number of edges in F.

Assume that P is flag. Let $(F_1, \dots, F_k)$, $k \geqslant 4$, be the sequence of facets adjacent to F, written in a cyclic order. The claim is that it forms a k-belt. Indeed, otherwise there is a pair of facets F_i, F_j with $F_i \cap F_j \neq \varnothing$ and $j - i \not\equiv \pm 1 \mod k$. Then each pair of facets in the triple F, F_i, F_j has nonempty intersection, but $F \cap F_i \cap F_j = \varnothing$. This contradicts the assumption that P is flag.

Now assume that P is not flag. Then either $P = \Delta^3$, or P has a 3-belt, and none of the facets in this 3-belt is surrounded by a belt. □

Lemma 4.3. *For any two facets* F_i *and* F_j *in a polytope* $P \in \mathcal{P}$*, there is a vertex* $x \notin F_i \cup F_j$.

Proof. Take any facet F_ℓ different from F_i and F_j. Then F_ℓ has at most two common vertices with F_i and at most two common vertices with F_j. On the other hand, F_ℓ has at least 5 vertices by the previous corollary. Thus, at least one vertex of F_ℓ does not lie in $F_i \cup F_j$. □

Lemma 4.4. *In a flag* 3*-polytope* P*, for any facet* F_i *there is a facet* F_j *such that* $F_i \cap F_j = \varnothing$.

Proof. By Lemma 4.2 the facet F_i is surrounded by a k-belt $\mathcal{B}_k$ with $k \geqslant 4$. Then $\partial P \setminus \mathcal{B}_k$ consists of two connected components: one of them is the interior of F_i, and the other contains the interior of a facet F_j that we look for. □

Now we consider cohomology of moment-angle manifolds $\mathcal{Z}_P$ with coefficients in $\mathbb{Z}$. We recall from Proposition 2.18 that $H^3(\mathcal{Z}_P)$ has a basis of cohomology classes $[u_i v_j] = [u_j v_i]$ corresponding to pairs of non-adjacent facets F_i, F_j.

Proposition 4.5. *Let* P *be a simple* 3*-polytope with* m *facets and let* $\mathcal{K} = \mathcal{K}_P$ *be its dual simplicial complex. In the notation of Theorem 2.17, we have*

$$H^\ell(\mathcal{Z}_P) = \begin{cases} \widetilde{H}^{-1}(\mathcal{K}_\varnothing) = \mathbb{Z} & \text{for } \ell = 0, \\ \bigoplus_{|I|=\ell-1} \widetilde{H}^0(\mathcal{K}_I) \oplus \bigoplus_{|I|=\ell-2} \widetilde{H}^1(\mathcal{K}_I) & \text{for } 3 \leqslant \ell \leqslant m, \\ \widetilde{H}^2(\mathcal{K}) = \mathbb{Z} & \text{for } \ell = m+3, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $H^*(\mathcal{Z}_P)$ *does not have torsion. Furthermore, all nontrivial products in* $H^*(\mathcal{Z}_P)$ *are of the form*

$$\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^0(\mathcal{K}_J) \to \widetilde{H}^1(\mathcal{K}_{I \cup J}), \quad I \cap J = \varnothing,$$

or

$$\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_{[m] \setminus I}) \to \widetilde{H}^2(\mathcal{K}) = \mathbb{Z}.$$

For the multigraded components of $H^*(\mathcal{Z}_P)$*, these two cases correspond to*

$$H^{-(|I|-1),\,2I}(\mathcal{Z}_P) \otimes H^{-(|J|-1),\,2J}(\mathcal{Z}_P) \to H^{-(|I|+|J|-2),\,2(I \sqcup J)}(\mathcal{Z}_P),$$
$$H^{-(|I|-1),\,2I}(\mathcal{Z}_P) \otimes H^{-(m-|I|-2),\,2([m]\setminus I)}(\mathcal{Z}_P) \to H^{-(m-3),\,2[m]}(\mathcal{Z}_P) = \mathbb{Z},$$

where the latter is the Poincaré duality pairing.

Proof. This follows from Theorem 2.14, Theorem 2.16 and Theorem 2.17. □

An element in a graded ring is called *decomposable* if it can be written as a sum of nontrivial products of elements of nonzero degree.

Lemma 4.6 ([28, Proposition 6.3])**.** *Let P be a flag 3-polytope and $\mathcal{K}$ its dual simplicial complex. Then the ring $H^*(\mathcal{Z}_P) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J)$ is multiplicatively generated by $\bigoplus_{J\subset[m]} \widetilde{H}^0(\mathcal{K}_J)$.*

To prove this lemma it is enough to show that each nontrivial cohomology class in $\widetilde{H}^1(\mathcal{K}_I) \subset H^*(\mathcal{Z}_P)$ is decomposable or, equivalently, the product map

$$\bigoplus_{I=I_1\sqcup I_2} \widetilde{H}^0(\mathcal{K}_{I_1}) \otimes \widetilde{H}^0(\mathcal{K}_{I_2}) \to \widetilde{H}^1(\mathcal{K}_I)$$

is surjective. This proof is quite technical. We include it in Appendix B for the reader's convenience.

Lemma 4.7. *A simple 3-polytope $P \neq \Delta^3$ with m facets is flag if and only if any nontrivial cohomology class in $H^{m-2}(\mathcal{Z}_P)$ is decomposable. In particular, if $H^{m-2}(\mathcal{Z}_P) = 0$ then either P is flag or $P = \Delta^3$.*

Proof. Suppose that P is not flag. Since $P \neq \Delta^3$, it has a 3-belt $\{F_{j_1}, F_{j_2}, F_{j_3}\}$. Equivalently, the dual complex $\mathcal{K}$ has a missing 3-face $J = \{j_1, j_2, j_3\}$. It gives a nonzero cohomology class $\alpha \in H^{-1,2J}(\mathcal{Z}_P) \subset H^5(\mathcal{Z}_P)$. Consider the Poincaré duality pairing

$$H^{m-2}(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \to H^{m+3}(\mathcal{Z}_P) = \mathbb{Z},$$

which specifies to

$$H^{-(m-4),2([m]\setminus J)}(\mathcal{Z}_P) \otimes H^{-1,2J}(\mathcal{Z}_P) \to H^{-(m-3),\,2[m]}(\mathcal{Z}_P) = \mathbb{Z}$$

(see Proposition 4.5). Take $\beta \in H^{-(m-4),2([m]\setminus J)}(\mathcal{Z}_P) \subset H^{m-2}(\mathcal{Z}_P)$ such that $\alpha \cdot \beta$ is a generator of $H^{-(m-3),\,2[m]}(\mathcal{Z}_P) = \mathbb{Z}$. By Theorem 2.17, $H^{-(m-4),2([m]\setminus J)}(\mathcal{Z}_P) = \widetilde{H}^0(\mathcal{K}_{[m]\setminus J})$, and any element of $\widetilde{H}^0(\mathcal{K}_{[m]\setminus J})$ is indecomposable by Proposition 4.5. We have therefore found an indecomposable element $\beta \in H^{m-2}(\mathcal{Z}_P)$.

Now suppose that P is flag. By Proposition 4.5,

$$H^{m-2}(\mathcal{Z}_P) = \bigoplus_{|I|=m-3} \widetilde{H}^0(\mathcal{K}_I) \oplus \bigoplus_{|I|=m-4} \widetilde{H}^1(\mathcal{K}_I),$$

Consider the Poincaré duality pairing $\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_{[m]\setminus I}) \to \mathbb{Z}$. Since $\mathcal{K}$ is flag, $\widetilde{H}^1(\mathcal{K}_{[m]\setminus I}) = 0$ for $|I| = m-3$ (as there are no missing 2-faces). Hence, $\bigoplus_{|I|=m-3} \widetilde{H}^0(\mathcal{K}_I)=0$ by Poincaré duality, and $H^{m-2}(\mathcal{Z}_P) = \bigoplus_{|I|=m-4} \widetilde{H}^1(\mathcal{K}_I)$. Then each nonzero element of $H^{m-2}(\mathcal{Z}_P)$ is decomposable by Lemma 4.6. □

Theorem 4.8. *Let P be a flag 3-polytope, and assume given a ring isomorphism $H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$ for another simple 3-polytope P'. Then P' is also flag.*

In other words, the property of being a flag 3-polytope is B-rigid.

Proof. This follows from Lemma 4.7 and the fact that Δ^3 is B-rigid. □

Remark. Theorem 4.8 also follows from [28, Theorem 6.6].

Proposition 4.9. *Let P be a simple 3-polytope.*

(a) *The product $H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \to H^6(\mathcal{Z}_P)$ is trivial if and only if P does not have 4-belts.*
(b) *The triple Massey product $H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \to H^8(\mathcal{Z}_P)$ is trivial if P does not have 4-belts.*

Proof. We first prove (a). Suppose P has a 4-belt (F_1, F_2, F_3, F_4). It corresponds to a chordless 4-cycle $\{1, 2, 3, 4\}$ in $\mathcal{K} = \mathcal{K}_P$, i. e. a cycle with $\{1, 3\} \notin \mathcal{K}$ and $\{2, 4\} \notin \mathcal{K}$. Hence, we have a nontrivial product $\widetilde{H}^0(\mathcal{K}_{\{1,3\}}) \otimes \widetilde{H}^0(\mathcal{K}_{\{2,4\}}) \to \widetilde{H}^1(\mathcal{K}_{\{1,2,3,4\}})$, and a nontrivial product $H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \to H^6(\mathcal{Z}_P)$.

Now suppose there is a nontrivial product $H^3(\mathcal{Z}_P)\otimes H^3(\mathcal{Z}_P)\to H^6(\mathcal{Z}_P)$. We have $H^6(\mathcal{Z}_P)=\bigoplus_{|I|=5}\widetilde{H}^0(\mathcal{K}_I)\oplus\bigoplus_{|I|=4}\widetilde{H}^1(\mathcal{K}_I)$. Elements of $\widetilde{H}^0(\mathcal{K}_I)$ are indecomposable. An element of $\widetilde{H}^1(\mathcal{K}_I)$ with $|I|=4$ can be decomposed into a product if and only if I can be split into two pairs of non-adjacent vertices, which means that I is a chordless 4-cycle. It corresponds to a 4-belt in P.

To prove (b), assume that there is a nontrivial Massey product $\langle\alpha,\beta,\gamma\rangle\in H^8(\mathcal{Z}_P)$. Then, by Theorem 2.20, the graph $\mathcal{K}_P^1$ contains an induced subgraph isomorphic to one of the five graphs in Figure 1. By inspection, each of these five graphs has a chordless 4-cycle (the outer cycle for the first four graphs, and the left cycle for the last one). Hence, the polytope P has a 4-belt. □

It is not known whether moment-angle manifolds of polytopes from the Pogorelov class $\mathcal{P}$ have nontrivial Massey products of cohomology classes of dimension >3 or of order >3, or whether these moment-angle manifolds are formal in the sense of rational homotopy theory. For general polytopes P, there are examples of nontrivial Massey products of any order in $H^*(\mathcal{Z}_P)$, see [34].

Theorem 4.10. *Let P be a simple 3-polytope without 4-belts, and assume given a ring isomorphism $H^*(\mathcal{Z}_P)\cong H^*(\mathcal{Z}_{P'})$ for another simple 3-polytope P'. Then P' also does not have 4-belts.*

It other words, the property of being a 3-polytope without 4-belts is B-rigid.

Proof. This follows from Proposition 4.9 (a). □

Recently Fan, Ma and Wang proved that any polytope $P\in\mathcal{P}$ is B-rigid, see [27, Theorem 3.1]. The proof builds upon the following crucial lemma:

Lemma 4.11 ([27, Corollary 3.4])**.** *Consider the set of cohomology classes*

$$\mathcal{T}(P)=\{\pm[u_iv_j]\in H^3(\mathcal{Z}_P),\quad F_i\cap F_j=\varnothing\}.$$

If $P\in\mathcal{P}$, then for any cohomology ring isomorphism $\psi\colon H^*(\mathcal{Z}_P)\xrightarrow{\cong}H^*(\mathcal{Z}_{P'})$, we have $\psi(\mathcal{T}(P))=\mathcal{T}(P')$.

Note that the lemma above does not hold for all simple 3-polytopes. For example, if P is a 3-cube with the pairs of opposite facets $\{F_1,F_4\}$, $\{F_2,F_5\}$, $\{F_3,F_6\}$, then $\mathcal{Z}_P\cong S^3\times S^3\times S^3$ and there is an isomorphism $\psi\colon H^*(\mathcal{Z}_P)\xrightarrow{\cong}H^*(\mathcal{Z}_P)$ which maps $[u_1v_4]$ to $[u_1v_4]+[u_2v_5]$.

We include the proof of Lemma 4.11 in Appendix C for the reader's convenience, and also because some details were missing in the original argument. Note that this proof uses Theorem 4.8 and Theorem 4.10.

5. Main results

Here we prove the cohomological rigidity for small covers and quasitoric manifolds over 3-polytopes from the Pogorelov class $\mathcal{P}$. We start with a crucial lemma.

Lemma 5.1. *In the notation of Theorem 2.3, consider the set of cohomology classes*

$$\mathcal{D}(M)=\{\pm[v_i]\in H^2(M),\quad i=1,\dots,m\}.$$

If P is a 3-polytope from the Pogorelov class $\mathcal{P}$ and M' is a quasitoric manifold over P', then for any cohomology ring isomorphism $\varphi\colon H^*(M)\xrightarrow{\cong}H^*(M')$ we have $\varphi(\mathcal{D}(M))=\mathcal{D}(M')$. Moreover, the set $\mathcal{D}(M)$ is mapped bijectively under φ.

Proof. The idea is to reduce the statement to Lemma 4.11. The ring isomorphism φ is determined uniquely by the isomorphism $H^2(M)\xrightarrow{\cong}H^2(M')$ of free abelian groups. Let $\varphi([v_i])=\sum_{j=1}^m A_{ij}[v_j']$ for some $A_{ij}\in\mathbb{Z}$, $1\leqslant i,j\leqslant m$. The elements A_{ij} are not defined uniquely as there are linear relations between the classes $[v_j']$ in

$H^2(M')$. To get rid of this indeterminacy, one can choose a vertex $x = F_{i_1} \cap F_{i_2} \cap F_{i_3}$ of P and a vertex $x' = F'_{p_1} \cap F'_{p_2} \cap F'_{p_3}$ of P'. Then the complementary cohomology classes $[v_i]$ with $i \notin \{i_1, i_2, i_3\}$ form a basis in $H^2(M)$ and the cohomology classes $[v'_p]$ with $p \notin \{p_1, p_2, p_3\}$ form a basis in $H^2(M')$, so we have

$$\varphi([v_i]) = \sum_{p \notin \{p_1,p_2,p_3\}}^{m} B_{ip}[v'_p], \qquad i \in [m] \setminus \{i_1, i_2, i_3\}, \tag{5.1}$$

with uniquely defined $B_{ip} \in \mathbb{Z}$, $i \in [m] \setminus \{i_1, i_2, i_3\}$, $p \in [m] \setminus \{p_1, p_2, p_3\}$.

As we have seen in the proof of Proposition 3.5, the isomorphism φ gives an isomorphism $\psi\colon H^*(\mathcal{Z}_P) \xrightarrow{\cong} H^*(\mathcal{Z}_{P'})$, which is obtained from (3.2) by passing to the cohomology. We write (3.2) as $\widetilde{\varphi}\colon C(P, \Lambda) \xrightarrow{\cong} C(P', \Lambda')$. This isomorphism is defined on the exterior generators u_i and the polynomial generators v_i of the Koszul algebra $C(P, \Lambda)$ by the same formulae as φ.

Now take a cohomology class $[u_i v_j] \in H^3(\mathcal{Z}_P)$. By Lemma 4.11, it is mapped under ψ to an element $\varepsilon[u'_k v'_l] \in H^3(\mathcal{Z}_{P'})$, $\varepsilon = \pm 1$. Choose vertices $x = F_{i_1} \cap F_{i_2} \cap F_{i_3}$ of P and $x' = F'_{p_1} \cap F'_{p_2} \cap F'_{p_3}$ of P' such that $x \notin F_i \cup F_j$ and $x' \notin F'_k \cup F'_l$ (see Lemma 4.3). We use the vertices x and x' to choose bases in the groups $H^2(M)$ and $H^2(M')$ as described in the first paragraph of the proof. Then we have

$$\psi[u_i v_j] = \sum_{p,q \notin \{p_1,p_2,p_3\}} B_{ip} B_{jq} [u'_p v'_q].$$

On the other hand, we have $\psi[u_i v_j] = \varepsilon[u'_k v'_l]$ by Lemma 4.11. Hence,

$$a = \sum_{p,q \notin \{p_1,p_2,p_3\}} B_{ip} B_{jq} u'_p v'_q - \varepsilon u'_k v'_l$$

is a coboundary in $C(P', \Lambda')$, so there exists

$$c = \sum_{p,q \notin \{p_1,p_2,p_3\},\ p<q} L_{pq} u'_p u'_q, \qquad dc = a.$$

We have

$$dc = \sum_{p,q \notin \{p_1,p_2,p_3\},\ p<q} L_{pq} (u'_q v'_p - u'_p v'_q).$$

Comparing this with the expression for a we obtain the following relations between the coefficients:

$$\begin{aligned} &B_{ip} B_{jq} = -B_{iq} B_{jp} = -L_{pq} \quad \text{for } p < q \text{ and } \{p, q\} \neq \{k, l\}; \\ &B_{ik} B_{jl} - \varepsilon = -B_{il} B_{jk} = \begin{cases} -L_{kl} & \text{if } k < l, \\ L_{lk} & \text{if } l < k; \end{cases} \\ &B_{ip} B_{jp} = 0. \end{aligned} \tag{5.2}$$

From the third equation of (5.2) we have, for any $p \in [m] \setminus \{p_1, p_2, p_3\}$, either $B_{ip} = 0$ or $B_{jp} = 0$. The first equation of (5.2) implies that for $\{p, q\} \neq \{k, l\}$ the vectors $\begin{pmatrix} B_{ip} \\ B_{jp} \end{pmatrix}$ and $\begin{pmatrix} B_{iq} \\ -B_{jq} \end{pmatrix}$ are linearly dependent. Hence, for $\{p, q\} \neq \{k, l\}$, either one of the vectors $\begin{pmatrix} B_{ip} \\ B_{jp} \end{pmatrix}$ and $\begin{pmatrix} B_{iq} \\ B_{jq} \end{pmatrix}$ is zero, or both vectors are nonzero and have a zero entry on the same place. From the second equation $B_{ik} B_{jl} + B_{il} B_{jk} = \varepsilon$ we see that both vectors $b_k = \begin{pmatrix} B_{ik} \\ B_{jk} \end{pmatrix}$ and $b_l = \begin{pmatrix} B_{il} \\ B_{jl} \end{pmatrix}$ are nonzero. If there is a nonzero vector $b_p = \begin{pmatrix} B_{ip} \\ B_{jp} \end{pmatrix}$ for some $p \notin \{k, l\}$, then by considering the pairs (b_p, b_k) and

(b_p, b_l) we see that both b_k and b_l have zero on the same place, which contradicts the second equation of (5.2). It follows that $B_{ip} = B_{jp} = 0$ for any $p \notin \{k, l\}$, and

$$\begin{pmatrix} B_{ik} & B_{il} \\ B_{jk} & B_{jl} \end{pmatrix} = \begin{pmatrix} B_{ik} & 0 \\ 0 & \frac{\varepsilon}{B_{ik}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B_{ik} & B_{il} \\ B_{jk} & B_{jl} \end{pmatrix} = \begin{pmatrix} 0 & B_{il} \\ \frac{\varepsilon}{B_{il}} & 0 \end{pmatrix}.$$

Since all entries are integer, we have $B_{ik} = \pm 1$ and $B_{il} = \pm 1$. Then (5.1) gives $\varphi([v_i]) \in \{\pm[v'_k], \pm[v'_l]\}$. It follows that $\varphi(\mathcal{D}(M)) \subset \mathcal{D}(M')$.

It remains to show that the set $\mathcal{D}(M)$ is mapped bijectively under φ. For this we note that $[v_i] \neq \pm[v_j]$ in $H^2(M)$ for any $i \neq j$. Indeed, by Lemma 4.3 we can choose a vertex $x \notin F_i \cup F_j$. Then both $[v_i]$ and $[v_j]$ belong to a basis of $H^2(M)$. Now, since φ is an isomorphism, we also have $\varphi([v_i]) \neq \pm\varphi([v_j])$ in $H^2(M')$. Thus, both sets $\mathcal{D}(M)$ and $\mathcal{D}(M')$ consist of $2m$ elements and $\varphi(\mathcal{D}(M)) = \mathcal{D}(M')$. □

It follows from the Steinitz Theorem that any toric manifold of complex dimension 3 is a quasitoric manifold. Also, the family of quasitoric manifolds agrees with that of topological toric manifolds in real dimension 6 if we forget the actions.

Now we state the first main result.

Theorem 5.2. *Let $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$ be quasitoric manifolds over 3-dimensional simple polytopes P and P', respectively. Assume that P belongs to the Pogorelov class $\mathcal{P}$. Then the following conditions are equivalent:*

(a) *there is a cohomology ring isomorphism $\varphi\colon H^*(M) \xrightarrow{\cong} H^*(M')$;*
(b) *there is a diffeomorphism $M \cong M'$;*
(c) *there is an equivalence of characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

Proof. The implication (b)⇒(a) is obvious. The implication (c)⇒(b) follows from Proposition 2.21. We need to prove (a)⇒(c).

By Lemma 5.1, $\varphi([v_i]) = \pm[v'_{\sigma(i)}]$, where σ is a permutation of the set $[m]$. Renumbering the facets and multiplying the matrix Λ from the right by a matrix B as in Definition 2.6, we may assume that $\varphi([v_i]) = v'_i$; this does not change the equivalence class of the pair (P, Λ). Then $\varphi[v_i v_j] = [v'_i v'_j]$. By Corollary 2.4, $[v_i v_j] = 0$ in $H^*(M)$ if and only if $F_i \cap F_j = \varnothing$ and $[v_i v_j v_k] = 0$ in $H^*(M)$ if and only if $F_i \cap F_j \cap F_k = \varnothing$ in P, and the same holds for $H^*(M')$ and P'. It follows that $\mathcal{K}_P$ is isomorphic to $\mathcal{K}_{P'}$. Hence, P and P' are combinatorially equivalent.

Now consider the $(3 \times m)$-matrices Λ and Λ'. First, by changing the order of facets in P and P' if necessary we may assume that $F_1 \cap F_2 \cap F_3 \neq \varnothing$ in P and $F'_1 \cap F'_2 \cap F'_3 \neq \varnothing$ in P'. Then, by multiplying the matrices Λ and Λ' from the left by appropriate matrices from $GL(3, \mathbb{Z})$ we may assume that

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & \lambda_{14} & \cdots & \lambda_{1m} \\ 0 & 1 & 0 & \lambda_{24} & \cdots & \lambda_{2m} \\ 0 & 0 & 1 & \lambda_{34} & \cdots & \lambda_{3m} \end{pmatrix}, \qquad \Lambda' = \begin{pmatrix} 1 & 0 & 0 & \lambda'_{14} & \cdots & \lambda'_{1m} \\ 0 & 1 & 0 & \lambda'_{24} & \cdots & \lambda'_{2m} \\ 0 & 0 & 1 & \lambda'_{34} & \cdots & \lambda'_{3m} \end{pmatrix}.$$

This does not change the equivalence class of pairs (P, Λ) and (P', Λ'). Now the entries λ_{jk}, $4 \leqslant k \leqslant m$, are the coefficients in the expression of $[v_j]$, $1 \leqslant j \leqslant 3$, via the basis $[v_4], \ldots, [v_m]$ of $H^2(M)$. The same holds for the λ'_{jk}. Since $\varphi([v_i]) = v'_i$, it follows that $\lambda_{jk} = \lambda'_{jk}$. Thus, the pairs (P, Λ) and (P', Λ') are equivalent. □

Remark. Any smooth structure on a quasitoric manifold M over a polytope $P \in \mathcal{P}$ is equivalent to the standard one defined on the canonical model $M(P, \Lambda)$ via Proposition 2.21. This follows from the general classification results for 6-dimensional manifolds, see Corollary 6.4.

Corollary 5.3. *Toric, quasitoric and topological toric manifolds over polytopes from the Pogorelov class $\mathcal{P}$ are cohomologically rigid.*

The family of quasitoric (or topological toric) manifolds over 3-polytopes from the Pogorelov class $\mathcal{P}$ is large enough, as there is at least one quasitoric manifold over any such polytope by Proposition 2.8 (recall that it uses the Four Colour Theorem). There are fewer toric manifolds in this family. In fact, there are no *projective* toric manifolds over combinatorial flag 3-polytopes without 4-belts. The reason is that a Delzant 3-polytope must have at least one triangular or quadrangular face by the result of C. Delaunay [23] (see also [2]). On the other hand, there are many non-projective toric manifolds in this family, see [42].

Remark. In general, a (non-equivariant) diffeomorphism $M \cong M'$ does not imply an equivalence of characteristic pairs. For example consider the family of toric manifolds (*Hirzebruch surfaces*) $H_k = \mathbb{C}P(\mathcal{O}(k) \oplus \underline{\mathbb{C}})$, where $\mathcal{O}(k)$ is the kth power of the canonical line bundle over $\mathbb{C}P^1$, $\underline{\mathbb{C}}$ is a trivial line bundle, and $\mathbb{C}P(-)$ denotes the complex projectivisation. Then each H_k is a quasitoric manifold over a combinatorial 4-gon with characteristic matrix

$$\begin{pmatrix} 1 & 0 & -1 & k \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Manifolds H_k with even k are all diffeomorphic to $S^2 \times S^2$, but the characteristic matrices corresponding to different positive k are not equivalent. Similar examples exist in all dimensions, see e.g. [36].

Our second main result is about small covers (or hyperbolic 3-manifolds).

Theorem 5.4. *Let $N = N(P, \Lambda)$ and $N' = N(P', \Lambda')$ be small covers of 3-dimensional simple polytopes P and P', respectively. Assume that P belongs to the Pogorelov class $\mathcal{P}$, so N is a hyperbolic 3-manifold of Löbell type. Then the following conditions are equivalent:*

(a) *there is a cohomology ring isomorphism $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$;*
(b) *there is a diffeomorphism $N \cong N'$;*
(c) *there is an equivalence of $\mathbb{Z}_2$-characteristic pairs $(P, \Lambda) \sim (P', \Lambda')$.*

Proof. We only need to prove the implication (a)⇒(c). Using Proposition 2.10 we upgrade (P, Λ) and (P', Λ') to $\mathbb{Z}$-characteristic pairs and consider the corresponding quasitoric manifolds $M = M(P, \Lambda)$ and $M' = M(P', \Lambda')$. Since the cohomology ring $H^*(M;\mathbb{Z}_2)$ is obtained from $H^*(N;\mathbb{Z}_2)$ by doubling the grading (see Theorem 2.9), we have an isomorphism $H^*(M;\mathbb{Z}_2) \xrightarrow{\cong} H^*(M';\mathbb{Z}_2)$. Now the equivalence of characteristic pairs follows from Theorem 5.2 (with coefficients in $\mathbb{Z}_2$). □

6. Classification of 6-dimensional manifolds and related problems

The classification of smooth simply connected 6-dimensional manifolds with torsion-free homology was done in the works of Wall [44] and Jupp [32]. They also stated a result in the topological category, whose proof was corrected in the work of Zhubr [45]. The latter work also treated the case of homology with torsion. We only give the following result, which will be enough for our purposes (the cohomology is with integer coefficients, unless otherwise specified).

Theorem 6.1 ([44], [32])**.** *Let $\varphi\colon H^*(N) \xrightarrow{\cong} H^*(N')$ be an isomorphism of the cohomology rings of smooth closed simply connected 6-dimensional manifolds N, N' with $H^3(N) = H^3(N') = 0$. Suppose that*

(a) *$\varphi(w_2(N)) = w_2(N')$, where $w_2(N) \in H^2(N;\mathbb{Z}_2)$ is the second Stiefel–Whitney class;*
(b) *$\varphi(p_1(N)) = p_1(N')$, where $p_1(N) \in H^4(N)$ is the first Pontryagin class.*

Then the manifolds N and N' are diffeomorphic.

The following lemma is proved using Steenrod squares:

Lemma 6.2 ([17, Lemma 8.1]). *Suppose that the ring $H^*(N;\mathbb{Z}_2)$ is generated by $H^k(N;\mathbb{Z}_2)$ for some $k > 0$. Then any cohomology ring isomorphism $\varphi\colon H^*(N;\mathbb{Z}_2) \xrightarrow{\cong} H^*(N';\mathbb{Z}_2)$ preserves the total Stiefel–Whitney class, i. e. $\varphi(w(N)) = w(N')$.*

Lemma 6.2 applies to toric or quasitoric manifolds, whose cohomology is generated in degree two. From Theorem 6.1 we obtain

Corollary 6.3. *Let $\varphi\colon H^2(M) \xrightarrow{\cong} H^2(M')$ be an isomorphism of second cohomology groups of* 6*-dimensional smooth quasitoric manifolds. Suppose that*

(a) *φ preserves the cubic form $H^2(M)\otimes H^2(M)\otimes H^2(M) \to \mathbb{Z} = H^6(M)$ given by the cohomology multiplication;*
(b) *φ preserves the first Pontryagin class.*

Then the manifolds M and M' are diffeomorphic.

From the topological invariance of rational Pontryagin classes (proved in general by S. P. Novikov) we obtain

Corollary 6.4. *Let M and M' be* 6*-dimensional smooth quasitoric manifolds. If M and M' are homeomorphic, then they are diffeomorphic.*

The characteristic classes of quasitoric manifolds are given as follows:

Proposition 6.5 ([22, Corollary 6.7]). *In the notation of Theorem 2.3, the total Stiefel–Whitney and Pontryagin classes of a quasitoric manifold M are given by*

$$w(M) = \prod_{i=1}^{m}(1 + v_i) \mod 2, \qquad p(M) = \prod_{i=1}^{m}(1 + v_i^2).$$

In particular, $w_2(M) = v_1 + \cdots + v_m \mod 2$, and $p_1(M) = v_1^2 + \cdots + v_m^2$.

Corollary 6.6. *A family of* 6*-dimensional quasitoric manifolds is cohomologically rigid if any cohomology ring isomorphism between manifolds from the family preserves the first Pontryagin class.*

This reduces cohomological rigidity for 6-dimensional quasitoric manifolds M to a problem of combinatorics and linear algebra, as both the cohomology ring $H^*(M)$ and the first Pontryagin class $p_1(M) = v_1^2 + \cdots + v_m^2$ are defined entirely in terms of the characteristic pair (P, Λ).

Our result on cohomological rigidity for quasitoric manifolds over polytopes from the Pogorelov class $\mathcal{P}$ (Theorem 5.2) gives a complete classification for this particular class of simply connected 6-manifolds, and its proof is indepenent of the general classification results of Wall and Jupp. The invariance of the first Pontryagin class for quasitoric manifolds over polytopes from the Pogorelov class follows directly from Lemma 5.1. It would be interesting to find a direct (combinatorial?) proof of this fact. Bott towers (of any dimension) form another family of toric manifolds for which the invariance of Pontryagin classes is known, see [15].

Remark. In dimension 4 we have the identity $\langle p_1(M), [M]\rangle = 3\,\mathrm{sign}(M)$, where $[M] \in H_4(M)$ is the fundamental class and $\mathrm{sign}(M)$ is the signature of M. Therefore, p_1 is invariant under cohomology ring isomorphisms. When M is a toric manifold, the signature is equal to $4-m$, where m is the number of vertices in the quotient polygon P (see e.g. [11, Example 9.5.3]). The identity $\langle p_1(M), [M]\rangle = 3\,\mathrm{sign}(M)$ then becomes

$$\langle v_1^2 + \cdots + v_m^2, [M]\rangle = 12 - 3m,$$

which can be seen directly from Theorem 2.2.

Appendix A. Belts in 3-polytopes

Here we give proofs of two combinatorial lemmata on belts in flag 3-polytopes, originally due to [28] and [27] respectively. These proofs are included mainly for the sake of completeness, but we also fill in some details missing in the original works. Lemma A.1 is used in the proof of the product decomposition lemma in Appendix B, while Lemma A.3 is used in the proof of B-rigidity of the set of canonical generators of $H^3(\mathcal{Z}_P)$ in Appendix C.

Recall that a belt in a simple polytope P corresponds to a chordless cycle in the dual simplicial complex $\mathcal{K}_P$, or to a full subcomplex $(\mathcal{K}_P)_I$ isomorphic to the boundary of a polygon.

Lemma A.1. *Let P be a flag simple 3-polytope. Then for every three facets F_i, $F_{i'}$, F_k with $F_i \cap F_{i'} = \varnothing$, there exists a belt $\mathcal{B}$ such that $F_i, F_{i'} \in \mathcal{B}$ and $F_k \notin \mathcal{B}$.*

We reformulate this lemma in the dual notation; this is how the lemma was stated and proved in [28]:

Lemma A.2 ([28, Lemma 6.1])**.** *Let $\mathcal{K}$ be a flag triangulation of the disk D^2 with m vertices, and let S be the set of vertices of the boundary $\partial\mathcal{K}$. Assume that $\mathcal{K}_S = \partial\mathcal{K}$. Then for every missing face $\{i, i'\}$ of $\mathcal{K}$ there exists a subset $I \subset [m]$ such that $\{i, i'\} \subset I$ and $\mathcal{K}_I$ is a chordless cycle (boundary of a polygon).*

To obtain Lemma A.1 from Lemma A.2 we take $\mathcal{K} = (\mathcal{K}_P)_{[m]\setminus\{k\}}$, the simplicial complement to the vertex of $\mathcal{K}_P$ corresponding to the facet $F_k \subset P$. Then $\mathcal{K}$ is a flag triangulation of D^2 (as a full subcomplex in the flag complex $\mathcal{K}_P$), and $\mathcal{K}_S = \partial\mathcal{K}$ because $\mathcal{K}_P$ is flag. Lemma A.2 gives a chordless cycle I in $\mathcal{K} \subset \mathcal{K}_P$, which corresponds to the required belt in P.

The *star* and *link* of a vertex $\{i\} \in \mathcal{K}$ are the subcomplexes

$$\mathrm{star}_{\mathcal{K}}\{i\} = \{I \in \mathcal{K}\colon \{i\} \cup I \in \mathcal{K}\}, \quad \mathrm{link}_{\mathcal{K}}\{i\} = \{I \in \mathcal{K}\colon \{i\} \cup I \in \mathcal{K}, i \notin I\}.$$

Proof of Lemma A.2. We use the induction on m, the number of vertices of $\mathcal{K}$. Since $\mathcal{K}$ is flag, $\partial\mathcal{K}$ has at least 4 vertices, that is, $|S| \geqslant 4$ and $m \geqslant 5$. If $m = 5$, then $|S| = 4$ and $\mathcal{K}$ is the cone over a square. Then we can take $I = S$, as every missing face of $\mathcal{K}$ is contained in the chordless boundary cycle $\mathcal{K}_S = \partial\mathcal{K}$.

Now assume that the statement holds for simplicial complexes with $< m$ vertices. If both vertices i and i' lie in $\partial\mathcal{K}$, then $I = S$ is the required chordless cycle. Hence, we only need to consider the case $\{i, i'\} \not\subset S$. If $|S| \geqslant m-1$, then there is no missing face $\{i, i'\}$ such that $\{i, i'\} \not\subset S$. Hence, $|S| < m - 1$. For a vertex $j \in S$, denote by m_j the number of vertices in $\mathrm{star}_{\mathcal{K}}\{j\}$. Then $m_j \geqslant 4$ for any $j \in S$, since $\mathcal{K}_S = \partial\mathcal{K}$.

I. Suppose that there is a vertex $j \in S \setminus \{i, i'\}$ such that $m_j = 4$. Then the set of vertices of $\mathrm{star}_{\mathcal{K}}\{j\}$ is $\{j, j', j'', k\}$, where $j, j', j'' \in S$ and $k \notin S$, see Figure 2.

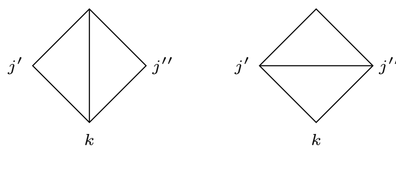


Figure 2. $\mathrm{star}_{\mathcal{K}}\{j\}$ and its bistellar 1-move

(i) If there is no vertex $k' \in S \setminus \{j, j', j''\}$ such that $\{k, k'\}$ is an edge of $\mathcal{K}$, then the simplicial complex $\mathcal{K}' = \mathcal{K}_{[m]\setminus\{j\}}$ satisfies the hypothesis of the lemma. Hence, there is a subset I' of $[m] \setminus \{j\}$ such that $\{i, i'\} \subset I'$ and $\mathcal{K}'_{I'}$ is a chordless cycle. Then $I = I'$ is the required set, as $\mathcal{K}_{I'} = \mathcal{K}'_{I'}$.

(ii) Now assume that there exists a vertex $k' \in S \setminus \{j, j', j''\}$ such that $\{k, k'\}$ is an edge in $\mathcal{K}$. Let $\mathcal{K}'$ be the simplicial complex obtained from $\mathcal{K}$ by applying a bistellar 1-move at $\mathrm{star}_{\mathcal{K}}\{j\}$, see Figure 2. Then $\mathcal{K}'' := \mathcal{K}'_{[m]\setminus\{j\}}$ satisfies the hypothesis of the lemma. By induction, there is a subset I'' of $[m] \setminus \{j\}$ such that $\{i, i'\} \subset I''$ and $\mathcal{K}''_{I''}$ is a chordless cycle. If j' or j'' is not in I'', then $I = I''$ is the required set. If both j' and j'' are in I'', then $I = I'' \cup \{j\}$ is the required set.

II. Suppose that $m_j > 4$ for every $j \in S \setminus \{i, i'\}$. Let $S = \{j_1, \ldots, j_n\}$, ordered counterclockwise, and assume that $j_1 \notin \{i, i'\}$. Let $\mathcal{V}_{j_p}$ denote the set of vertices of $\mathrm{star}_{\mathcal{K}}(j_p)$, so $|\mathcal{V}_{j_p}| = m_{j_p}$, for $1 \leqslant p \leqslant n$. Note that if $j_p \in S \setminus \{i, i'\}$, then $m_{j_p} > 4$ and $|\mathcal{V}_{j_p} \setminus S| > 1$.

(i) Assume that, for some $j_p \in S \setminus \{i, i'\}$, there is no edge $\{k, k'\}$ in $\mathcal{K}$ such that

$$(*) \qquad k \in \mathcal{V}_{j_p} \setminus S \quad \text{and} \quad k' \in S \setminus \{j_{p-1}, j_p, j_{p+1}\}, \quad \text{where } j_0 = j_n.$$

Then $\mathcal{K}' := \mathcal{K}_{[m]\setminus\{j_p\}}$ satisfies the hypothesis of the lemma, so we can find the required subset I of $[m] \setminus \{j_p\}$.

(ii) Assume that, for every $j_p \in S \setminus \{i, i'\}$, there is an edge $\{k_p, j_{q_p}\}$ in $\mathcal{K}$ satisfying $(*)$ for $k = k_p$ and $k' = j_{q_p}$. We shall lead this case to a contradiction. Set $I_1 = \{j_1, k_1, j_{q_1}\}$. Then $\mathcal{K}_{I_1}$ divides $\mathcal{K}$ into two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$, where $\mathcal{K}_1$ has boundary vertices $j_1, \ldots, j_{q_1}, k_1$, and $\mathcal{K}_2$ has boundary vertices $j_{q_1}, \ldots, j_n, k_1$, see Figure 3.

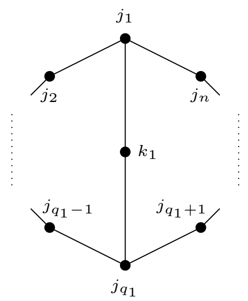


FIGURE 3. $\mathcal{K}_{I_1}$ divides $\mathcal{K}$ into two simplicial complexes.

Since $\{i, i'\} \not\subset S$, either $\{i, i'\} \cap \{j_1, \ldots, j_{q_1-1}\} = \varnothing$ or $\{i, i'\} \cap \{j_{q_1}, \ldots, j_n\} = \varnothing$. Without loss of generality, assume that $\{i, i'\} \cap \{j_1, \ldots, j_{q_1-1}\} = \varnothing$. Then $m_{j_p} > 4$ for $1 \leqslant p \leqslant q_1 - 1$. By the flagness of $\mathcal{K}$ and the condition for the existence of an edge satisfying $(*)$, there is no vertex $k \in [m] \setminus S$ such that k is connected to the vertices j_p and j_{p+2} for $1 \leqslant p \leqslant q_1 - 2$. This implies in particular that $q_1 > 3$.

Now consider the path from j_2 to k_2 and to j_{q_2}. If $k_2 = k_1$, then we may assume that $j_{q_2} = j_{q_1}$. Otherwise, k_2 must be contained in the simplicial complex $\mathcal{K}_1$. In either case, the the path $j_2 - k_2 - j_{q_2}$ is contained in the simplicial subcomplex $\mathcal{K}_1$ with boundary vertices $j_1, \ldots, j_{q_1}, k_1$. Proceeding inductively, we obtain that the path $j_p - k_p - j_{q_p}$ is contained in the simplicial subcomplex whose boundary vertices are $j_{p-1}, \ldots, j_{q_{p-1}}, k_{p-1}$, see Figure 4. It follows that $p < q_p \leqslant q_{p-1} \leqslant \cdots \leqslant q_1$. Eventually we obtain p such that $q_p = p + 2$, so the vertex k_p is connected to the vertices j_p and j_{p+2}. This is a contradiction.

From I and II, the lemma is proved. □

Lemma A.3 ([27, Lemma 3.2])**.** *Let P be a flag 3-polytope without 4-belts. Then for every three different facets F_i, $F_{i'}$, F_k with $F_i \cap F_{i'} = \varnothing$ there is a belt $\mathcal{B}$ such that $F_i, F_{i'} \in \mathcal{B}$, $F_k \notin \mathcal{B}$, and F_k does not intersect at least one of the two connected components of $\mathcal{B} \setminus \{F_i, F_{i'}\}$.*

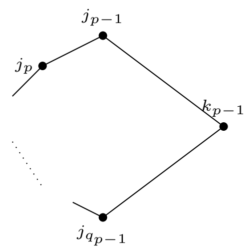


FIGURE 4. The path $j_p - k_p - j_{q_p}$ is contained in the above simplicial complex.

Proof. We work with the dual simplicial complex $\mathcal{K} = \mathcal{K}_P$, which is a triangulated 2-sphere. We need to find a subset $I \subset [m] \setminus \{k\}$ such that $\{i, i'\} \subset I$, $\mathcal{K}_I$ is a chordless cycle, and $\widetilde{H}^0(\mathcal{K}_{(I\setminus\{i,i'\})\cup\{k\}}) \neq 0$. By Lemma A.2, there is a subset I_0 of $[m] \setminus \{k\}$ such that $\{i, i'\} \subset I_0$ and $\mathcal{K}_{I_0}$ is a chordless cycle. We construct the required subset I by modifying I_0.

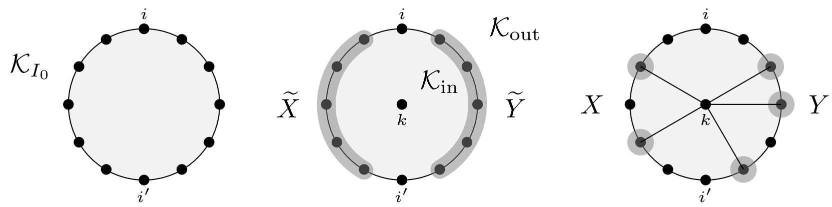


FIGURE 5. Examples of $\mathcal{K}_{I_0}$, $\mathcal{K}_{\text{in}}$, $\mathcal{K}_{\text{out}}$, $\widetilde{X}$, $\widetilde{Y}$, X and Y

Since $\mathcal{K}_{I_0}$ is a cycle, it divides $\mathcal{K}$ into two polygons (triangulated discs) $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ with the common boundary $\mathcal{K}_{I_0}$. Assume that the vertex k is contained in $\mathcal{K}_{\text{in}}$. The vertices i and i' divide the cycle $\mathcal{K}_{I_0}$ into two arcs, and we denote by $\widetilde{X}$ and $\widetilde{Y}$ the sets of vertices in $I_0 \setminus \{i, i'\}$ contained in these arcs, so $I_0 \setminus \{i, i'\} = \widetilde{X} \sqcup \widetilde{Y}$. We set $X = \operatorname{link}_{\mathcal{K}}\{k\} \cap \widetilde{X}$ and $Y = \operatorname{link}_{\mathcal{K}}\{k\} \cap \widetilde{Y}$, see Figure 5. If either X or Y is empty, then $\widetilde{H}^0(\mathcal{K}_{(I_0\setminus\{i,i'\})\cup\{k\}}) \neq 0$, so $I = I_0$ is the required subset. In what follows we assume that both X and Y are nonempty.

We consider the links of all $x \in X$ in $\mathcal{K}_{\text{out}}$. Since $\mathcal{K}_{I_0}$ is a chordless cycle, every such link has at least three vertices, that is, there is a vertex in $\operatorname{link}_{\mathcal{K}_{\text{out}}}\{x\}$ which is not in I_0. To simplify notation, for $X \subset [m]$, we write $\operatorname{link}_{\mathcal{K}} X$ instead of $\bigcup_{x\in X} \operatorname{link}_{\mathcal{K}}\{x\}$. Now define

$$\mathcal{K}_X := \text{ the full subcomplex of } \mathcal{K} \text{ induced on the set } \widetilde{X} \cup \{i, i'\} \cup \operatorname{link}_{\mathcal{K}_{\text{out}}} X.$$

We take the outermost path $\mathcal{P}_X$ between i and i' in $\mathcal{K}_X$ with respect to the vertex k, so that all vertices of $\mathcal{K}_X$ not in $\mathcal{P}_X$ are on the side of k, see Figure 6. Let I_X be the vertex set of $\mathcal{P}_X$.

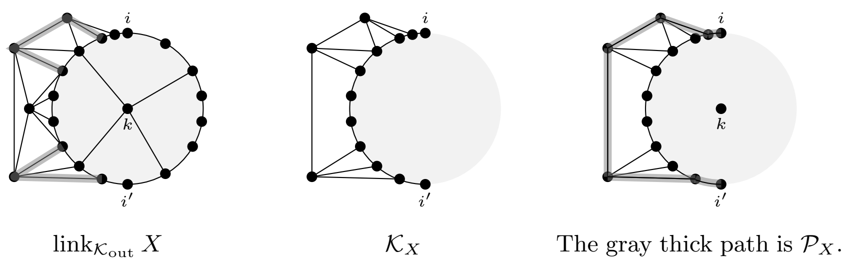
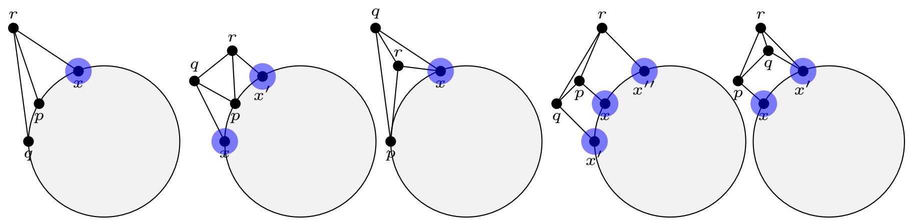


FIGURE 6. Examples of $\mathcal{K}_X$ and $\mathcal{P}_X$

Claim. The full subcomplex of $\mathcal{K}$ induced on I_X is the path $\mathcal{P}_X$, i.e., $\mathcal{K}_{I_X} = \mathcal{P}_X$.

Proof of Claim. Suppose to the contrary that there is a subset $\{p,q,r\}$ of I_1 such that $\mathcal{K}_{\{p,q,r\}}$ is a triangle. Consider the intersection $\{p,q,r\} \cap \widetilde{X}$. Note that $|\{p,q,r\} \cap \widetilde{X}| < 3$ because $\mathcal{K}_{\widetilde{X}}$ is a part of a chordless cycle $\mathcal{K}_{I_0}$. We have the following cases, shown in Figure 7.



(1) (2)-(a) (2)-(b) (3)-(a) (3)-(b)

FIGURE 7. $\mathcal{K}_X$ has no triangle.

(1) If $|\{p,q,r\} \cap \widetilde{X}| = 2$, say $\{p,q,r\} \cap \widetilde{X} = \{p,q\}$, then p and q are consecutive vertices in X, and r is in $\mathrm{link}_{\mathcal{K}_{\mathrm{out}}}\{x\}$ for some $x \in X$. Then, p or q is on the side of k in $\mathcal{K}_X$. This is a contradiction.
(2) If $|\{p,q,r\} \cap \widetilde{X}| = 1$, say $\{p,q,r\} \cap \widetilde{X} = \{p\}$, then $q \in \mathrm{link}_{\mathcal{K}_2} x$ and $r \in \mathrm{link}_{\mathcal{K}_2}\{x'\}$ for some $x, x' \in X$.
   (a) If $x \neq x'$, then p must be on the side of k in $\mathcal{K}_X$, which contradicts the assumption that $p \in \mathcal{P}_X$.
   (b) If $x = x'$, then either q or r is on the side of k in $\mathcal{K}_X$, and we obtain a contradiction again.
(3) If $|\{p,q,r\} \cap \widetilde{X}| = 0$, then there are x, x', x'' in X such that $p \in \mathrm{link}_{\mathcal{K}_{\mathrm{out}}}\{x\}$, $q \in \mathrm{link}_{\mathcal{K}_{\mathrm{out}}}\{x'\}$, and $r \in \mathrm{link}_{\mathcal{K}_{\mathrm{out}}}\{x''\}$. Since p,q,r are in the outermost path $\mathcal{P}_X$, the case $x = x' = x''$ is impossible. Hence, we may assume that $x \neq x'$ or $x \neq x''$.
   (a) If x, x', x'' are all distinct, then one of p, q, and r must be on the side of k in $\mathcal{K}_X$, which contradicts the assumption that p,q,r are on $\mathcal{P}_X$ and $\mathcal{P}_X$ is the outermost path with respect to k.
   (b) If $x' = x''$, then either q or r is on the side of k in $\mathcal{K}_X$. This final contradiction finishes the proof of the claim. □

We return to the proof of Lemma A.3. The endpoints of the path $\mathcal{P}_X = \mathcal{K}_{I_X}$ are i, i' and there is no edge connecting k and I_X. Therefore, if $\mathcal{K}_{I_X \cup \widetilde{Y}}$ is a chordless cycle, then $I_X \cup \widetilde{Y}$ is the required set I.

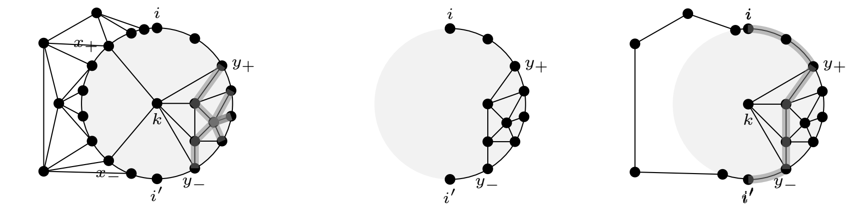
Suppose that $\mathcal{K}_{I_X \cup \widetilde{Y}}$ has a chord. Then the chord must be an edge in $\mathcal{K}_{\mathrm{out}}$. Note that since $\mathcal{K}$ has no chordless 4-cycles, there is no edge connecting $\mathrm{link}_{\mathcal{K}_{\mathrm{out}}} X$ and Y. We consider the vertices $x_+ \in X$ and $x_- \in X$ that are closest to i and i', respectively, on the arc containing $\widetilde{X}$. Similarly, consider the vertices $y_+ \in Y$ and $y_- \in Y$ that are closest to i and i', respectively, on the arc containing $\widetilde{Y}$. Denote by X_+ the subset of vertices in $\widetilde{X}$ lying strictly between i and x_+. Define the subsets $X_- \subset \widetilde{X}$, $Y_+ \subset \widetilde{Y}$ and $Y_- \subset \widetilde{Y}$ similarly. See Figure 8, left.

We consider two cases.

Case 1. There is no edge connecting I_X and $Y_- \cup Y_+$ in $\mathcal{K}_{\mathrm{out}}$.

We define

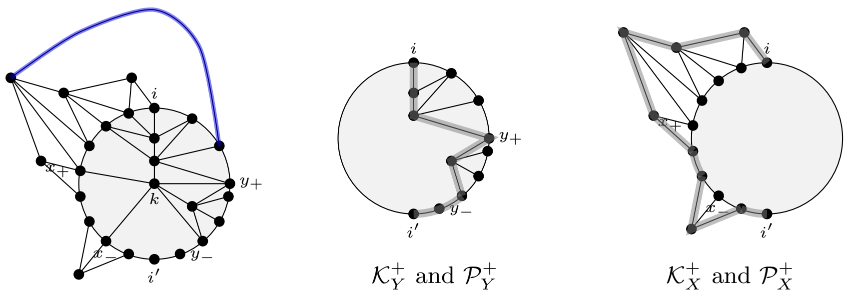
$$\mathcal{K}_Y := \text{ the full subcomplex of } \mathcal{K} \text{ induced on } \widetilde{Y} \cup \{i,i'\} \cup \mathrm{link}_{\mathcal{K}_{\mathrm{in}}}(\widetilde{Y} \setminus (Y \cup Y_- \cup Y_+)).$$



$\operatorname{link}_{\mathcal{K}_{\text{in}}}(\widetilde{Y} \setminus (Y \cup Y_- \cup Y_+))$ $\mathcal{K}_Y$ The gray thick path is $\mathcal{P}_Y$.

FIGURE 8. Example of Case 1

We take the innermost path $\mathcal{P}_Y$ connecting i and i' in $\mathcal{K}_Y$ with respect to k, see Figure 8, and let I_Y be the vertex set of $\mathcal{P}_Y$. Then $\mathcal{K}_{I_Y} = \mathcal{P}_Y$ by the same argument as the claim above, and $I_X \cup I_Y$ is the required subset I.



$\mathcal{K}_Y^+$ and $\mathcal{P}_Y^+$ $\mathcal{K}_X^+$ and $\mathcal{P}_X^+$

FIGURE 9. Example of Case 2

Case 2. There is an edge connecting I_X and Y_+ or Y_- in $\mathcal{K}_{\text{out}}$.
Suppose that I_X is connected by an edge in $\mathcal{K}_{\text{out}}$ to only one of Y_+ and Y_-, say to Y_+. We define

$$\mathcal{K}_Y^+ = \text{the full subcomplex of } \mathcal{K} \text{ induced on } \widetilde{Y} \cup \{i, i'\} \cup \operatorname{link}_{\mathcal{K}_{\text{in}}}(\widetilde{Y} \setminus (Y \cup Y_-)).$$

We take the innermost path $\mathcal{P}_Y^+$ connecting i and i' in $\mathcal{K}_{\text{in}}$ with respect to the vertex k, and let I_Y^+ be the vertex set of $\mathcal{P}_Y^+$. See Figure 9, middle. Then $\mathcal{K}_{I_Y^+} = \mathcal{P}_Y^+$ by the same reason as the claim above. If $\mathcal{K}_{I_X \cup I_Y^+}$ is a chordless cycle, then $I_X \cup I_Y^+$ is the required subset I.

If $\mathcal{K}_{I_X \cup I_Y^+}$ has a chord, then it must be an edge in $\mathcal{K}_{\text{in}}$ connecting $\operatorname{link}_{\mathcal{K}_{\text{in}}} Y_+$ and $X_+ \cap I_X$. In this case we modify I_X as follows. We define

$$\mathcal{K}_X^+ := \text{ the full subcomplex of } \mathcal{K} \text{ induced on } \widetilde{X} \cup \{i, i'\} \cup \operatorname{link}_{\mathcal{K}_{\text{out}}}(X \cup X_+).$$

We take the outermost path $\mathcal{P}_X^+$ connecting i and i' in $\mathcal{K}_X^+$ with respect to the vertex k, see Figure 9, right. Let I_X^+ be the vertex set of $\mathcal{P}_X^+$. Then $\mathcal{K}_{I_X^+} = \mathcal{P}_X^+$ by the same argument as the claim above, and we can see that $I_X^+ \cup I_Y$ is the required subset I. Indeed, we only need to check that there is no edge connecting $\operatorname{link}_{\mathcal{K}_{\text{out}}} X_+$ and Y in $\mathcal{K}_{\text{out}}$. This is because there is an edge connecting I_X and Y_+.

It remains to consider the case when I_X is connected to both Y_+ and Y_- by edges in $\mathcal{K}_{\text{out}}$. Here the same argument as above works if we consider

$$\mathcal{K}_Y^{\pm} := \text{the full subcomplex of } \mathcal{K} \text{ induced on } \widetilde{Y} \cup \{i, i'\} \cup \operatorname{link}_{\mathcal{K}_{\text{in}}}(\widetilde{Y} \setminus Y)$$

and

$$\mathcal{K}_X^{\pm} := \text{the full subcomplex of } \mathcal{K} \text{ induced on } \widetilde{X} \cup \{i, i'\} \cup \operatorname{link}_{\mathcal{K}_{\text{out}}}(X \cup X_+ \cup X_-)$$

instead of $\mathcal{K}_Y^+$ and $\mathcal{K}_X^+$, respectively. ∎

Appendix B. Proof of Lemma 4.6.

Here we give a proof which is different from the original proof of [28]. It uses a reformulation of the description of product in the cohomology of a moment-angle complex (Theorem 2.17) in terms of the polytope P. A detailed description of this approach can be found in [9, §5.8].

We need to prove that the product map

$$\bigoplus_{I=I_1\sqcup I_2} \widetilde{H}^0(\mathcal{K}_{I_1})\otimes \widetilde{H}^0(\mathcal{K}_{I_2}) \to \widetilde{H}^1(\mathcal{K}_I) \tag{B.1}$$

is surjective for any flag 3-polytope P and $I \subset [m]$. We first restate this in terms of the polytope P rather than its dual simplical complex $\mathcal{K}$. The decomposition of ∂P into facets $F_1,\ldots,F_m$ defines a cell decomposition of ∂P which is Poincaré dual to the simplicial decomposition $\mathcal{K}$. The two decompositions have the same barycentric subdivision, $(\partial P)'\cong\mathcal{K}'$. We identify the set of facets $\{F_1,\ldots,F_m\}$ with $[m]$, and for each $I\subset[m]$ define

$$P_I=\bigcup_{i\in I}F_i\subset\partial P.$$

Note that P_I is the combinatorial neighbourhood of $(\mathcal{K}_I)'$ in $\mathcal{K}'$, so there is a deformation retraction $P_I \xrightarrow{\simeq} \mathcal{K}_I$. We have Poincaré duality isomorphisms

$$H_{2-i}(P_I,\partial P_I)\cong H^i(\mathcal{K}_I),\quad i=0,1,2, \tag{B.2}$$

where the boundary ∂P_I consists of points $x\in P_I$ such that $x\in F_j$ for some $j\notin I$. Topologically, P_I is a disjoint union of several discs with holes, and ∂P_I is a disjoint union of edge cycles.

The cellular homology groups $H_i(P_I,\partial P_I)$ have the following description. Let $P_I=P_{I^1}\sqcup\cdots\sqcup P_{I^s}$ be the decomposition into connected components. Then

(a) $H_2(P_I,\partial P_I)$ is a free abelian group with basis of homology classes $[P_{I^k}]=\sum_{i\in I^k}[F_i]$, $k=1,\ldots,s$;

(b) $H_1(P_I,\partial P_I)=\bigoplus_{k=1}^{s}H_1(P_{I^k},\partial P_{I^k})$, where $H_1(P_{I^k},\partial P_{I^k})$ is a free abelian group of rank one less the number of cycles in ∂P_{I^k}. A basis of $H_1(P_{I^k},\partial P_{I^k})$ is given by any set of edge paths in P_{I^k} connecting one fixed boundary cycle with the other boundary cycles.

As the product map (B.1) is stated in terms of the reduced cohomology groups $\widetilde{H}^i(\mathcal{K}_I)$, we introduce the corresponding "reduced" homology groups

$$\widehat{H}_i(P_I,\partial P_I)=\begin{cases}H_i(P_I,\partial P_I), & i=0,1;\\ H_2(P_I,\partial P_I)/\big(\sum_{i\in I}[F_i]\big), & i=2.\end{cases}$$

Then we can rewrite (B.2) as

$$\widehat{H}_{2-i}(P_I,\partial P_I)\cong\widetilde{H}^i(\mathcal{K}_I),\quad i=0,1,2. \tag{B.3}$$

With this interpretation in mind, we can rewrite the product map (B.1) as the "intersection pairing"

$$\begin{aligned}\bigoplus_{I=I_1\sqcup I_2}\widehat{H}_2(P_{I_1},\partial P_{I_1})\otimes\widehat{H}_2(P_{I_2},\partial P_{I_2})&\to\widehat{H}_1(P_I,\partial P_I),\\ [P_{I_1^p}]\otimes[P_{I_2^q}]&\mapsto[P_{I_1^p}\cap P_{I_2^q}]=[\gamma_1]+\cdots+[\gamma_r],\end{aligned} \tag{B.4}$$

where $P_{I_1^p}$ is a connected component of P_{I_1}, $P_{I_2^q}$ is a connected component of P_{I_2}, and $\gamma_1,\ldots,\gamma_r$ are edge paths in P which form the connected components of the intersection $P_{I_1^p}\cap P_{I_2^q}$. (There is a sign involved in the transition from (B.1) to (B.4), but it does not affect our subsequent considerations.)

Proof of Lemma 4.6. To see that (B.4) is surjective for a flag 3-polytope P, we recall that $\widehat{H}_1(P_I, \partial P_I) = \bigoplus_{k=1}^{s} \widehat{H}_1(P_{I^k}, \partial P_{I^k})$ and consider for each connected component P_{I^k} of P_I the decomposition $\partial P_{I^k} = \eta_1 \sqcup \dots \sqcup \eta_{t_k}$ into boundary cycles. We may assume that $t_k \geqslant 2$, as otherwise P_{I^k} is a disc and $\widehat{H}_1(P_{I^k}, \partial P_{I^k}) = 0$. For each pair of boundary cycles η_p and η_q among $\eta_1, \dots, \eta_{t_k}$, we shall decompose the generator g_{pq} of $\widehat{H}_1(P_{I^k}, \partial P_{I^k})$ corresponding to an edge path from η_p to η_q into a product of elements of $\widehat{H}_2(P_{I_1}, \partial P_{I_1})$ and $\widehat{H}_2(P_{I_2}, \partial P_{I_2})$, $I_1 \sqcup I_2 = I$. This will prove the surjectivity of (B.4).

We choose facets F_p and F_q in $\partial P \setminus P_{I^k}$ adjacent to η_p and η_q respectively, see Figure 10. By Lemma A.1, there is a belt $\mathcal{B} = (F_{j_1}, \dots, F_{j_l})$ with $F_{j_1} = F_p$ and

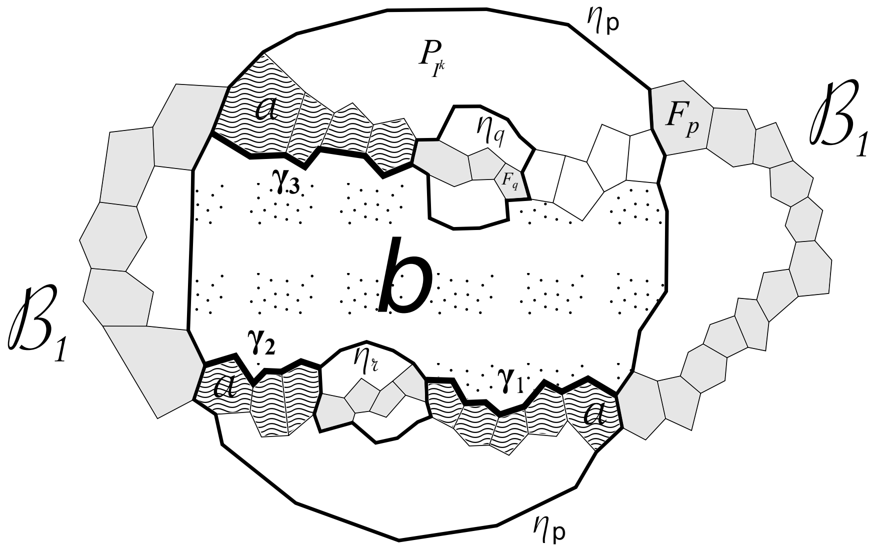


Figure 10. A belt crossing a disc with holes.

$F_{j_r} = F_q$, where $3 \leqslant r \leqslant l-1$. Let $\mathcal{B}_1 = (F_{j_1}, \dots, F_{j_r})$ be a part of the belt between F_p and F_q (there are two such parts, and we can take any of them). The complement $\partial P \setminus \mathcal{B}$ is a union of two open discs. We denote the closures of these discs by $\mathcal{W}_1$ and $\mathcal{W}_2$; each of them is a union of facets not in $\mathcal{B}$. Now set

$$I_1 = \{i \colon F_i \in P_{I^k} \cap \mathcal{B}\}, \qquad I_2 = I \setminus I_1,$$

$$a = \sum_{F_i \in P_{I^k} \cap \mathcal{B}_1} [F_i] \in \widehat{H}_2(P_{I_1}, \partial P_{I_1}), \qquad b = \sum_{F_j \in P_{I^k} \cap \mathcal{W}_1} [F_j] \in \widehat{H}_2(P_{I_2}, \partial P_{I_2}).$$

Then $a \cdot b = [\gamma_1] + \dots + [\gamma_s] \in H_1(P_I, \partial P_I)$, where each γ_i is an edge path in P_{I^k} that begins at some boundary cycle $\eta_{j_{i-1}}$ and ends at η_{j_i}. We may assume that γ_1 begins at η_p and γ_s ends at η_q (where η_p, η_q is the pair of boundary cycles chosen above). The homology class $[\gamma_1] + \dots + [\gamma_s] \in \widehat{H}_1(P_I, \partial P_I)$ is then equal to the chosen generator g_{pq} of $\widehat{H}_1(P_{I^k}, \partial P_{I^k})$ corresponding to an edge path from η_p to η_q. We have therefore decomposed g_{pq} into a product $a \cdot b$, as needed. $\square$

Appendix C. Proof of Lemma 4.11

The proof uses the combinatorial result of Lemma A.3 and an algebraic "annihilator lemma" of Fan, Ma and Wang.

Recall that the *annihilator* of an element r in a ring R is defined as

$$\operatorname{Ann}_R(r) = \{s \in R \colon rs = 0\}.$$

Lemma C.1 ([27, Lemma 3.3]). *Let P be a 3-polytope from the Pogorelov class $\mathcal{P}$, with the dual complex $\mathcal{K} = \mathcal{K}_P$. Let $R = H^*(\mathcal{Z}_P; \mathbf{k})$, where $\mathbf{k}$ is a field. In the notation of Lemma 4.11, consider a $\mathbf{k}$-linear combination of elements of $\mathcal{T}(P)$,*

$$\alpha = \sum_{\{i,j\}\notin\mathcal{K}} r_{ij}[u_i v_j]$$

with at least two nonzero $r_{ij} \in \mathbf{k}$. Then, for any $\{k,l\}$ such that $r_{kl} \neq 0$,

$$\dim \operatorname{Ann}_R[u_k v_l] > \dim \operatorname{Ann}_R \alpha.$$

Proof. In view of the isomorphisms (B.3), we can rewrite the isomorphism of Theorem 2.17 as

$$R = H^*(\mathcal{Z}_P) \cong \bigoplus_{I\subset[m]} \widehat{H}_*(P_I, \partial P_I)$$

(we omit the coefficient field $\mathbf{k}$ in the notation for homology).

Take a complementary subspace L_{kl} to $\operatorname{Ann}_R[u_k v_l]$ in R, so that $L_{kl} \oplus \operatorname{Ann}_R[u_k v_l] = R$. For any $\beta \in L_{kl} \setminus \{0\}$ we have $\beta \cdot [u_k v_l] \neq 0$. Furthermore, we can choose L_{kl} respecting the multigrading, so that the Ith multigraded component of L_{kl} is a complementary subspace to $\operatorname{Ann}_R[u_k v_l] \cap \widehat{H}_*(P_I, \partial P_I)$ in $\widehat{H}_*(P_I, \partial P_I)$. Then we can write $\beta = \sum_{I\subset[m]\setminus\{k,l\}} \beta_I$, where β_I denotes the Ith multigraded component of $\beta \in L_{kl} \setminus \{0\}$. (Note that $\beta_I = 0$ whenever $I \cap \{k,l\} \neq \varnothing$, as such β_I would annihilate $[u_k v_l]$.) We can choose $I \subset [m] \setminus \{k,l\}$ such that $\beta_I \cdot [u_k v_l] \neq 0$. Now consider $\alpha = \sum r_{ij}[u_i v_j]$. We claim that the $(I \cup \{k,l\})$th multigraded component of $\beta \cdot \alpha$ consists of $\beta_I \cdot [u_k v_l]$ only. Indeed, for any other component $\beta_{I'}$ of β with $I' \neq I$ and any summand $r_{ij}[u_i v_j]$ of α, we have $I' \cup \{i,j\} \neq I \cup \{k,l\}$, as $I' \in [m] \setminus \{k,l\}$. Then $(\beta \cdot \alpha)_{I\cup\{k,l\}} = \beta_I \cdot [u_k v_l] \neq 0$. Hence, $L_{kl} \cap \operatorname{Ann}_R \alpha = \{0\}$, which implies that $\dim \operatorname{Ann}_R[u_k v_l] \geqslant \dim \operatorname{Ann}_R \alpha$.

In order to show that the strict inequality holds, we shall find an element $\xi \in \operatorname{Ann}_R[u_k v_l]$ such that $(L_{kl} \oplus \langle\xi\rangle) \cap \operatorname{Ann}_R \alpha = \{0\}$. Take a summand $r_{st}[u_s v_t]$ of α different from $r_{kl}[u_k v_l]$. That is, $\{s,t\} \neq \{k,l\}$ and $r_{st} \neq 0$. We can assume without loss of generality that $l \notin \{s,t\}$. By Lemma A.3, there is a belt $\mathcal{B}$ in P such that $F_s, F_t \in \mathcal{B}$, $F_l \notin \mathcal{B}$, and F_l does not intersect one of the two connected components B_1 and B_2 of $\mathcal{B} \setminus \{F_s, F_t\}$, say B_1. In the dual language, there is a chordless cycle $\mathcal{C}$ in $\mathcal{K}_P$ such that $s, t \in \mathcal{C}$, $l \notin \mathcal{C}$, and the vertex l is not joined by an edge to any vertex of the connected component L_1 of $\mathcal{C} \setminus \{s,t\}$.

Now we observe that $\mathcal{C} \setminus \{s,t\}$ is a full subcomplex of $\mathcal{K}_P$ and take ξ to be the cohomology class in $R = H^*(\mathcal{Z}_P)$ given by a generator of $\widetilde{H}^0(\mathcal{C} \setminus \{s,t\}) \cong \mathbb{Z}$. Such a generator is represented by the 0-cocycle $\sum_{i\in L_1} \alpha_{\{i\}}$ (see Example 2.19). We have $\xi \cdot [u_k v_l] = 0$ because we can write $\xi = \sum_{i\in L_1} \pm[u_{J_i} v_i]$ (see Example 2.19) and $v_i v_l = 0$ for any $i \in L_1$ by the choice of the cycle $\mathcal{C}$. On the other hand, the product $\xi \cdot [u_s v_t]$ corresponds to a generator of $\widetilde{H}^1(\mathcal{C}) \cong \mathbb{Z}$. Therefore, $\xi \in \operatorname{Ann}_R[u_k v_l]$ and $\xi \cdot \alpha \neq 0$ (the latter is because the multigraded component of $\xi \cdot \alpha$ corresponding to $\mathcal{C}$ is $\xi \cdot r_{st}[u_s v_t] \neq 0$). Take $\beta = \sum_{I\subset[m]\setminus\{k,l\}} \beta_I \in L_{kl} \setminus \{0\}$ and choose $I \subset [m] \setminus \{k,l\}$ such that $(\beta \cdot \alpha)_{I\cup\{k,l\}} = \beta_I \cdot r_{kl}[u_k v_l] \neq 0$, as in the beginning of the proof. The multigrading of ξ does not contain l, so we have $(\xi \cdot \alpha)_{I\cup\{k,l\}} = \xi \cdot r_{jl}[u_j v_l]$ for some $j \in [m]$. Now, $\xi \cdot r_{jl}[u_j v_l] = 0$ because $\xi = \sum_{i\in L_1} \pm[u_{J_i} v_i]$ and $v_i v_l = 0$ for any $i \in L_1$, as i and l are not joined by an edge. Hence, $((\beta+\xi)\cdot\alpha)_{I\cup\{k,l\}} = (\beta\cdot\alpha)_{I\cup\{k,l\}} \neq 0$. Thus, $(\beta + \xi) \cdot \alpha \neq 0$ and we have proved that $(L_{kl} \oplus \langle\xi\rangle) \cap \operatorname{Ann}_R \alpha = \{0\}$. This implies that $\dim \operatorname{Ann}_R[u_k v_l] > \dim \operatorname{Ann}_R \alpha$. □

Proof of Lemma 4.11. We are given a 3-polytope P from the Pogorelov class $\mathcal{P}$ and a ring isomorphism $\psi\colon R = H^*(\mathcal{Z}_P) \xrightarrow{\cong} H^*(\mathcal{Z}_{P'}) = R'$. We defined the set

$$\mathcal{T}(P) = \{\pm[u_iv_j] \in H^3(\mathcal{Z}_P), \quad F_i \cap F_j = \varnothing\},$$

and the corresponding set for P',

$$\mathcal{T}(P') = \{\pm[u'_iv'_j] \in H^3(\mathcal{Z}_{P'}), \quad F'_i \cap F'_j = \varnothing\}.$$

We need to show that $\psi(\mathcal{T}(P)) = \mathcal{T}(P')$, in other words, $\psi([u_pv_q]) = \pm[u'_rv'_s]$. We first use Theorems 4.8 and 4.10 to conclude that P' also belongs to the class $\mathcal{P}$. Now suppose that $\psi([u_pv_q]) = \alpha' = \sum r_{ij}[u'_iv'_j]$ with at least two nonzero r_{ij}. We are then in the situation of Lemma C.1, which we can apply to P'. We obtain that $\dim \operatorname{Ann}_{R'} \alpha' < \dim \operatorname{Ann}_R[u'_kv'_l]$ for any nonzero summand $r_{kl}[u'_kv'_l]$ of α'. Considering the inverse isomorphism $\psi^{-1}\colon R' \to R$, we can choose $[u'_kv'_l]$ such that $\psi^{-1}([u'_kv'_l]) = \alpha = \sum r_{ab}[u_av_b]$ where $[u_pv_q]$ appears in the latter sum. As an isomorphism preserves the dimension of the annihilator subspace, we obtain

$$\begin{aligned}\dim \operatorname{Ann}_R[u_pv_q] = \dim \operatorname{Ann}_{R'} \alpha' < \dim \operatorname{Ann}_R[u'_kv'_l] = \dim \operatorname{Ann}_R \alpha \\ < \dim \operatorname{Ann}_R[u_pv_q],\end{aligned}$$

which is a contradiction. It follows that $\psi([u_pv_q])$ is a multiple of a single $[u'_rv'_s]$. Since ψ is an isomorphism over $\mathbb{Z}$, we have $\psi([u_pv_q]) = \pm[u'_rv'_s]$. □

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