Note on non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$ II

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Abstract: A complex hyperbolic triangle group is a group generated by three complex involutions fixing complex lines in complex hyperbolic space. In our previous papers [4, 5, 6, 7, 8] we discussed complex hyperbolic triangle groups. In particular, in [5, 8] we considered complex hyperbolic triangle groups of type $(n, n, \infty; k)$ and proved that for $n \geq 22$ these groups are not discrete. In this paper we show that if $n \geq 14$, then complex hyperbolic triangle groups of type $(n, n, \infty; k)$ are not discrete and give a new list of non-discrete groups of type $(n, n, \infty; k)$.

Keywords: complex hyperbolic triangle group; complex involution

1. Introduction. Let n and k be integers greater than 2. Let I_1, I_2, I_3 be the following matrices:

$$I_1 = \begin{bmatrix} 1 & \rho & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\rho} & 1 & \sigma \\ 0 & 0 & -1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\sigma} & 1 \end{bmatrix}.$$

Assume that ρ, σ, τ satisfy the conditions $|\rho| = 2\cos(\pi/n)$, $|\sigma| = 2$, $|\tau| = 2\cos(\pi/n)$, $|\rho\tau - \overline{\sigma}| = 2\cos(\pi/k)$. Then we have that $I_1^2 = I_2^2 = I_3^2 = (I_1 I_2)^n = (I_3 I_1)^n = (I_1 I_2 I_1 I_3)^k = E$(the identity matrix) and $I_2 I_3$ is a unipotent element. We call the group generated by I_1, I_2 and I_3 a *complex hyperbolic triangle group of type* $(n, n, \infty; k)$ and denote it by $\Gamma(n, n, \infty; k)$. Up to conjugation, there is a one-parameter family of these groups parametrized by k.

It is interesting to ask which values of the parameter correspond to discrete groups as mentioned in [12].

The purpose of this paper is to show the following theorem, which improves our previous result in [5] and gives a new list of non-discrete groups of type $(n, n, \infty; k)$.

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Theorem 1. *Let* $\Gamma =< I_1, I_2, I_3 >$ *be a complex hyperbolic triangle group of type* $(n, n, \infty; k)$ *with* $k \geq [n/2] + 1$*. The following groups are non-discrete.*

(1) $\Gamma(5, 5, \infty; 3)$

(2) $\Gamma(6, 6, \infty; 5)$.

(3) $\Gamma(7, 7, \infty; 4), \Gamma(7, 7, \infty; 5), \Gamma(7, 7, \infty; 6)$.

(4) $\Gamma(8, 8, \infty; 5), \Gamma(8, 8, \infty; 7)$.

(5) $\Gamma(9, 9, \infty; k)$ *for* $5 \leq k \leq 8$.

(6) $\Gamma(10, 10, \infty; k)$ *for* $6 \leq k \leq 9$.

(7) $\Gamma(11, 11, \infty; k)$ *for* $6 \leq k \leq 11$.

(8) $\Gamma(12, 12, \infty; k)$ *for* $7 \leq k \leq 16$.

(9) $\Gamma(13, 13, \infty; k)$ *for* $7 \leq k \leq 38$.

(10) $\Gamma(14, 14, \infty; k)$ *for* $k \geq 8$.

(11) $\Gamma(n, n, \infty; k)$ *for any* n (> 15).

In [12] Schwartz classified complex hyperbolic triangle groups into two types. It is said that $\Gamma(n, n, \infty)$ has *type B* if $I_1I_2I_3$ becomes regular elliptic before $I_1I_2I_1I_3$. If $n \geq 14$, then $\Gamma(n, n, \infty)$ has type B. Thus we have:

Corollary 2. *If* $\Gamma(n, n, \infty)$ *has type B, then* $\Gamma(n, n, \infty; k)$ *is not discrete.*

Details for background material on complex hyperbolic space will be found in [2]. For material on complex hyperbolic triangle groups see [3], [6], [7], [9], [12] and [13].

2. Proof of Theorem 1. To show a group of type $(n, n, \infty; k)$ to be non-discrete, we find regular elliptic elements of infinite order.

Lemma 1. *Let* g *be an element of* $\Gamma(n, n, \infty; k)$*. If trace*(g) *is real and contained in (-1,3), then* g *is regular elliptic and trace*$(g) = 1 + 2\cos\phi\pi$*. Moreover,* g *has finite order if and only if* ϕ *is a rational number.*

In our previous paper [5, 8] we used the result by Conway and Jones in [1]. Parker extended their results as follows (see [9] and [10]).

Lemma 2 ([10; Theorem A.1.1]). *Suppose that we have at most six distinct rational multiples of* π *lying strictly between 0 and* $\pi/2$*, for which some rational linear combination of their cosines is zero but no proper subset has this property, then the appropriate linear combination is propositional to one of the following:*

$$
\begin{aligned}
0 &= \sum_{k=0}^{2} \cos(\phi + \frac{2k\pi}{3}), \;\; \phi \in (0, \pi), \;\; \phi \neq \frac{m\pi}{6}; \\
0 &= \sum_{k=0}^{4} \cos(\phi + \frac{2k\pi}{5}), \;\; \phi \in (0, \pi), \;\; \phi \neq \frac{n\pi}{10};
\end{aligned}
$$

$$\begin{aligned}
0 &= \sum_{k=1}^{2}\cos(\phi+\frac{2k\pi}{3})-\sum_{k=1}^{4}\cos(\phi+\frac{2k\pi}{5}),\ \ \phi\in(0,\pi),\ \ \phi\neq\frac{m\pi}{6},\ \ \phi\neq\frac{n\pi}{10};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{5}+\cos\frac{2\pi}{5};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{11}+\cos\frac{2\pi}{11}-\cos\frac{3\pi}{11}+\cos\frac{4\pi}{11}-\cos\frac{5\pi}{11};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{5}+\cos\frac{\pi}{15}-\cos\frac{4\pi}{15};\\
0 &= \cos\frac{\pi}{3}+\cos\frac{2\pi}{5}-\cos\frac{2\pi}{15}+\cos\frac{7\pi}{15};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{2\pi}{21}+\cos\frac{5\pi}{21};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{\pi}{21}-\cos\frac{8\pi}{21};\\
0 &= \cos\frac{\pi}{3}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{4\pi}{21}-\cos\frac{10\pi}{21};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{\pi}{7}+\cos\frac{\pi}{21}-\cos\frac{2\pi}{21}+\cos\frac{5\pi}{21}-\cos\frac{8\pi}{21};\\
0 &= \cos\frac{\pi}{3}+\cos\frac{2\pi}{7}-\cos\frac{2\pi}{21}-\cos\frac{4\pi}{21}+\cos\frac{5\pi}{21}-\cos\frac{10\pi}{21};\\
0 &= \cos\frac{\pi}{3}-\cos\frac{3\pi}{7}+\cos\frac{\pi}{21}-\cos\frac{4\pi}{21}-\cos\frac{8\pi}{21}-\cos\frac{10\pi}{21};\\
0 &= \cos\frac{\pi}{5}-\cos\frac{2\pi}{5}-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7};\\
0 &= \cos\frac{\pi}{5}-\cos\frac{2\pi}{5}-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{2\pi}{21}+\cos\frac{5\pi}{21};\\
0 &= \cos\frac{\pi}{5}-\cos\frac{2\pi}{5}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{2\pi}{21}-\cos\frac{8\pi}{21};\\
0 &= \cos\frac{\pi}{5}-\cos\frac{2\pi}{5}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{4\pi}{21}-\cos\frac{10\pi}{21}.
\end{aligned}$$

We consider the element $I_3I_1I_3I_1I_2I_1$ in $\Gamma(n,n,\infty;k)$, which is denoted by $I_{(31)^2 21}$. It is seen that

$$\begin{aligned}
trace(I_{(31)^2 21}) &= (\rho\sigma\tau+\overline{\rho}\,\overline{\sigma}\,\overline{\tau})(1-|\tau|^2)-1+|\rho|^2+|\tau|^2(-2|\rho|^2+|\sigma|^2+|\rho|^2|\tau|^2)\\
&= 2\cos\frac{2\pi}{k}+4\cos\frac{2\pi}{k}\cos\frac{2\pi}{n}-2\cos\frac{4\pi}{n}-2\cos\frac{2\pi}{n}+1\\
&= 2\cos\frac{2\pi}{k}+2\cos\left(\frac{2\pi}{k}+\frac{2\pi}{n}\right)+2\cos\left(\frac{2\pi}{k}-\frac{2\pi}{n}\right)\\
&- 2\cos\frac{4\pi}{n}-2\cos\frac{2\pi}{n}+1.
\end{aligned}$$

We note that $trace(I_{(31)^2 21})$ is real. It can be shown that the element $I_{(31)^2 21}$ is unipotent in $\Gamma(3,3,\infty;k)$ for any $k \geq 4$. Also we see that for $n \geq 4$ the element $I_{(31)^2 21}$ is regular elliptic in $\Gamma(n,n,\infty;k)$ with $k \leq n-1$ and it is unipotent in $\Gamma(n,n,\infty;n)$.

We are ready to prove our theorem.

Proof of Thorem 1. First we consider the group $\Gamma(21,21,\infty;18)$. From the above, the element $I_{(31)^2 21}$ is regular elliptic in $\Gamma(21,21,\infty;18)$. Lemma 1 implies that $I_{(31)^2 21}$ has finite order if there is a rational number ϕ satisfying

$$2\cos\frac{\pi}{9}+2\cos\frac{13\pi}{63}+2\cos\frac{\pi}{63}-2\cos\frac{4\pi}{21}-2\cos\frac{2\pi}{21}+1=1+2\cos\phi\pi,$$

that is,

$$\cos\frac{\pi}{9}+\cos\frac{13\pi}{63}+\cos\frac{\pi}{63}-\cos\frac{4\pi}{21}-\cos\frac{2\pi}{21}-\cos\phi\pi=0.$$

Lemma 2 lists all possible trigonometric Diopantine equations with up to six. We use this result to conclude that there is no rational number ϕ satisfying the equation above. It follows from Lemma 1 that $I_{(31)^2 21}$ has infinite order in the group $\Gamma(21,21,\infty;18)$, which implies that $\Gamma(21,21,\infty;18)$ is not discrete.

In the same manner as above, we show that there is no rational ϕ satisfying the following each equation.

$$\cos\frac{2\pi}{17}+\cos\frac{76\pi}{357}+\cos\frac{8\pi}{357}-\cos\frac{4\pi}{21}-\cos\frac{2\pi}{21}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{8}+\cos\frac{9\pi}{40}+\cos\frac{\pi}{40}-\cos\frac{\pi}{5}-\cos\frac{\pi}{10}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{17}+\cos\frac{37\pi}{170}+\cos\frac{3\pi}{170}-\cos\frac{\pi}{5}-\cos\frac{\pi}{10}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{7}+\cos\frac{33\pi}{133}+\cos\frac{5\pi}{133}-\cos\frac{4\pi}{19}-\cos\frac{2\pi}{19}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{15}+\cos\frac{68\pi}{285}+\cos\frac{8\pi}{285}-\cos\frac{4\pi}{19}-\cos\frac{2\pi}{19}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{8}+\cos\frac{35\pi}{152}+\cos\frac{3\pi}{152}-\cos\frac{4\pi}{19}-\cos\frac{2\pi}{19}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{13}+\cos\frac{31\pi}{117}+\cos\frac{5\pi}{117}-\cos\frac{2\pi}{9}-\cos\frac{\pi}{9}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{7}+\cos\frac{16\pi}{63}+\cos\frac{2\pi}{63}-\cos\frac{2\pi}{9}-\cos\frac{\pi}{9}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{15}+\cos\frac{11\pi}{45}+\cos\frac{\pi}{45}-\cos\frac{2\pi}{9}-\cos\frac{\pi}{9}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{6}+\cos\frac{29\pi}{102}+\cos\frac{5\pi}{102}-\cos\frac{4\pi}{17}-\cos\frac{2\pi}{17}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{13}+\cos\frac{60\pi}{221}+\cos\frac{8\pi}{221}-\cos\frac{4\pi}{17}-\cos\frac{2\pi}{17}-\cos\phi\pi=0.$$

$$\cos\frac{\pi}{7}+\cos\frac{31\pi}{119}+\cos\frac{3\pi}{119}-\cos\frac{4\pi}{17}-\cos\frac{2\pi}{17}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{11}+\cos\frac{27\pi}{88}+\cos\frac{5\pi}{88}-\cos\frac{\pi}{4}-\cos\frac{\pi}{8}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{6}+\cos\frac{7\pi}{24}+\cos\frac{\pi}{24}-\cos\frac{\pi}{4}-\cos\frac{\pi}{8}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{13}+\cos\frac{29\pi}{104}+\cos\frac{3\pi}{104}-\cos\frac{\pi}{4}-\cos\frac{\pi}{8}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{11}+\cos\frac{52\pi}{165}+\cos\frac{8\pi}{165}-\cos\frac{4\pi}{15}-\cos\frac{2\pi}{15}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{6}+\cos\frac{3\pi}{10}+\cos\frac{\pi}{30}-\cos\frac{4\pi}{15}-\cos\frac{2\pi}{15}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{9}+\cos\frac{23\pi}{63}+\cos\frac{5\pi}{63}-\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{5}+\cos\frac{12\pi}{35}+\cos\frac{2\pi}{35}-\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{11}+\cos\frac{25\pi}{77}+\cos\frac{3\pi}{77}-\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{4}+\cos\frac{21\pi}{52}+\cos\frac{5\pi}{52}-\cos\frac{4\pi}{13}-\cos\frac{2\pi}{13}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{9}+\cos\frac{44\pi}{117}+\cos\frac{8\pi}{117}-\cos\frac{4\pi}{13}-\cos\frac{2\pi}{13}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{5}+\cos\frac{23\pi}{65}+\cos\frac{3\pi}{65}-\cos\frac{4\pi}{13}-\cos\frac{2\pi}{13}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{11}+\cos\frac{48\pi}{143}+\cos\frac{4\pi}{143}-\cos\frac{4\pi}{13}-\cos\frac{2\pi}{13}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{4}+\cos\frac{5\pi}{12}+\cos\frac{\pi}{12}-\cos\frac{\pi}{3}-\cos\frac{\pi}{6}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{9}+\cos\frac{7\pi}{18}+\cos\frac{\pi}{18}-\cos\frac{\pi}{3}-\cos\frac{\pi}{6}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{5}+\cos\frac{11\pi}{30}+\cos\frac{\pi}{30}-\cos\frac{\pi}{3}-\cos\frac{\pi}{6}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{7}+\cos\frac{36\pi}{77}+\cos\frac{8\pi}{77}-\cos\frac{4\pi}{11}-\cos\frac{2\pi}{11}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{4}+\cos\frac{19\pi}{44}+\cos\frac{3\pi}{44}-\cos\frac{4\pi}{11}-\cos\frac{2\pi}{11}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{9}+\cos\frac{40\pi}{99}+\cos\frac{4\pi}{99}-\cos\frac{4\pi}{11}-\cos\frac{2\pi}{11}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{7}+\cos\frac{17\pi}{35}+\cos\frac{3\pi}{35}-\cos\frac{2\pi}{5}-\cos\frac{\pi}{5}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{4}+\cos\frac{9\pi}{20}+\cos\frac{\pi}{20}-\cos\frac{2\pi}{5}-\cos\frac{\pi}{5}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{7}-\cos\frac{31\pi}{63}+\cos\frac{4\pi}{63}-\cos\frac{4\pi}{9}-\cos\frac{2\pi}{9}-\cos\phi\pi=0.$$
$$\cos\frac{\pi}{4}+\cos\frac{17\pi}{36}+\cos\frac{\pi}{36}-\cos\frac{4\pi}{9}-\cos\frac{2\pi}{9}-\cos\phi\pi=0.$$

$$\cos\frac{2\pi}{7}-\cos\frac{13\pi}{28}+\cos\frac{\pi}{28}-\cos\frac{\pi}{4}-\cos\phi\pi=0.$$
$$\cos\frac{2\pi}{5}-\cos\frac{11\pi}{35}+\cos\frac{4\pi}{35}+\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}-\cos\phi\pi=0.$$

These correspond to the equations $trace(I_{(31)^2 21})=1+2\cos\phi\pi$ in the following groups, respectively:

$\Gamma(21,21,\infty;17)$;
$\Gamma(20,20,\infty;16),\Gamma(20,20,\infty;17)$;
$\Gamma(19,19,\infty;14),\Gamma(19,19,\infty;15),\Gamma(19,19,\infty;16)$;
$\Gamma(18,18,\infty;13),\Gamma(18,18,\infty;14),\Gamma(18,18,\infty;15)$;
$\Gamma(17,17,\infty;12),\Gamma(17,17,\infty;13),\Gamma(17,17,\infty;14)$;
$\Gamma(16,16,\infty;11),\Gamma(16,16,\infty;12),\Gamma(16,16,\infty;13)$;
$\Gamma(15,15,\infty;11),\Gamma(15,15,\infty;12)$;
$\Gamma(14,14,\infty;9),\Gamma(14,14,\infty;10),\Gamma(14,14,\infty;11)$;
$\Gamma(13,13,\infty;8),\Gamma(13,13,\infty;9),\Gamma(13,13,\infty;10),\Gamma(13,13,\infty;11)$;
$\Gamma(12,12,\infty;8),\Gamma(12,12,\infty;9),\Gamma(12,12,\infty;10)$;
$\Gamma(11,11,\infty;7),\Gamma(11,11,\infty;8),\Gamma(11,11,\infty;9)$;
$\Gamma(10,10,\infty;7),\Gamma(10,10,\infty;8)$;
$\Gamma(9,9,\infty;7),\Gamma(9,9,\infty;8)$;
$\Gamma(8,8,\infty;7)$;
$\Gamma(7,7,\infty;5)$.

It follows that in each group above, $I_{(31)^2 21}$ is a regular elliptic element with infinite order. Thus the groups above are not discrete. Together with Theorem 2.3 in [5], we obtain our theorem.

Remark. (1) In [11] Parker, Wang and Xie showed that $\Gamma(3,3,\infty;k)$ is discrete for $k\geq 4$.
(2) In $\Gamma(13,13,\infty;8),\Gamma(11,11,\infty;7),\Gamma(7,7,\infty;5)$ among the groups above, $I_{(12)^2 32}$ is also regular elliptic.
(3) In $\Gamma(8,8,\infty;6)$, $I_{(31)^2 21}$ is a regular elliptic element of order 6.

References

[1] J.H. Conway and A.J. Jones, Trigonometric diophantine equations (On vanishing sums of roots of unity), *Acta Arithmetica* **30** (1976), 229-240.
[2] W.M. Goldman, Complex hyperbolic geometry, Oxford University Press, New York, 1999.
[3] W.M. Goldman and J.R. Parker, Complex hyperbolic ideal triangle groups, *J. reine angew. Math.* **425** (1992), 71-86.

[4] S. Kamiya, Remarks on complex hyperbolic triangle groups, *Complex Analysis and its Applications*, OCAMI Stud. 2, Osaka Munic. Univ. Press, Osaka (2007), 219-223.
[5] S. Kamiya, Note on non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$, *Proc. Japan Acad.* **89** (2013), 100-102.
[6] S. Kamiya, Complex hyperbolic triangle groups of type (n, n, ∞), *Math. Newsl.* **24** (2014), 97-103.
[7] S. Kamiya, J.R. Parker and J.M. Thompson, Notes on complex hyperbolic triangle groups. *Conform. Geom. and Dyn.* **14** (2010), 202-218.
[8] S. Kamiya, J.R. Parker and J.M. Thompson, Non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$, *Canad. Math. Bull.* **55** (2012), 329-338.
[9]A. Monagham, Complex hyperbolic triangle groups, Ph.D. thesis, University of Liverpool, 2013.
[10] J.R. Parker, 2-Generator Möbius Group, Ph.D. thesis, University of Cambridge, 1989.
[11] J.R. Parker, J. Wang and B. Xie, Complex hyperbolic (3,3,n)-triangle groups, *Pacific J. Math.* **280** (2016), 433-453.
[12] R.E. Schwartz, Complex hyperbolic triangle groups, Proc. of the International Cong. of Math. Vol. II (Beijing, 2002) 339-349, Higher Ed. Press Beijing, 2000.
[13] J. Wyss-Gallifent, Complex Hyperbolic Triangle Groups, Ph.D. thesis, University of Maryland, 2000.