Enumeration of ribbon 2-knots presented by virtual arcs with up to four crossings

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ABSTRACT

We enumerate ribbon 2-knots presented by virtual arc diagrams with up to four classical crossings. We use a linear Gauss diagram for a virtual arc diagram.

Keywords: Ribbon 2-knot; virtual arc; welded arc; linear Gauss diagram; ribbon handlebody.

Mathematics Subject Classification 2000: 57Q45

1. Intoduction

A ribbon 2-knot is a knotted 2-sphere in $\boldsymbol{R}^4$ that bounds a ribbon 3-disk, which is an immersed 3-disk with only ribbon singularities. The *ribbon crossing number* of a ribbon 2-knot is the minimal number of the ribbon singularities of any ribbon 3-disk bounding the knot [24]. The enumeration of the classical knots are based on the crossing number of a knot diagram. So, it is natural to enumerate ribbon 2-knots based on the ribbon crossing number. Yasuda has listed ribbon 2-knots with ribbon crossing number up to four [22,23,25,26,27].

Moreover, a ribbon 2-knot is presented by a virtual arc diagram, which was introduced by Satoh [19]; a virtual arc diagram has classical and virtual crossings and a classical crossing corresponds to a ribbon singularity. If a ribbon 2-knot is presented by a virtual arc diagram with n classical crossings, then its ribbon crossing number is at most n. In this paper we enumerate ribbon 2-knots based on the virtual arc presentation. Our main result is the table of ribbon 2-knots which are presented by virtual arc diagram with up to 4 crossings (Tables 2, 3 and 4). Therefore, the ribbon 2-knots in our table are contained in that of Yasuda. However, in the course of this enumeration we discovered an oversight in Yasuda's table, that is, we found 6 ribbon 2-knots whose Alexander polynomials are not listed in Yasuda's table.

From a virtual arc diagram we may obtain a *linear Gauss diagram*, which is an immersing interval with the preimages of each double point connected with a chord equipped with the sign of the crossing. Then an oriented linear Gauss diagram determines an associated ribbon 2-knot (Proposition 3.1). Also there are several moves for a linear Gauss diagram which do not change the isotopy class of the associated ribbon 2-knot (Subsec. 3.2); they are induced from the Reidemeister moves (moves $\mathrm{A_I}$, $\mathrm{A_{II}}$, $\mathrm{A_{III}}$), welded moves (moves D, $\mathrm{E_{II}}$), negative-amphicheirality of a ribbon 2-knot (reversing move), and composition of knots (Eq. (3.1)). Furthermore, they yield some simplified moves (moves Ω_i, $i = 1, \ldots, 8$), which are useful for investigating the equivalence of Gauss diagrams.

Using these moves, we enumerate linear Gauss diagrams. First, we show a ribbon 2-knot presented by linear Gauss diagrams with up to 4 chords is presented by one of the 13 linear Gauss diagrams Γ^4_l, $1 \leq l \leq 13$, given in Fig. 29 (Lemma 5.1). Then, for each diagram we examine the corresponding ribbon 2-knot, where we calculate the knot group and Alexander polynomial (Sects. 5 and 7). We apply the above mentioned moves for considering the equivalence. However, in some cases we consider a ribbon handlebody (Lemma 7.3) or a virtual arc diagram (Lemma 7.5).

This paper is organized as follows: In Sec. 2, we introduce a ribbon 2-knot and its virtual arc presentation, where we use a ribbon handlebody to present a ribbon 3-disk. In Sec. 3, we introduce a linear Gauss diagram which presents a virtual arc diagram and so a ribbon 2-knot. In Sec. 4, we give our main result, tables of ribbon 2-knots presented by virtual arc diagrams with up to 4 crossings (Tables 2, 3 and 4). Secs. 5, 6, and 7 are devoted to the proof of our main result. Most of the results in this paper are from the master's thesis of one of the authors [14].

2. Ribbon 2-Knot and Its Virtual Arc Presentation

In this section, we introduce a ribbon knot, virtual arc, welded arc, and virtual arc presentation of a ribbon 2-knot.

2.1. *Ribbon knot*

A *ribbon $(n+1)$-disk*, $n \geq 1$, is an immersed $(n+1)$-disk D^{n+1} into $\boldsymbol{R}^{n+2}$ with only transverse double points such that the singular set consists of ribbon singularities, that is, the preimage of each ribbon singularity consists of a properly embedded n-disk in D^{n+1} and an embedded n-disk interior to D^{n+1}. An n-knot is a *ribbon n-knot* if it bounds a ribbon $(n+1)$-disk in $\boldsymbol{R}^{n+2}$.

The *ribbon crossing number* of a ribbon n-knot K is the smallest number of the ribbon singularities of any ribbon $(n+1)$-disk bounding the knot K; cf. [24]. Yasuda [22,23,25,26,27] enumerated ribbon 2-knots with ribbon crossing number up to four.

A *ribbon handlebody* [3] H is a ribbon 2-disk, which is a 2-dimensional handlebody in $\boldsymbol{R}^3$ consisting of $(p+1)$ 0-handles D_1, $D_2, \ldots, D_{p+1}$, p 1-handles B_1, $B_2, \ldots, B_p$ and no 2-handle such that the preimage of each ribbon singularity consists of an arc in the interior of a 0-handle and a cocore of a 1-handle. We put $\mathcal{D} = D_1 \cup D_2 \cup \cdots \cup D_{p+1}$ and $\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_p$. Then the set of ribbon singularities in H is the connected components of the intersection $\mathcal{D} \cap \mathcal{B}$. Note that in order to present a ribbon handlebody we may use a ribbon regular projection defined in [6, Sect. 1] or [7, Sect. 3], .

We define the associated ribbon 2-knot in $\boldsymbol{R}^4 = \boldsymbol{R}^3 \times \boldsymbol{R}$ as the ribbon 2-knot that bounds the immersed 3-disk $\mathcal{D} \times [-2, 2] \cup \mathcal{B} \times [-1, 1]$. Conversely, for any ribbon 2-knot K, there exists a ribbon handlebody whose associated ribbon 2-knot is K; see [10,21]. Thus we may represent a ribbon 2-knot by a ribbon handlebody.

2.2. *Virtual arc and welded arc*

A *virtual knot diagram* is an immersed circle and a *virtual arc diagram* is an immersed interval in the plane $\boldsymbol{R}^2$ such that each crossing is either a classical crossing or a virtual crossing as shown in Fig. 1.

Fig. 1. (a) Classical crossing; (b) vitrtual crossing.

A *virtual knot* is an equivalence class of virtual knot diagram under the classical Reidemeister moves $\mathrm{A_I}$, $\mathrm{A_{II}}$, $\mathrm{A_{III}}$ as shown in Fig. 2 and the virtual Reidemeister moves $\mathrm{B_I}$, $\mathrm{B_{II}}$, $\mathrm{B_{III}}$, C, $\mathrm{E_I}$ as shown in Fig. 3. Similarly, a *virtual arc* is an equivalence class of virtual arc diagram under the classical Reidemeister moves $\mathrm{A_I}$, $\mathrm{A_{II}}$, $\mathrm{A_{III}}$ and the virtual Reidemeister moves $\mathrm{B_I}$, $\mathrm{B_{II}}$, $\mathrm{B_{III}}$, C as well as $\mathrm{E_I}$ as shown in Fig. 3.

We introduce the *welded moves* D and $\mathrm{E_{II}}$ as shown in Fig. 4; the move D is known as one of the "forbidden moves"; cf. [8,17]. Two virtual knot diagrams or virtual arc diagrams A and A' are *w-equivalent* if there exist a finite sequence of the classical Reidemeister moves, virtual Reidemeister moves, and welded moves which transforms A into A'. A *welded knot* is an equivalence class under the w-equivalence of virtual knot diagrams, and a *welded arc* is an equivalence class under the w-equivalence of virtual arc diagrams. A welded knot or arc is *trivial* if it is an equivalence class containing a virtual knot or virtual arc diagram without any crossing. Note that a welded knot and a welded arc are called a *weakly virtual knot* and a *weakly virtual knot*, respectively, in [9,19].

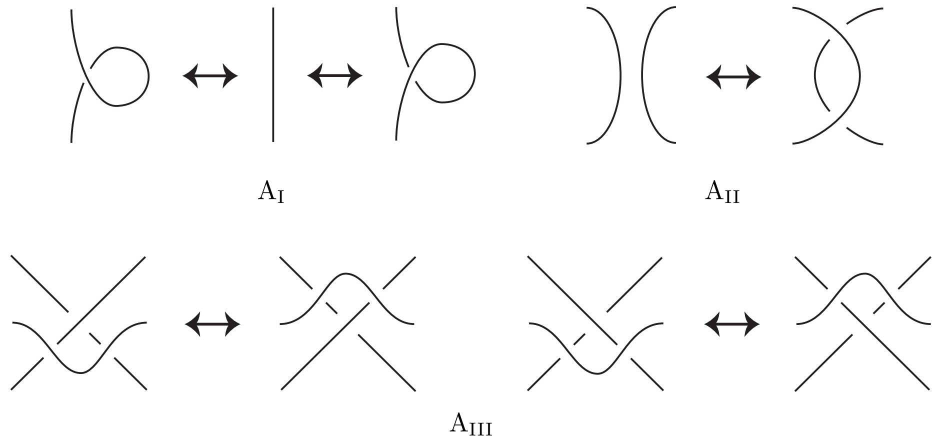


Fig. 2. Reidemeister moves A_I, A_{II}, A_{III}.

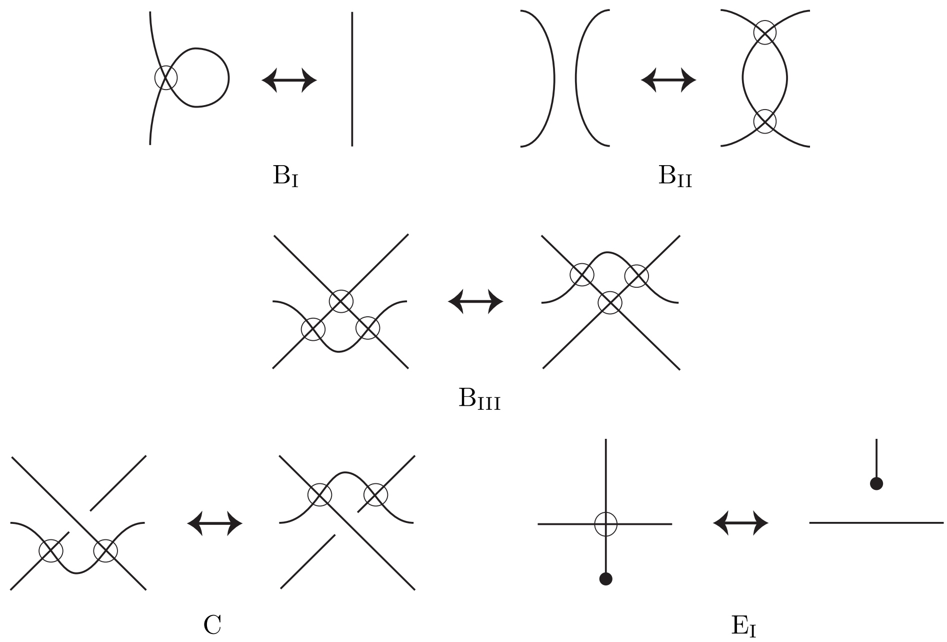


Fig. 3. Virtual Reidemeister moves B_I, B_{II}, B_{III}, C, E_I.

2.3. *Virtual arc presentation of a ribbon 2-knot*

Given an oriented virtual arc diagram A, we can construct a ribbon handlebody H_A as shown in Fig. 5, where an overpass at a classical crossing corresponds to a 0-handle containing a ribbon singularity, a virtual crossing corresponds to two 1-

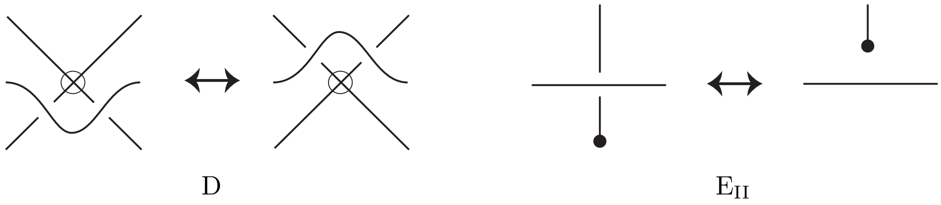


Fig. 4. Welded moves D, E_{II}.

handles passing over/under each other, and an end point corresponds to a 0-handle. Then we denote the associated ribbon 2-knot of H_A by Tube(A). If two virtual arc diagrams A and A' are w-equivalent, then the corresponding ribbon 2-knots Tube(A) and Tube(A') are ambient isotopic, and thus a welded arc determines a ribbon 2-knot. Conversely, any ribbon 2-knot is associated with some virtual arc diagram; see [19].

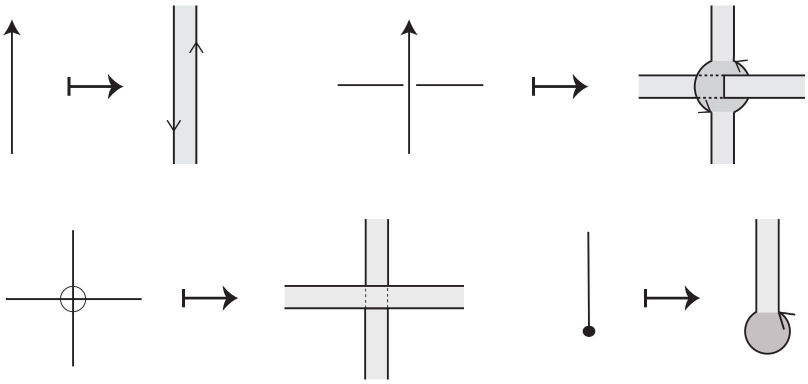


Fig. 5. Correspondence of an oriented virtual arc diagram to a ribbon handlebody.

Remark 2.1. Satoh [19] constructs the associated ribbon 2-knot Tube(A) using a surface diagram, which is a projection image of a ribbon 2-knot in $\boldsymbol{R}^4$; see [1], cf. [10].

Furthermore, we define two oriented virtual arc diagrams A and A' are *r-equivalent*, denoted by $A \overset{\mathrm{r}}{\sim} A'$, if they correspond to the isotopic ribbon 2-knots.

Let A be an oriented virtual arc diagram. We denote the horizontal mirror image by $A^\dagger$, which is obtained from A by reflecting the diagram across a vertical plane. We denote the orientation-reversion of A by $-A$. See Fig. 6; cf. [5, Sec. 2.2].

Then the both associated ribbon handlebodies $H_{A^\dagger}$ and H_{-A} are obtained from H_A by reflecting across the vertical plane and the projection plane, respectively.

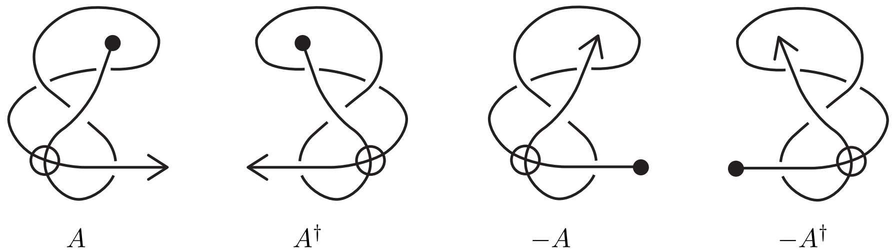


Fig. 6. Virtual arc diagrams A, $A^\dagger$, $-A$, and $-A^\dagger$.

Alternatively, both $H_{A^\dagger}$ and H_{-A} are obtained from H_A by reversing the orientation of the immersed surface. This shows that a ribbon 2-knot is negative-amphicheiral; see [20, Theorem 2.18], [19, Proposition 4.1]. Thus we have:

Proposition 2.2. *Let A be an oriented virtual arc diagram. Then the associated ribbon* 2*-knots* Tube($A^\dagger$) *and* Tube($-A$) *are isotopic, which are the mirror images of* Tube(A)*. In other words, A and $-A^\dagger$ are r-equivalent.*

We define the *product* of two oriented virtual arc diagrams A and A', denoted by $A \cdot A'$, as follows: First, we deform the diagrams A and A' so that the the terminal point of A and the initial point of A' lie in the outermost region by the move $\mathrm{E_I}$. We denote the resulting diagrams by $\tilde{A}$ and $\tilde{A}'$. Then the product of A and A' is obtained by connecting the terminal point of $\tilde{A}$ to the initial point of $\tilde{A}'$ as shown in Fig. 7. It is easy to see that the product of oriented virtual arc diagrams is well defined up to w-equivalence.

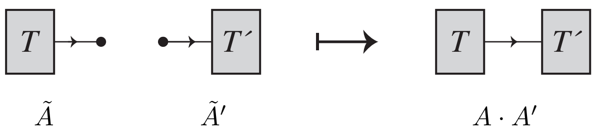


Fig. 7. Product of virtual arc diagrams.

The associated ribbon 2-knot Tube($A \cdot A'$) is the composition of Tube(A) and Tube(A'); Tube($A \cdot A'$) = Tube(A)#Tube(A'). Since the composition of oriented 2-knots is well defined up to isotopy, we have the following:

Proposition 2.3. *Let A and A' be oriented virtual arc diagrams. Then the associated ribbon* 2*-knots* Tube($A \cdot A'$) *and* Tube($A' \cdot A$) *are isotopic, $A \cdot A' \overset{\mathrm{r}}{\sim} A' \cdot A$.*

Note that $A \cdot A'$ and $A' \cdot A$ may not be w-equivalent. We can see Proposition 2.3 by the associated ribbon handlebody as follows: $H_{A \cdot A'}$ is the boundary connected sum of H_A and $H_{A'}$. Then by sliding H_A along the boundary of $H_{A'}$ we can isotope $H_{A\cdot A'}$ into $H_{A'\cdot(-A^\dagger)}$. This implies Tube$(A \cdot A')$ is isotopic to Tube$(A' \cdot (-A^\dagger))$. From this we have further Tube$(A' \cdot (-A^\dagger))$ is isotopic to Tube$((-A^\dagger) \cdot (-A'^\dagger))$. Since the virtual arc $(-A^\dagger) \cdot (-A'^\dagger)$ is the same as $-(A' \cdot A)^\dagger$, by Proposition 2.2 Tube$((-A^\dagger) \cdot (-A'^\dagger))$ is isotopic to Tube$(A' \cdot A)$, completing the proof.

3. Gauss diagram

In this section, we first review a Gauss diagram of a virtual knot or a welded knot, and then introduce a linear Gauss diagram of a virtual arc. We give several moves of a linear Gauss diagram which do not change the isotopy class of the associated ribbon 2-knot.

3.1. *Gauss diagrams of virtual and welded knots*

The Gauss diagram of an oriented virtual knot diagram is an oriented circle as the preimage of the immersed circle with chords connecting the preimages of each classical crossing. We specify crossing information on each chord by directing the chord toward the under crossing equipped with the sign of the crossing. We call such a Gauss diagram a *circular Gauss diagram.* A circular Gauss diagram is considered up to orientation preserving homeomorphism of the underlying oriented circle. A circular Gauss diagram determines a virtual knot diagram up to virtual Reidemeister moves; see [2, Theorem 1.A]; cf. [12]. Namely, if two virtual knot diagrams have the same circular Gauss diagram, then there exist a finite sequence of virtual Reidemeister moves B_{I}, B_{II}, B_{III}, C which transforms one into the other.

The classical Reidemeister moves on a knot give rise to moves on circular Gauss diagrams. Depending on orientations of strings involved in the Reidemeister moves, one may distinguish four different versions of each of the A_{I} and A_{II} moves, and eight versions of the A_{III} move as shown in Figs. 8–10 [18, Figs. 2–4].

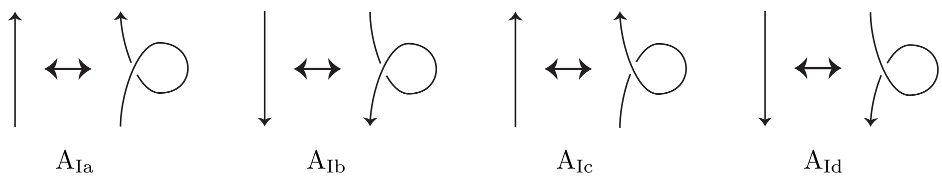


Fig. 8. Oriented Reidemeister moves of type I.

We give the corresponding moves of circular Gauss diagrams in Figs. 11–14. For each of the moves $A_{IIIa}, \ldots, A_{IIIh}$ for a knot, there are two different possibilities according to the structure of the diagram outside the local part given in the

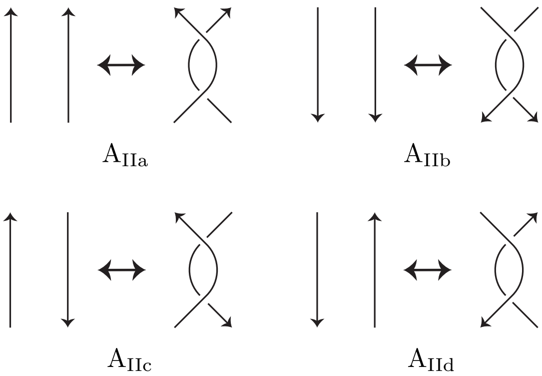


Fig. 9. Oriented Reidemeister moves of type II.

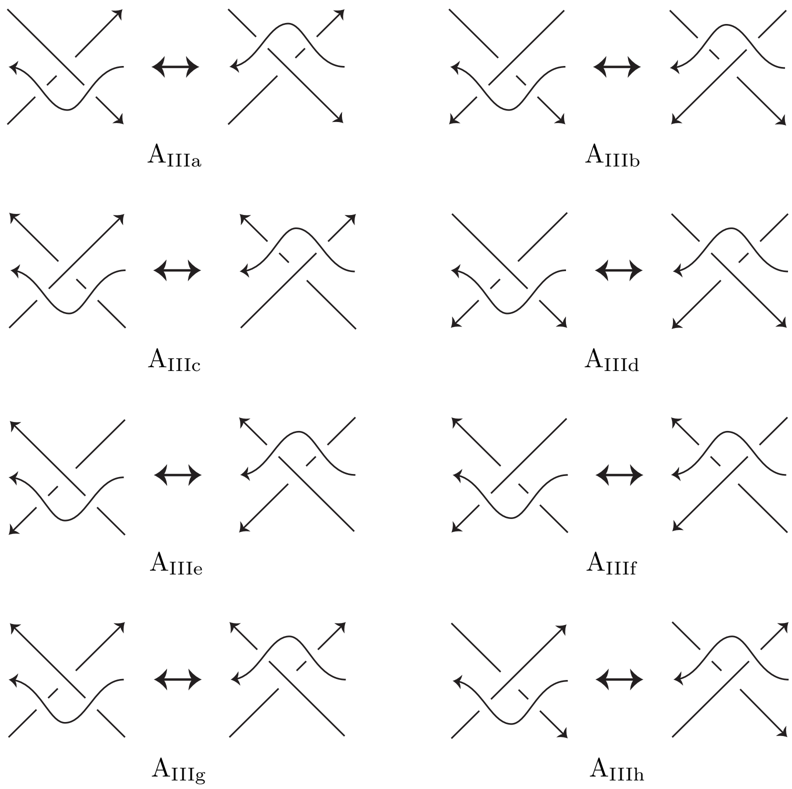


Fig. 10. Oriented Reidemeister moves of type III.

diagrams in Fig. 10. By abuse of notation, we denote these moves by the same notation; for the moves $A_{IIIa}, \ldots, A_{IIIh}$, we denote the corresponding moves by A_{IIIa}, $A_{IIIa'}, \ldots$, A_{IIIh}, $A_{IIIh'}$; cf. [2,17]. Therefore, an oriented virtual knot (that is, an oriented virtual knot diagram modulo Reidemeister moves A_{Ia}, A_{Ib}, A_{Ic}, A_{Id}, A_{IIa}, A_{IIb}, A_{IIc}, A_{IId}, $A_{IIIa}, \ldots$, A_{IIIh}, and virtual Reidemeister moves B_I, B_{II}, B_{III}, C) is equivalent to the corresponding circular Gauss diagram considered up to moves given in Figs. 11–14.

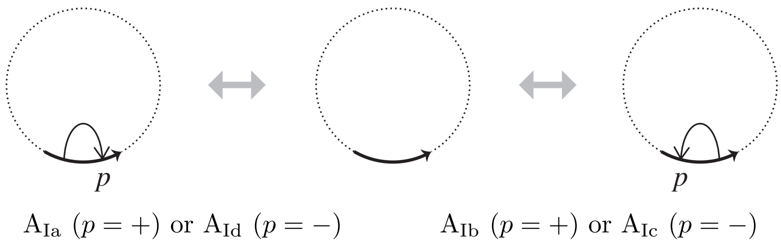


Fig. 11. Moves of circular Gauss diagrams corresponding to the oriented Reidemeister moves of type I.

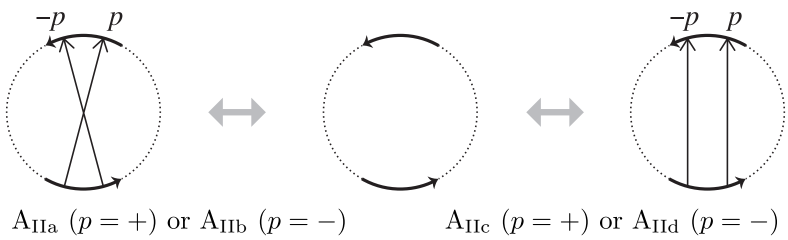


Fig. 12. Moves of circular Gauss diagrams corresponding to the oriented Reidemeister moves of type II.

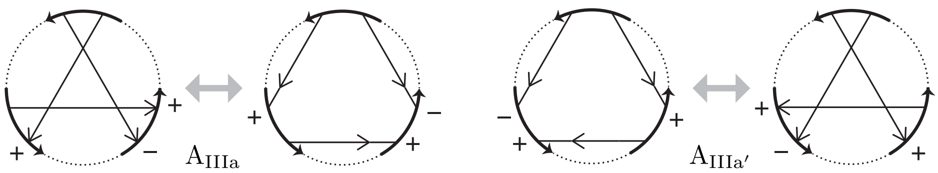


Fig. 13. Moves of circular Gauss diagrams corresponding to the oriented Reidemeister moves of type III.

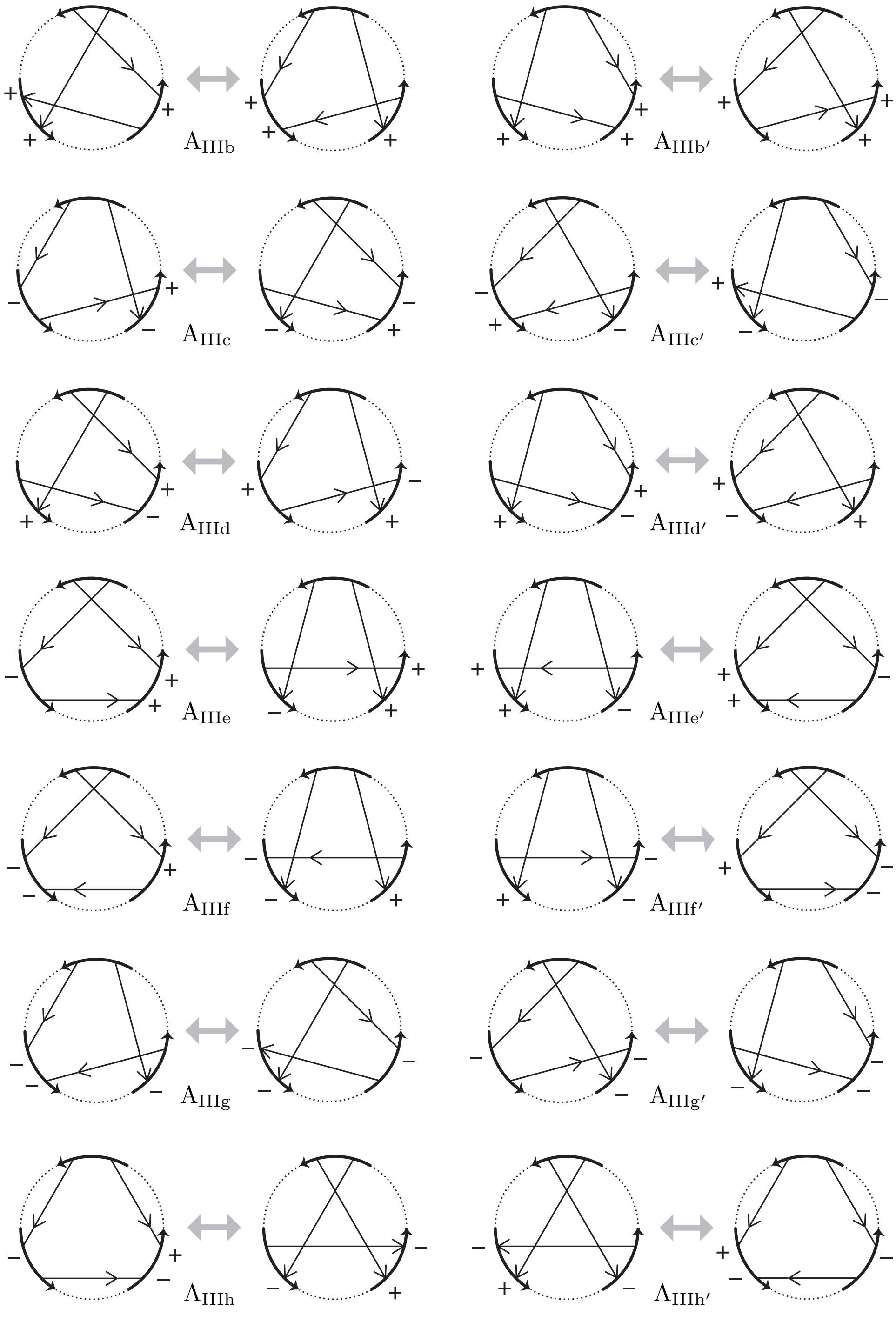


Fig. 14. Moves of circular Gauss diagrams corresponding to the oriented Reidemeister moves of type III.

Now we consider a circular Gauss diagram for a welded knot. Depending on orientations of strings, there are four different versions of the welded move D as shown in Fig. 15, where p, $q = \pm$.

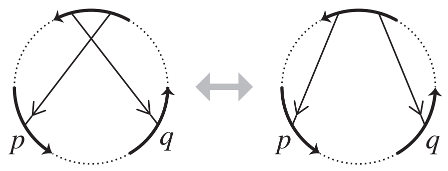


Fig. 15. Moves of circular Gauss diagrams corresponding to the move D, where p, $q = \pm$.

Therefore, if we allow the move D, then the moves A_{IIa} and A_{IIc} are equivalent, and the moves A_{IIb} and A_{IId} are equivalent. Also, the moves A_{IIIa}, $A_{\mathrm{IIIa'}}, \dots, A_{\mathrm{IIIh}}$, $A_{\mathrm{IIIh'}}$ are put together into the moves A_{IIIx}, A_{IIIy}, A_{IIIz}, A_{IIIw} as shown in Fig. 16, where p, $q = \pm$. We give the equivalent pairs in Table 1, where, for example, it shows the move A_{IIIa} is equivalent to the move A_{IVa} with $(p, q) = (+, +)$. Thus, an oriented welded knot (that is, an oriented virtual knot diagram modulo Reidemeister moves, virtual Reidemeister moves, and welded move D) is equivalent to the corresponding circular Gauss diagram considered up to moves given in Figs. 11, 12, 15, and 16.

Table 1. Equivalent moves of circular Gauss diagrams allowing the move D.

(p, q)	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
A_{IVa}	A_{IIIa}	$A_{\mathrm{IIIf'}}$	A_{IIIe}	A_{IIIh}
A_{IVb}	$A_{\mathrm{IIIe'}}$	$A_{\mathrm{IIIh'}}$	$A_{\mathrm{IIIa'}}$	A_{IIIf}
A_{IVc}	$A_{\mathrm{IIIb'}}$	A_{IIId}	A_{IIIc}	$A_{\mathrm{IIIg'}}$
A_{IVd}	A_{IIIb}	$A_{\mathrm{IIId'}}$	$A_{\mathrm{IIIc'}}$	A_{IIIg}

3.2. *Linear Gauss diagram*

We can define the Gauss diagram for an oriented virtual arc in a similar way to a circular Gauss diagram, which we call a *linear Gauss diagram*; see Fig. 17. Similarly, a linear Gauss diagram is considered up to orientation preserving homeomorphism of the underlying oriented interval. Then a linear Gauss diagram determines a virtual arc diagram, that is, if two virtual arc diagrams have the same linear Gauss diagram, then there exist a finite sequence of virtual Reidemeister moves B_{I}, B_{II}, B_{III}, C, E_{I} which transforms one into the other. Therefore, since a welded arc determines a ribbon 2-knot [19, Proposition 5.2], we have:

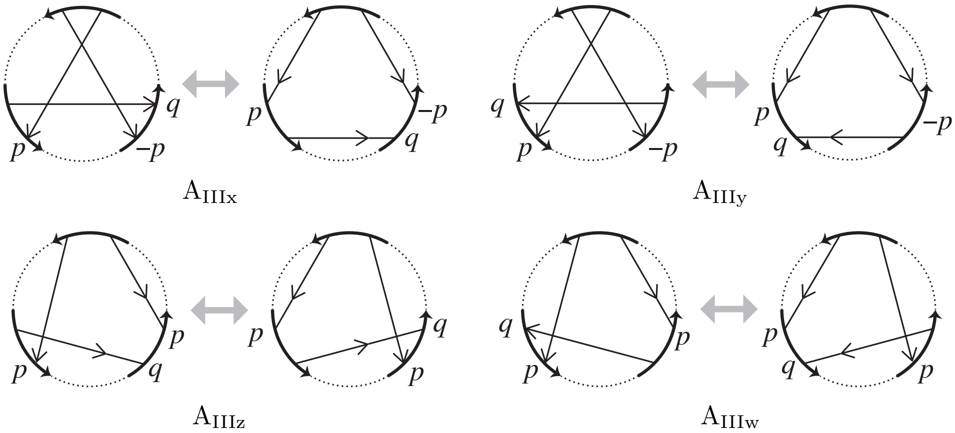


Fig. 16. Moves of circular Gauss diagrams equivalent to the moves $\mathrm{A_{IIIa}}, \ldots, \mathrm{A_{IIIh'}}$ allowing the move D, where p, $q = \pm$.

Proposition 3.1. *A linear Gauss diagram determines an associated ribbon* 2*-knot.*

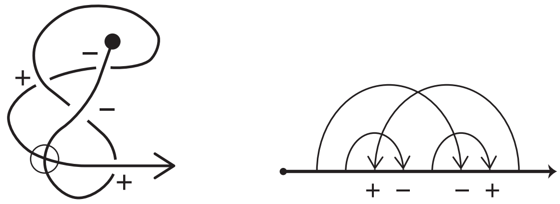


Fig. 17. Virtual arc diagram A in Fig. 6 and its linear Gauss diagram γ.

By abuse of notation, we denote by $\mathrm{Tube}(\gamma)$ the ribbon 2-knot obtained from a virtual arc diagram with Gauss diagram γ. Also, we define two Gauss diagrams γ and γ' are *r-equivalent*, denoted by $\gamma \overset{\mathrm{r}}{\sim} \gamma'$, if they correspond to the isotopic ribbon 2-knots; $\mathrm{Tube}(\gamma) \approx \mathrm{Tube}(\gamma')$.

We define the moves $\mathrm{A_I}$, $\mathrm{A_{II}}$, $\mathrm{A_{III}}$, D, and $\mathrm{E_{II}}$ for a linear Gauss diagrams as follows:

$\mathrm{A_I}$: One of the 2 types of moves as shown in Fig. 18.
$\mathrm{A_{II}}$: One of the 4 types of moves as shown in Fig. 19.
$\mathrm{A_{III}}$: One of the 12 types of moves as shown in Fig. 20.
D: One of the 3 types of moves as shown in Fig. 21.
$\mathrm{E_{II}}$: One of the 2 types of moves as shown in Fig. 22.

Fig. 18. Moves A_{I} for a linear Gauss diagram, where $p = \pm$.

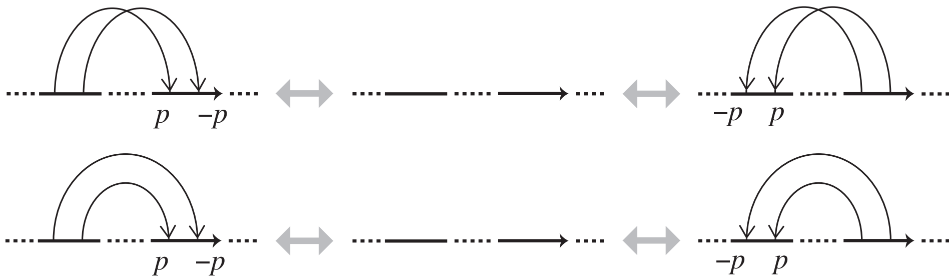


Fig. 19. Moves A_{II} for a linear Gauss diagram, where $p = \pm$.

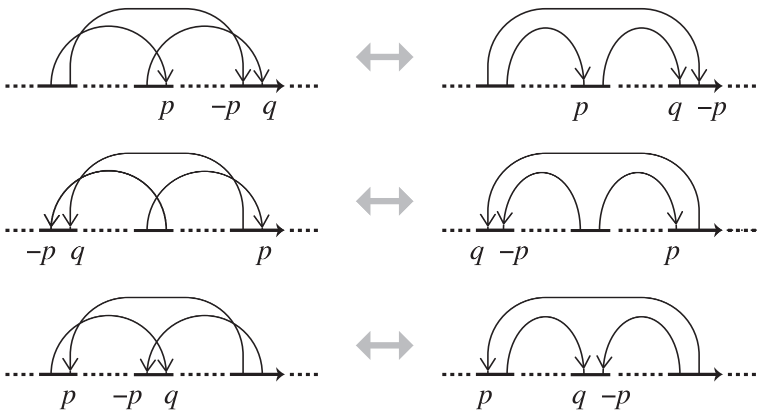


Fig. 20. Moves A_{III} for a linear Gauss diagram, where p, $q = \pm$.

The moves A_{I}, A_{II}, A_{III}, and D are yielded from those of a circular Gauss diagram given in Figs. 11, 12, 16, and 15, respectively, and the move E_{II} corresponds to the welded move E_{II} as shown in Fig. 4. So, we have:

Proposition 3.2. *If two linear Gauss diagrams are related by the moves* A_{I}, A_{II}, A_{III}, D, *or* E_{II}, *then they are r-equivalent.*

Furthermore, we define the move Ω_i, $i = 1, 2, \ldots, 8$, for a linear Gauss diagram as shown in Fig. 23. Then we have:

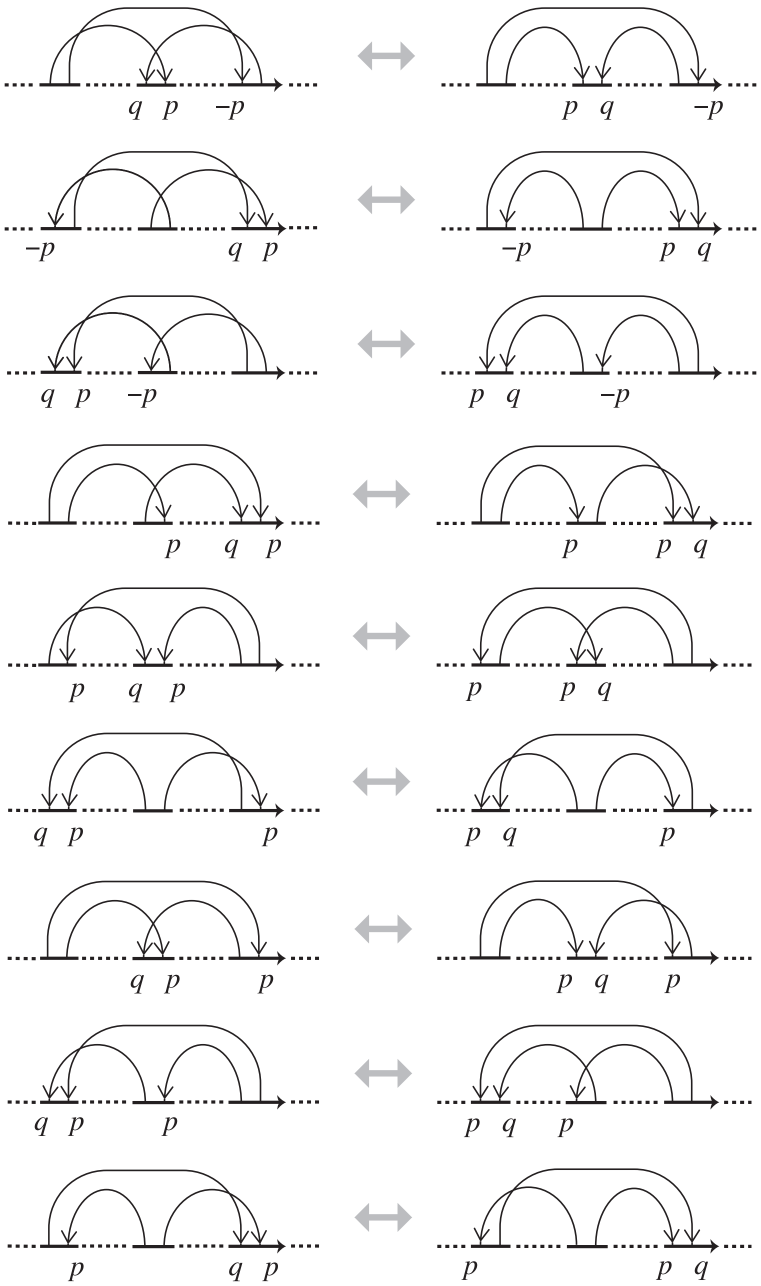


Fig. 20. Cont'd.

Proposition 3.3. *If two linear Gauss diagrams are related by the move Ω_i, $i = 1, \dots, 8$, then they are r-equivalent.*

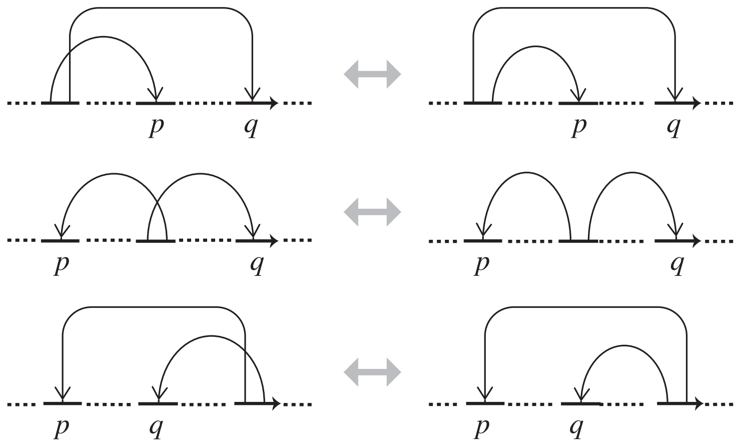


Fig. 21. Moves D for a linear Gauss diagram, where p, $q = \pm$.

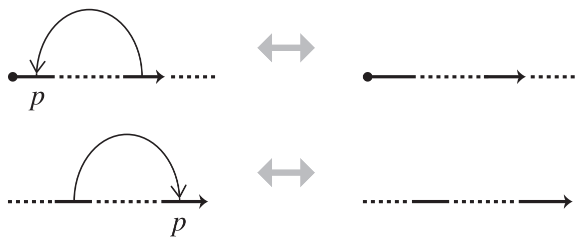


Fig. 22. Moves $\mathrm{E_{II}}$ for a linear Gauss diagram, where $p = \pm$.

Proof. Each of the moves Ω_1, Ω_2, Ω_3, Ω_4 is realized by a sequence of the moves $\mathrm{A_I}$ and $\mathrm{A_{III}}$; for the move Ω_1 see Fig. 24.

Next, the move Ω_1 implies the move Ω_5. In fact, the Ω_5 is realized by a sequence of the moves D and $\mathrm{E_{II}}$ as shown in Fig. 25. Similarly, the moves Ω_1, Ω_3, Ω_4 imply the moves Ω_6, Ω_7, Ω_8, respectively. This completes the proof. □

Let γ be the Gauss diagram of a virtual arc A. We denote by $\gamma^\dagger$ and $-\gamma$ the Gauss diagrams obtained from γ by changing the signs of chords and by reversing the orientation of the underlying interval, respectively. Then $\gamma^\dagger$ and $-\gamma$ are the Gauss diagrams of the virtual arcs $A^\dagger$ and $-A$, respectively. In particular, the Gauss diagram $-\gamma^\dagger$ is obtained from γ by changing the signs of chords and reversing the orientation of the interval, which is the Gauss diagrams of the virtual arc $-A^\dagger$.

Definition 3.4. We call the change of a linear Gauss diagram $\gamma \mapsto -\gamma^\dagger$ the *reversing move*; see Fig. 26.

Then by Proposition 2.2, we have

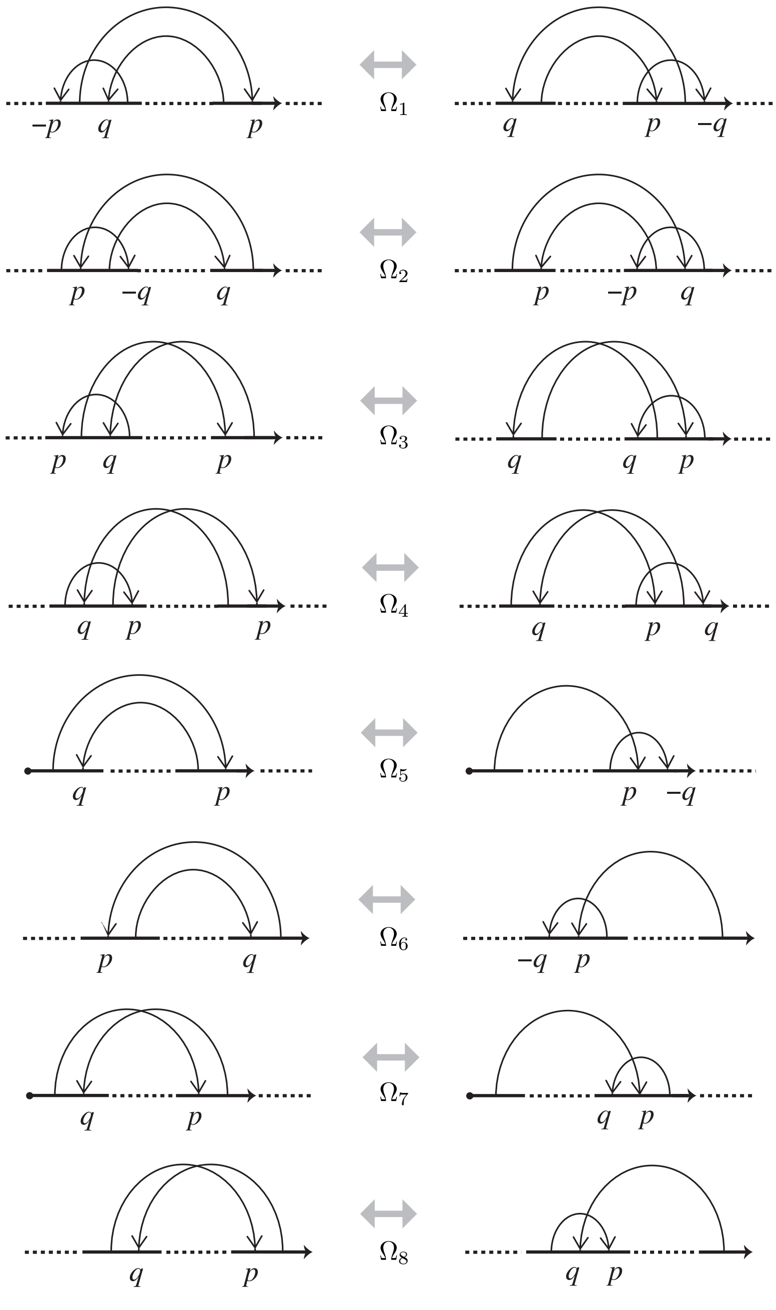


Fig. 23. Moves Ω_i, $i = 1, \ldots, 8$, which preserve the r-equivalence.

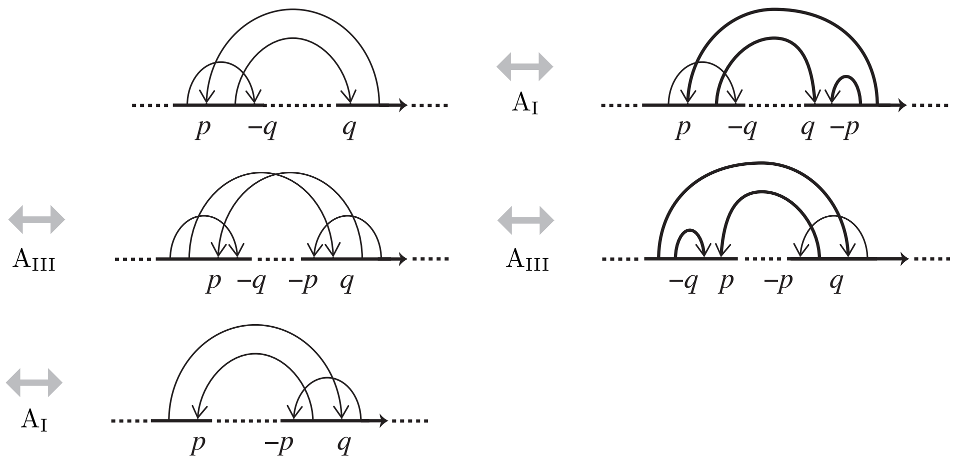


Fig. 24. Proof of Proposition 3.3 for the move Ω_1.

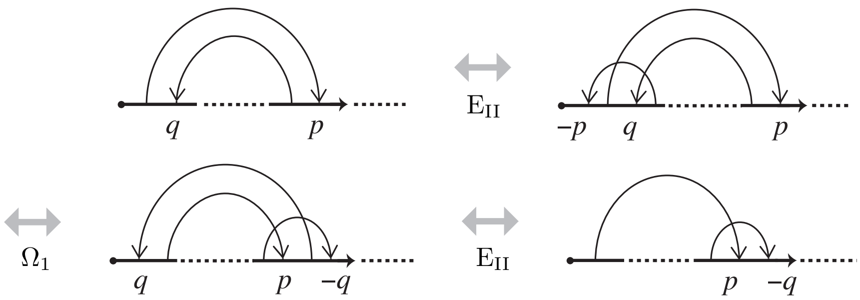


Fig. 25. Proof of Proposition 3.3 for the move Ω_5.

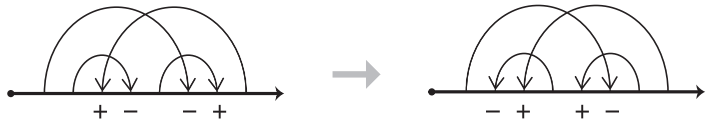


Fig. 26. Reversion move $\gamma \mapsto -\gamma^\dagger$, where γ and $-\gamma^\dagger$ are the linear Gauss diagrams for the virtual arcs A and $-A^\dagger$ given in Fig. 6.

Proposition 3.5. *The reversing move of a linear Gauss diagram preserves the r-equivalence,* $\gamma \overset{\mathrm{r}}{\sim} -\gamma^\dagger$.

Let γ and γ' be the Gauss diagrams of virtual arcs A and A'. We define the product $\gamma \cdot \gamma'$ by connecting the terminal point of the underlying interval of γ to the initial point of the underlying interval of γ'; see Fig. 27. Then $\gamma \cdot \gamma'$ is the Gauss diagram of the product virtual arc $A \cdot A'$.

The associated ribbon 2-knots Tube($\gamma \cdot \gamma'$) represents the composition of Tube(γ) and Tube(γ'). Since the composition of oriented 2-knots is well defined up to isotopy, we have the following:

$$\gamma \cdot \gamma' \overset{\mathrm{r}}{\sim} \gamma' \cdot \gamma \overset{\mathrm{r}}{\sim} (-\gamma^{\dagger}) \cdot \gamma'. \tag{3.1}$$

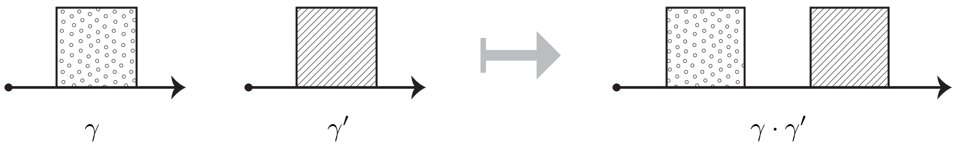


Fig. 27. Product of Gauss diagrams.

By Proposition 2.2 two ribbon 2-knots Tube($\gamma^{\dagger}$) and Tube($-\gamma$) are isotopic, which are the mirror images of Tube(γ), and thus, we have

$$\gamma \overset{\mathrm{r}}{\sim} -\gamma^{\dagger}. \tag{3.2}$$

If a virtual arc diagram A has a nugatory crossing as in Fig. 28(a), then A may be deformed into A' as in Fig. 28(b) by the Reidemeister moves and welded moves. Therefore, we have the following:

Lemma 3.6. *If a Gauss diagram γ has a separated single chord c as in Fig. 28(c) with any orientation and sign, then it is r-equivalent to the Gauss diagram obtained from γ by deleting the chord c.*

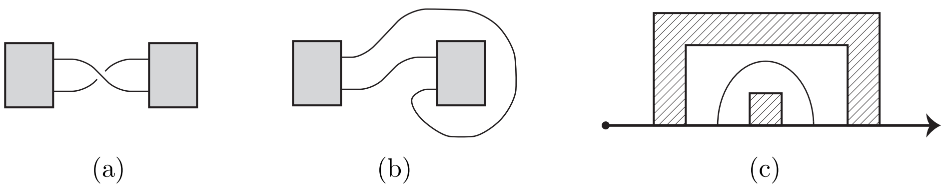


Fig. 28. (a) Virtual arc diagram A with a nugatory crossing; (b) virtual arc diagram A'; (c) Gauss diagram with a separated single chord.

4. Enumeration Result

Before stating the main result, we give some convention of a linear Gauss diagram which presents a ribbon 2-knot. Given a linear Gauss diagram we order its chords following the orientation of the underlying interval. More precisely, let γ be a linear

Gauss diagram with n chords such that $2n$ endpoints lie in the underlying interval $[0, 1]$, where 0 is the initial point and 1 is the terminal point. We order the chords $c_1, c_2, \ldots, c_n$ of γ so that $h(c_i) < h(c_{i+1})$, where $h(c_i)$ is the arrowhead point of c_i. If the chord c_i has the sign p_i, $p_i = \pm$, then we denote γ by $\gamma(p_1, p_2, \ldots, p_n)$.

The following is our main result.

Theorem 4.1. *A ribbon* 2*-knot presented by a virtual arc with up to* 4 *crossings is either the trivial one or one of those listed in Tables* 2*,* 3 *and* 4*.*

Each column in Tables 2, 3 and 4 shows as follows:

- The first column, Name, shows the name of the corresponding ribbon 2-knot, where $K!$ denotes the mirror image of the knot K, and $K\#K'$ the composition of two knots K and K'.

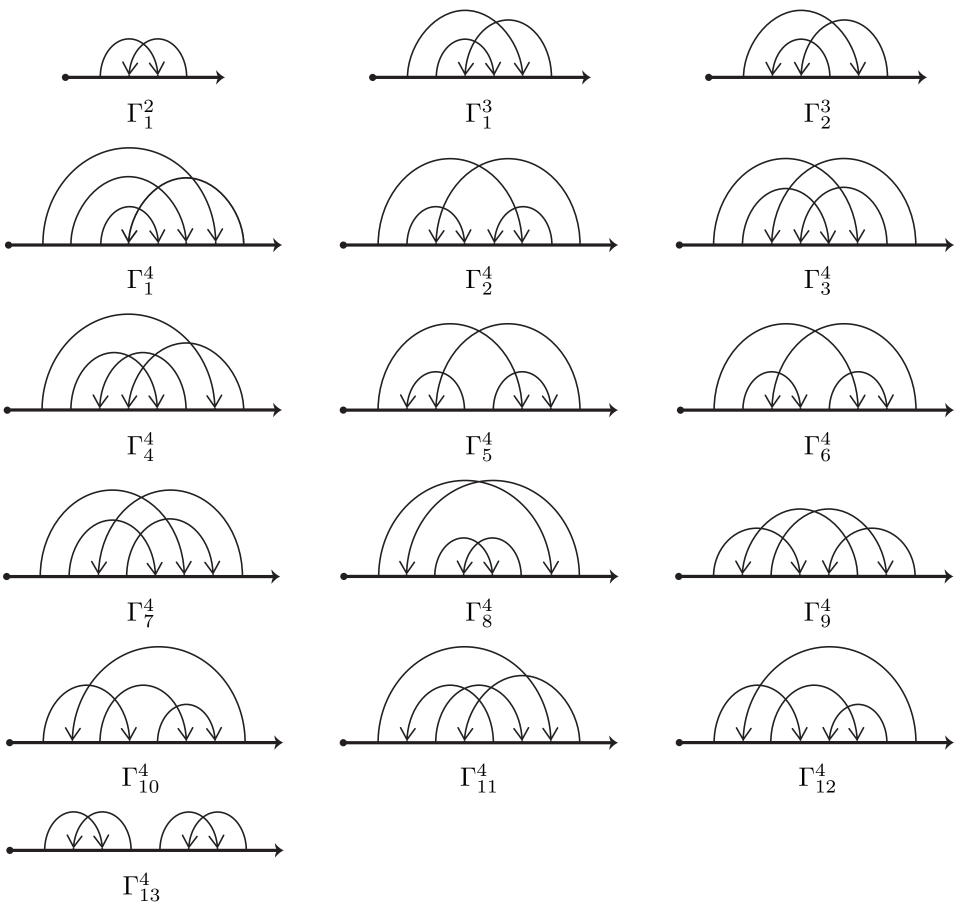


Fig. 29. Linear Gauss diagrams with up to 4 chords.

- The second column, Gauss diagram, shows the linear Gauss diagram presenting the ribbon 2-knot, which are shown as in Fig. 29. If there are more than one diagram, then they present isotopic ribbon 2-knots.
- The column, Group, in Tables 2 and 3 shows the group presentation of the ribbon 2-knot, that is, the fundamental group of the complement of the ribbon 2-knot, where

$$G(a_1, b_1, \dots, a_n, b_n) = \langle\, x, y\, ;\, y = wxw^{-1} \,\rangle, \quad w = x^{a_1} y^{b_1} \cdots x^{a_n} y^{b_n}. \tag{4.1}$$

  Note that x, y are meridians. In Table 4 there is no column listing the knot group because those ribbon 2-knots are composite and do not have such a group presentation.
- The column, $\Delta(t)$, shows the Alexander polynomial of the ribbon 2-knot which is normalized so that $\Delta(1) = 1$ and $(d/dt)\Delta(1) = 0$. We abbreviate $\Delta(t)$ as follows: for $c_i \in \boldsymbol{Z}$

$$(c_{-m}\ c_{-m+1}\ \cdots\ c_{-1}\ [c_0]\ c_1\ c_2\ \cdots\ c_n) = \sum_{i=-m}^{n} c_i t^i. \tag{4.2}$$

- The column, Det, shows the determinant of the ribbon 2-knot which is given by $|\Delta(-1)|$.
- In the last column if there is a mark "a", then the ribbon 2-knot is amphicheiral.

Remark 4.2. The group $G(a_1, b_1, \dots, a_n, b_n)$ is isomorphic to $G(-b_n, -a_n, \dots, -b_1, -a_1)$. In fact, $\langle\, x, y\, ;\, y = wxw^{-1} \,\rangle = \langle\, x, y\, ;\, x = w^{-1}yw \,\rangle$, where $w^{-1} = y^{-b_n} x^{-a_n} \cdots y^{-b_1} x^{-a_1}$ for $w = x^{a_1} y^{b_1} \cdots x^{a_n} y^{b_n}$. This is realized by the reversing move of a linear Gauss diagram.

Table 2. Ribbon 2-knots presented by linear Gauss diagrams with 2 or 3 chords.

Name	Gauss diagram	Group	$\Delta(t)$	Det	
R^2_1	$\Gamma^2_1(++)$, $\Gamma^2_1(--)$	G(1, 1)	(1 [-1] 1)	3	a
R^2_2	$\Gamma^2_1(+-)$	G(1, -1)	([0] 2 -1)	3	
$R^2_2!$	$\Gamma^2_1(-+)$	G(-1, 1)	(-1 2 [0])	3	
R^3_1	$\Gamma^3_1(+++)$	G(1, 2)	(1 -1 [0] 1)	1	
$R^3_1!$	$\Gamma^3_1(---)$	G(-1, -2)	(1 [0] -1 1)	1	
R^3_2	$\Gamma^3_1(+--)$	G(1, -2)	([0] 1 1 -1)	1	
$R^3_2!$	$\Gamma^3_1(-++)$	G(-1, 2)	(-1 1 1 [0])	1	
R^3_3	$\Gamma^3_2(+++)$, $\Gamma^3_2(+--)$	G(1, -1, 1, 1)	([2] -2 1)	5	
$R^3_3!$	$\Gamma^3_2(---)$, $\Gamma^3_2(-++)$	G(-1, 1, -1, -1)	(1 -2 [2])	5	
R^3_4	$\Gamma^3_2(++-)$	G(1, 1, 1, -1)	([1] -1 2 -1)	5	
$R^3_4!$	$\Gamma^3_2(--+)$	G(-1, -1, -1, 1)	(-1 2 -1 [1])	5	
R^3_5	$\Gamma^3_2(+-+)$, $\Gamma^3_2(-+-)$	G(1, -1, -1, 1)	(-1 [3] -1)	5	a

Table 3. Ribbon 2-knots presented by the linear Gauss diagrams Γ^4_l, $1 \leq l \leq 12$.

Name	Gauss diagram	Group	$\Delta(t)$	Det	
$R^4_{1,1}$	$\Gamma^4_1(++++)$	G(1, 3)	(1 -1 0 [0] 1)	3	
$R^4_{1,1}!$	$\Gamma^4_1(----)$	G(-1, -3)	(1 [0] 0 -1 1)	3	
$R^4_{1,2}$	$\Gamma^4_1(-+++)$	G(-1, 3)	(-1 1 0 1 [0])	3	
$R^4_{1,2}!$	$\Gamma^4_1(+---)$	G(1, -3)	([0] 1 0 1 -1)	3	
$R^4_{2,1}$	$\Gamma^4_2(++++)$, $\Gamma^4_2(----)$	G(1, 1, 1, 1)	(1 -1 [1] -1 1)	5	a
$R^4_{2,2}$	$\Gamma^4_2(+-+-)$	G(1, -1, 1, -1)	([0] 0 3 -2)	5	
$R^4_{2,2}!$	$\Gamma^4_2(-+-+)$	G(-1, 1, -1, 1)	(-2 3 0 [0])	5	
$R^4_{3,1}$	$\Gamma^4_3(++++)$, $\Gamma^4_3(----)$	G(2, 2)	(1 0 [-1] 0 1)	1	a
$R^4_{3,2}$	$\Gamma^4_3(++--)$	G(2, -2)	([0] 0 2 0 -1)	1	
$R^4_{3,2}!$	$\Gamma^4_3(--++)$	G(-2, 2)	(-1 0 2 0 [0])	1	
$R^4_{4,1}$	$\Gamma^4_4(++++)$	G(1, -1, 1, 2)	(2 [-2] 0 1)	3	
$R^4_{4,1}!$	$\Gamma^4_4(----)$	G(-1, 1, -1, -2)	(1 0 [-2] 2)	3	
$R^4_{4,2}$	$\Gamma^4_4(+-++)$	G(1, -1, -1, 2)	(-1 2 [0])	3	
$R^4_{4,2}!$	$\Gamma^4_4(-+--)$	G(-1, 1, 1, -2)	([0] 2 -1)	3	
$R^4_{4,3}$	$\Gamma^4_4(-+++)$	G(-1, -1, 1, 2)	(1 -2 1 [1])	3	
$R^4_{4,3}!$	$\Gamma^4_4(+---)$	G(1, 1, -1, -2)	([1] 1 -2 1)	3	
$R^4_{4,4}$	$\Gamma^4_4(++--)$	G(1, 1, 1, 2)	([0] 1 0 1 -1)	3	
$R^4_{4,4}!$	$\Gamma^4_4(--++)$	G(-1, -1, -1, -2)	(-1 1 0 1 [0])	3	
$R^4_{5,1}$	$\Gamma^4_5(++--)$	G(1, -1, -1, -2)	([1] 0 -1 2 -1)	3	
$R^4_{5,1}!$	$\Gamma^4_5(--++)$	G(-1, 1, 1, 2)	(-1 2 -1 0 [1])	3	
$R^4_{6,1}$	$\Gamma^4_6(++++)$	G(1, 1, -1, 1, 1, 1)	(1 -2 2 [-1] 1)	7	
$R^4_{6,1}!$	$\Gamma^4_6(----)$	G(-1, -1, 1, -1, -1, -1)	(1 [-1] 2 -2 1)	7	
$R^4_{6,2}$	$\Gamma^4_6(+++-)$, $\Gamma^4_6(---+)$	G(1, 1, -1, -1, 1, 1)	(2 [-3] 2)	7	a
$R^4_{6,3}$	$\Gamma^4_6(++-+)$, $\Gamma^4_6(-+--)$, $\Gamma^4_6(--++)$	G(1, -1, -1, 1, 1, 1)	(-1 3 [-2] 1)	7	
$R^4_{6,3}!$	$\Gamma^4_6(--+-)$, $\Gamma^4_6(+-++)$, $\Gamma^4_6(++--)$	G(-1, 1, 1, -1, -1, -1)	(1 [-2] 3 -1)	7	
$R^4_{6,4}$	$\Gamma^4_6(-+++)$	G(-1, 1, 1, 1, -1, 1)	(-1 2 -2 2 [0])	7	
$R^4_{6,4}!$	$\Gamma^4_6(+---)$	G(1, -1, -1, -1, 1, -1)	([0] 2 -2 2 -1)	7	
$R^4_{6,5}$	$\Gamma^4_6(+-+-)$	G(1, 1, -1, -1, 1, -1)	([0] 3 -3 1)	7	
$R^4_{6,5}!$	$\Gamma^4_6(-+-+)$	G(-1, -1, 1, 1, -1, 1)	(1 -3 3 [0])	7	
$R^4_{6,6}$	$\Gamma^4_6(-++-)$	G(-1, 1, 1, -1, -1, 1)	(-2 4 [-1])	7	
$R^4_{6,6}!$	$\Gamma^4_6(+--+)$	G(1, -1, -1, 1, 1, -1)	([-1] 4 -2)	7	
$R^4_{7,1}$	$\Gamma^4_7(++++)$	G(-1, 1, -2, -1)	(1 -1 -1 [2])	1	
$R^4_{7,1}!$	$\Gamma^4_7(----)$	G(1, -1, 2, 1)	([2] -1 -1 1)	1	
$R^4_{7,2}$	$\Gamma^4_7(+++-)$	G(1, 1, -2, -1)	(1 [0] -1 1)	1	
$R^4_{7,2}!$	$\Gamma^4_7(---+)$	G(-1, -1, 2, 1)	(1 -1 [0] 1)	1	
$R^4_{7,3}$	$\Gamma^4_7(-+++)$	G(-1, -1, -2, 1)	(-1 1 1 -1 [1])	1	
$R^4_{7,3}!$	$\Gamma^4_7(+---)$	G(1, 1, 2, -1)	([1] -1 1 1 -1)	1	
$R^4_{7,4}$	$\Gamma^4_7(-++-)$	G(1, -1, -2, 1)	(-1 1 [2] -1)	1	
$R^4_{7,4}!$	$\Gamma^4_7(+--+)$	G(-1, 1, 2, -1)	(-1 [2] 1 -1)	1	

Table 3. Cont'd.

Name	Gauss diagram	Group	$\Delta(t)$	Det	
$R^4_{8,1}$	$\Gamma^4_8(++++)$, $\Gamma^4_8(----)$	G(1, 1, 1, -1, -1, 1, 1, 1)	(1 -2 [3] -2 1)	9	
$R^4_{8,2}$	$\Gamma^4_8(+++-)$, $\Gamma^4_8(+---)$	G(1, -1, 1, 1, -1, 1, 1, -1)	([2] -3 3 -1)	9	
$R^4_{8,2}!$	$\Gamma^4_8(---+)$, $\Gamma^4_8(-+++)$	G(-1, 1, -1, -1, 1, -1, -1, 1)	(-1 3 -3 [2])	9	
$R^4_{8,4}$	$\Gamma^4_8(++--)$	G(1, -1, 1, -1, -1, -1, 1, -1)	([0] 0 4 -4 1)	9	
$R^4_{8,4}!$	$\Gamma^4_8(--++)$	G(-1, 1, -1, 1, 1, 1, -1, 1)	(1 -4 4 0 [0])	9	
$R^4_{8,5}$	$\Gamma^4_8(+-+-)$	G(1, -1, -1, 1, -1, 1, 1, -1)	(-2 [5] -2)	9	
$R^4_{8,5}!$	$\Gamma^4_8(-+-+)$	G(-1, 1, 1, -1, 1, -1, -1, 1)	(-2 [5] -2)	9	
$R^4_{8,6}$	$\Gamma^4_8(-++-)$, $\Gamma^4_8(+--+)$	G(-1, -1, 1, 1, 1, 1, -1, -1)	(1 -2 [3] -2 1)	9	a
$R^4_{9,1}$	$\Gamma^4_9(++--)$	G(1, 1, -1, 1, -1, -1)	(2 [-3] 2)	7	
$R^4_{9,1}!$	$\Gamma^4_9(--++)$	G(-1, -1, 1, -1, 1, 1)	(2 [-3] 2)	7	
$R^4_{9,2}$	$\Gamma^4_9(-++-)$, $\Gamma^4_9(+--+)$	G(-1, 1, 1, 1, 1, -1)	(-1 2 [-1] 2 -1)	7	a
$R^4_{10,1}$	$\Gamma^4_{10}(++++)$	G(1, 1, -1, 1, 1, -1, -1, 1, 1, 1)	(1 -3 4 [-2] 1)	11	
$R^4_{10,1}!$	$\Gamma^4_{10}(----)$	G(-1, -1, 1, -1, -1, 1, 1, -1, -1, -1)	(1 [-2] 4 -3 1)	11	
$R^4_{10,2}$	$\Gamma^4_{10}(+++-)$	G(1, 1, -1, 1, 1, -1, -1, -1, 1, 1)	(2 [-4] 4 -1)	11	
$R^4_{10,2}!$	$\Gamma^4_{10}(---+)$	G(-1, -1, 1, -1, -1, 1, 1, 1, 1, 1)	(-1 4 [-4] 2)	11	
$R^4_{10,3}$	$\Gamma^4_{10}(++-+)$	G(1, 1, -1, -1, 1, -1, -1, 1, 1, 1)	(-1 4 [-4] 2)	11	
$R^4_{10,3}!$	$\Gamma^4_{10}(--+-)$	G(-1, -1, 1, 1, -1, 1, 1, -1, -1, -1)	(2 [-4] 4 -1)	11	
$R^4_{10,4}$	$\Gamma^4_{10}(+-++)$	G(1, -1, -1, 1, 1, 1, -1, 1, 1, -1)	(-1 3 [-3] 3 -1)	11	
$R^4_{10,4}!$	$\Gamma^4_{10}(-+--)$	G(-1, 1, 1, -1, -1, -1, 1, -1, -1, 1)	(-1 3 [-3] 3 -1)	11	
$R^4_{10,5}$	$\Gamma^4_{10}(-+++)$	G(-1, 1, 1, 1, -1, -1, 1, 1, -1, 1)	(-1 3 -4 3 [0])	11	
$R^4_{10,5}!$	$\Gamma^4_{10}(+---)$	G(1, -1, -1, -1, 1, 1, -1, -1, 1, -1)	([0] 3 -4 3 -1)	11	
$R^4_{10,6}$	$\Gamma^4_{10}(++--)$	G(1, 1, -1, -1, 1, -1, -1, -1, 1, 1)	(1 [-2] 4 -3 1)	11	
$R^4_{10,6}!$	$\Gamma^4_{10}(--++)$	G(-1, -1, 1, 1, -1, 1, 1, 1, -1, -1)	(1 -3 4 [-2] 1)	11	
$R^4_{10,7}$	$\Gamma^4_{10}(+-+-)$	G(1, -1, -1, 1, 1, 1, -1, -1, 1, -1)	([-1] 5 -4 1)	11	
$R^4_{10,7}!$	$\Gamma^4_{10}(-+-+)$	G(-1, 1, 1, -1, -1, -1, 1, 1, -1, 1)	(1 -4 5 [-1])	11	
$R^4_{10,8}$	$\Gamma^4_{10}(-++-)$	G(-1, 1, 1, 1, -1, -1, 1, -1, -1, 1)	(-2 5 [-3] 1)	11	
$R^4_{10,8}!$	$\Gamma^4_{10}(+--+)$	G(1, -1, -1, -1, 1, 1, -1, 1, 1, -1)	(1 [-3] 5 -2)	11	

The proof of Theorem 4.1 is given in Sects. 5, 6 and 7, where we attempt to find isotopic ribbon 2-knot pairs which are presented by linear Gauss diagrams with up to 4 chords. Actually, Theorem 4.1 does not claim the ribbon 2-knots listed in Tables 2, 3 and 4 are mutually distinct. A pair of ribbon 2-knots with the same Alexander polynomial might be isotopic. In the forthcoming paper [11] we discuss the classification of these ribbon 2-knots.

Table 3. Cont'd.

Name	Gauss diagram	Group	$\Delta(t)$	Det
$R^4_{11,1}$	$\Gamma^4_{11}(++++)$	G(1, 1, -1, 1, 1, 1, -1, -1)	(1 -2 [3] -2 1)	9
$R^4_{11,1}!$	$\Gamma^4_{11}(----)$	G(-1, -1, 1, -1, -1, -1 , 1, 1)	(1 -2 [3] -2 1)	9
$R^4_{11,2}$	$\Gamma^4_{11}(+++-)$	G(1, 1, -1, 1, 1, -1, -1, -1)	([2] -3 3 -1)	9
$R^4_{11,2}!$	$\Gamma^4_{11}(---+)$	G(-1,-1, 1,-1,-1, 1, 1, 1)	(-1 3 -3 [2])	9
$R^4_{11,3}$	$\Gamma^4_{11}(++-+)$, $\Gamma^4_{11}(+-+-)$	G(1, -1, -1, 1, 1, 1, -1, -1)	(-1 [4] -3 1)	9
$R^4_{11,3}!$	$\Gamma^4_{11}(--+-)$, $\Gamma^4_{11}(-+-+)$	G(-1, 1, 1, -1, -1, -1, 1, 1)	(1 -3 [4] -1)	9
$R^4_{11,4}$	$\Gamma^4_{11}(+-++)$	G(-1, 1, 1, 1, -1, 1, 1, -1)	(-1 2 -2 [3] -1)	9
$R^4_{11,4}!$	$\Gamma^4_{11}(-+--)$	G(1, -1, -1, -1, 1, -1 -1, 1)	(-1 [3] -2 2 -1)	9
$R^4_{11,5}$	$\Gamma^4_{11}(-+++)$	G(1, 1, -1, -1, 1, 1, -1, 1)	(2 -4 [3])	9
$R^4_{11,5}!$	$\Gamma^4_{11}(+---)$	G(-1, -1, 1, 1, -1, -1, 1, -1)	([3] -4 2)	9
$R^4_{11,6}$	$\Gamma^4_{11}(++--)$	G(1, -1, -1, 1, 1, -1, -1, -1)	([1] -2 4 -2)	9
$R^4_{11,6}!$	$\Gamma^4_{11}(--++)$	G(-1, 1, 1, -1, -1, 1, 1, 1)	(-2 4 -2 [1])	9
$R^4_{11,7}$	$\Gamma^4_{11}(-++-)$	G(1, 1, -1, -1, 1, -1, -1, 1)	(-1 [4] -3 1)	9
$R^4_{11,7}!$	$\Gamma^4_{11}(+--+)$	G(-1, -1, 1, 1, -1, 1, 1, -1)	(1 -3 [4] -1)	9
$R^4_{12,1}$	$\Gamma^4_{12}(+++-)$	G(1, 1, -2, 1)	([1])	1
$R^4_{12,1}!$	$\Gamma^4_{12}(---+)$	G(-1, -1, 2, -1)	([1])	1
$R^4_{12,2}$	$\Gamma^4_{12}(-+++)$	G(-1, 1, 2, 1)	(1 [-1] 0 2 -1)	1
$R^4_{12,2}!$	$\Gamma^4_{12}(+---)$	G(1, -1, -2, -1)	(-1 2 0 [-1] 1)	1

Table 4. Ribbon 2-knots presented by the linear Gauss diagram Γ^4_{13}.

Name	Gauss diagram	$\Delta(t)$	Det	
$R^2_1\#R^2_1$	$\Gamma^4_{13}(++++)$, $\Gamma^4_{13}(----)$, $\Gamma^4_{13}(++--)$, $\Gamma^4_{13}(--++)$	(1 -2 [3] -2 1)	9	a
$R^2_1\#R^2_2$	$\Gamma^4_{13}(+++-)$, $\Gamma^4_{13}(--+-)$, $\Gamma^4_{13}(-+--)$, $\Gamma^4_{13}(-+++)$	([2] -3 3 -1)	9	
$R^2_1\#R^2_2!$	$\Gamma^4_{13}(++-+)$, $\Gamma^4_{13}(---+)$, $\Gamma^4_{13}(-+--)$, $\Gamma^4_{13}(-+++)$	(-1 3 -3 [2])	9	
$R^2_2\#R^2_2!$	$\Gamma^4_{13}(+--+)$, $\Gamma^4_{13}(-++-)$	(-2 [5] -2)	9	a
$R^2_2\#R^2_2$	$\Gamma^4_{13}(+-+-)$	([0] 0 4 -4 1)	9	
$R^2_2!\#R^2_2!$	$\Gamma^4_{13}(-+-+)$	(1 -4 4 0 [0])	9	

In the remaining of this section, we explain the calculation of the knot group and Alexander polynomial.

Let A be an oriented virtual arc diagram. Then we can define the group of A in the same way as a virtual knot, which uses a generalization of Wirtinger's algorithm; see [12]. It is also the fundamental group of the complement of the corresponding ribbon 2-knot Tube(A), $\pi_1(\boldsymbol{R}^4 - \text{Tube}(A))$ [19, Proposition 5.3]. Furthermore, we can obtain the group presentation of A (and so of Tube(A)) from a linear Gauss diagram γ which presents A as for a virtual knot explained in [2, p. 1049]; cf. [13, Sec. 2.3]. If we cut the interval at each arrowhead (forgetting arrowtails), the interval of γ is divided into a set of arcs. To each of these arcs there corresponds a generator of the group, which corresponds to a meridian generator of the group of Tube(A). Each arrow gives rise to a relation. Suppose the sign of an arrow is p, its tail lies on an arc labeled x, its head is the final point of an arc labeled y and the initial point of an arc labeled z. Then we assign to this arrow the relation $z = x^{-p}yx^p$, meaning:

$$x^{-p}yx^p = \begin{cases} x^{-1}yx & \text{if } p = +; \\ xyx^{-1} & \text{if } p = -. \end{cases} \tag{4.3}$$

The resulting group is the group of the virtual arc A and also the group of the corresponding ribbon 2-knot Tube(A), which we denote by $\pi\gamma$.

Example 4.3. We calculate the group $\pi\Gamma_6^4(p, q, r, s)$. Cutting the interval at each arrowhead, we obtain 5 arcs. Take generators x, y, z, u, v as shown in Fig. 30. Then we obtain the following relations:

$$y = v^{-p}xv^p, \tag{4.4}$$
$$z = x^{-q}yx^q, \tag{4.5}$$
$$u = x^{-r}zx^r, \tag{4.6}$$
$$v = z^{-s}uz^s. \tag{4.7}$$

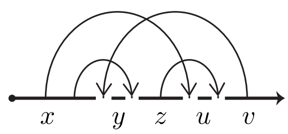


Fig. 30. Generators of the group $\pi\Gamma_6^4(p, q, r, s)$.

In fact, from the linear Gauss diagram $\Gamma_6^4(+ - - +)$ we obtain a virtual arc diagram as shown in Fig. 31, from which we obtain the above group presentation by Wirtinger's algorithm.

Substitute (4.4) and (4.6) into (4.5) and (4.7), respectively. Then we obtain

$$z = x^{-q}v^{-p}xv^px^q, \tag{4.8}$$
$$v = z^{-s}x^{-r}zx^rz^s. \tag{4.9}$$

Substitute (4.8) into (4.9). Then we obtain

$$\begin{aligned} v &= x^{-q}v^{-p}x^{-s}v^px^qx^{-r}x^{-q}v^{-p}xv^px^qx^rx^{-q}v^{-p}x^sv^px^q \\ &= (x^{-q}v^{-p}x^{-s}v^px^{-r}v^{-p})x(v^px^rv^{-p}x^sv^px^q). \end{aligned} \tag{4.10}$$

So, the group $\pi\Gamma_6^4(p, q, r, s)$ is presented by $G(-q, -p, -s, p, -r, -p)$.

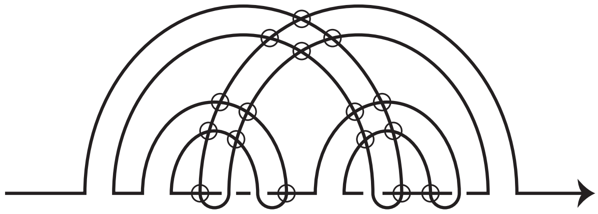


Fig. 31. Virtual arc diagram with Gauss diagram $\Gamma_6^4(+--+)$

Alternatively, substitute (4.9) into (4.4). Then we obtain

$$y = z^{-s}x^{-r}z^{-p}x^rz^sxz^{-s}x^{-r}z^px^rz^s \tag{4.11}$$

Then substitute (4.11) into (4.5). Then we obtain

$$z = (x^{-q}z^{-s}x^{-r}z^{-p}x^rz^s)x(z^{-s}x^{-r}z^px^rz^sx^q). \tag{4.12}$$

So, the group $\pi\Gamma_6^4(p,q,r,s)$ is also presented by $G(-q,-s,-r,-p,r,s)$. In Table 3 we list the former presentation. In particular, the group $\pi\Gamma_6^4(+--+)$ has two presentations, $G(1,-1,-1,1,1,-1)$ and $G(1,-1,1,-1,-1,1)$. It is not trivial that they present the same group.

Example 4.4. We calculate the group $\pi\Gamma_{11}^4(p,q,r,s)$. Take generators x, y, z, u, v as shown in Fig. 32. Then we obtain the following relations:

$$y = z^{-p}xz^p, \tag{4.13}$$
$$z = v^{-q}yv^q, \tag{4.14}$$
$$u = y^{-r}zy^r, \tag{4.15}$$
$$v = x^{-s}ux^s. \tag{4.16}$$

Removing x, y and u, we obtain:

$$v = (z^pv^qz^{-s}v^{-q}z^{-p})(v^qz^{-r}v^{-q}zv^qz^rv^{-q})(z^pv^qz^sv^{-q}z^{-p}). \tag{4.17}$$

So, the group $\pi\Gamma_{11}^4(p,q,r,s)$ is presented by $G(p,q,-s,-q,-p,q,-r,-q)$, which is isomorphic to $G(q,r,-q,p,q,s,-q,-p)$. In Table 3 we list this presentation.

Alternatively, removing y, u and v, we obtain:

$$z = (x^{-s}z^{-p}x^{-r}z^{-q}x^rz^px^sz^{-p})x(z^px^{-s}z^{-p}x^{-r}z^qx^rz^qx^s). \tag{4.18}$$

So, the group $\pi\Gamma_{11}^4(p,q,r,s)$ is also presented by $G(-s,-p,-r,-q,r,p,s,-p)$.

From the group presentation Eq. (4.1) we obtain the Alexander polynomial as follows:

$$\begin{aligned} 1 - t^{a_1} + t^{a_1+b_1} - t^{a_1+b_1+a_2} + t^{a_1+b_1+a_2+b_2} - \cdots \\ -t^{a_1+b_1+\cdots+b_{n-1}+a_n} + t^{a_1+b_1+\cdots+b_{n-1}+a_n+b_n}; \end{aligned} \tag{4.19}$$

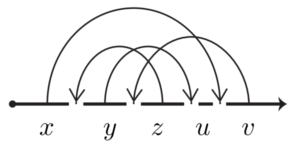


Fig. 32. Generators x, y, z, u, v of the group $\pi\Gamma_{11}^4(p,q,r,s)$.

see [15]. If $\Delta(t)$ is the normalized Alexander polynomial of a ribbon 2-knot, then the normalized Alexander polynomial of its mirror image is given by $\Delta(t^{-1})$, and thus their determinants coincide. Further, if $\Delta(t) \neq \Delta(t^{-1})$, then it is not amphicheiral.

Remark 4.5. We can define a sequence of invariants for a ribbon 2-knot by $\alpha_n(K) = (l/n!)d^n/dt^n\Delta_K(1)$, which are finite type invariants [3, Theorem2.2]; see also [4,10].

5. Ribbon 2-Knots Presented by Linear Gauss Diagrams with up to 3 Chords

In this section, we prove Theorem 4.1 for a linear Gauss diagram with up to 3 chords.

5.1. *Linear Gauss diagrams with up to 2 chords*

A ribbon 2-knot presented by a linear Gauss diagram with one chord is trivial by the move $\mathrm{A_I}$. In order to enumerate ribbon 2-knots presented by linear Gauss diagrams with 2 chords, by Lemma 3.6 and the move $\mathrm{E_{II}}$ we have only to consider the diagram Γ_1^2 as shown in Fig. 29. Then since $\Gamma_1^2(++) \overset{\mathrm{r}}{\sim} \Gamma_1^2(--)$ by the reversing move, we obtain the ribbon 2-knots R_1^2, R_2^2, $R_2^2!$ as listed in Table 2. The group is presented as follows:

$$\pi\Gamma_1^2(p,q) = G(p,q). \tag{5.1}$$

5.2. *Linear Gauss diagrams with 3 chords*

We enumerate ribbon 2-knots presented by linear Gauss diagrams with 3 chords.

Lemma 5.1. *Any linear Gauss diagram with* 3 *chords is r-equivalent to either the trivial diagram or the diagrams* Γ_1^2, Γ_1^3, *or* Γ_2^3 *given in Fig.* 29 *with some signs.*

Proof. By Lemma 3.6 we have only to consider three linear Gauss diagrams β_i, $i = 1, 2, 3$, as shown in Fig. 33, where we ignore the orientation of the interval, and the orientations and signs of the chords.

Then, by the move $\mathrm{E_{II}}$ and reversing move we have only to consider the linear Gauss diagrams Γ_1^3, Γ_2^3, β_{21}, β_{31} as shown in Fig. 34. By the move D we have

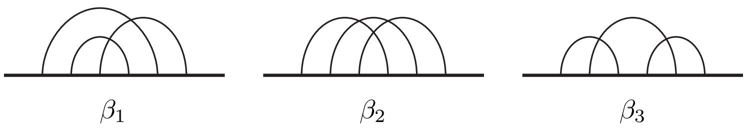


Fig. 33. Unoriented linear Gauss diagrams with 3 chords.

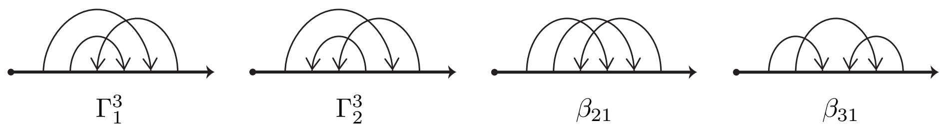


Fig. 34. Oriented linear Gauss diagrams with 3 chords.

$\beta_{21}(p, q, r) \overset{\mathrm{r}}{\sim} \Gamma_1^3(p, q, r)$, and by the moves D and $\mathrm{A_I}$ we have $\beta_{31}(p, q, r) \overset{\mathrm{r}}{\sim} \Gamma_1^2(q, r)$. This completes the proof. □

Now, we consider the diagrams $\Gamma_1^3(p, q, r)$ and $\Gamma_2^3(p, q, r)$. By the moves $\mathrm{A_{II}}$ and $\mathrm{A_I}$ the diagram $\Gamma_1^3(p, q, -q)$ is deformed into the trivial diagram. By the move Ω_5 and reversing move the diagram $\Gamma_2^3(p, q, r)$ is deformed into $\Gamma_2^3(p, -r, -q)$; see Fig. 35(a). This implies $\Gamma_2^3(+++) \overset{\mathrm{r}}{\sim} \Gamma_2^3(+--)$. Furthermore, by the reversing move and the move Ω_2 the diagram $\Gamma_2^3(p, -p, q)$ is deformed into $\Gamma_2^3(-q, q, -p)$; see Fig. 35(b). This implies $\Gamma_2^3(+-+) \overset{\mathrm{r}}{\sim} \Gamma_2^3(-+-)$. Then we obtain the ribbon 2-knots R_i^3, $R_i^3!$ ($i = 1, 2, 3, 4$) and R_5^3 as listed in Table 2. The groups are presented as follows:

$$\pi\Gamma_1^3(p, q, r) = G(p,\ q + r); \tag{5.2}$$

$$\pi\Gamma_2^3(p, q, r) = G(p,\ -r,\ q,\ r). \tag{5.3}$$

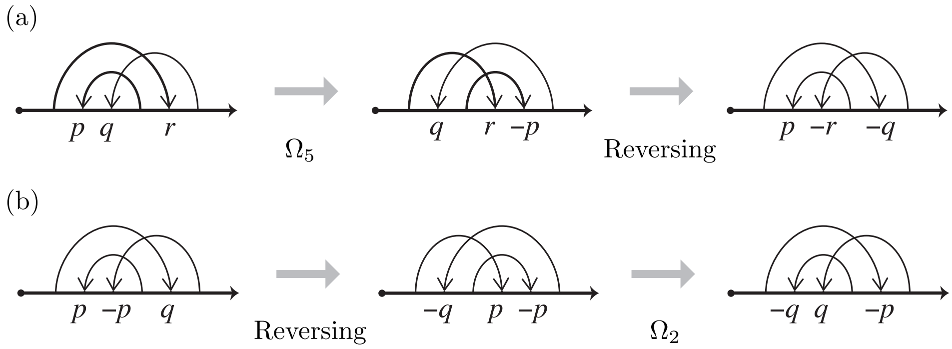


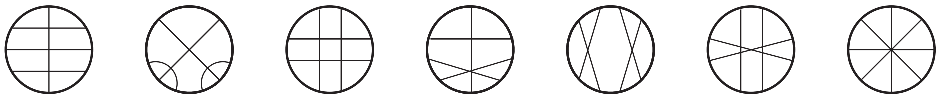
Fig. 35. r-Equivalent linear Gauss diagrams with 3 chords.

6. Linear Gauss diagrams with 4 chords

In this section, we prove the following lemma.

Lemma 6.1. *Any linear Gauss diagram with 4 chords is r-equivalent to either a linear Gauss diagram with up to 3 chords or Γ^4_l, $1 \le l \le 13$, given in Fig. 29 with some signs.*

Proof. First we note there are seven types of circular Gauss diagrams with 4 chords that has no separated single chord as shown in Fig. 36.



γ_1 γ_2 γ_3 γ_4 γ_5 γ_6 γ_7

Fig. 36. Circular Gauss diagrams with 4 chords.

From the circular Gauss diagram γ_i, $1 \le i \le 7$, we obtain linear Gauss diagrams γ_{ij} ignoring the orientation of the interval, and the orientations and signs of the chords as shown in Fig. 37.

From γ_{ij} we obtain oriented Gauss diagrams γ_{ijk} as shown in Fig. 38 up to r-equivalence, where:

- $\gamma_{ijk} = \Gamma^4_l$ means that the diagram γ_{ijk} is the same as Γ^4_l given in Fig. 29.
- $\gamma_{ijk} \to \Gamma^3$ means that by the move D the diagram $\gamma_{ijk}(p, q, r, s)$ with any signs (p, q, r, $s = \pm$) is deformed into a linear Gauss diagram having a single separated chord, which is r-equivalent to a Gauss diagram with up to 3 chords by Lemma 3.6.
- $\gamma_{ijk} \to \pm\Gamma^4_l$ means that by the move D the diagram $\gamma_{ijk}(p, q, r, s)$ is deformed into the Gauss diagram $\pm\Gamma^4_l(p', q', r', s')$ for some l and some signs p', q', r', $s' = \pm$, where Γ^4_l is given in Fig. 29.
- $\gamma_{ijk} \to \pm\gamma_{i'j'k'}$ means that by the move D the diagram $\gamma_{ijk}(p, q, r, s)$ is deformed into the Gauss diagram $\pm\gamma_{i'j'k'}(p', q', r', s')$.

The following are proved by the move Ω_i, $i = 5$, 6, 7, 8:

$$\gamma_{112}(p, q, r, s) \overset{\mathrm{r}}{\sim} \Gamma^4_7(q, r, s, -p) \quad \text{(by the move } \Omega_6\text{)}, \tag{6.1}$$

$$\gamma_{113}(p, q, r, s) \overset{\mathrm{r}}{\sim} \Gamma^4_6(q, r, s, -p) \quad \text{(by the move } \Omega_5\text{)}, \tag{6.2}$$

$$\gamma_{114}(p, q, r, s) \overset{\mathrm{r}}{\sim} \Gamma^4_5(q, r, s, -p) \quad \text{(by the move } \Omega_5\text{)}, \tag{6.3}$$

$$\gamma_{441}(p, q, r, s) \overset{\mathrm{r}}{\sim} \Gamma^4_6(p, s, q, r) \quad \text{(by the move } \Omega_8\text{)}, \tag{6.4}$$

$$\gamma_{442}(p, q, r, s) \overset{\mathrm{r}}{\sim} \Gamma^4_2(p, s, q, r) \quad \text{(by the move } \Omega_8\text{)}, \tag{6.5}$$

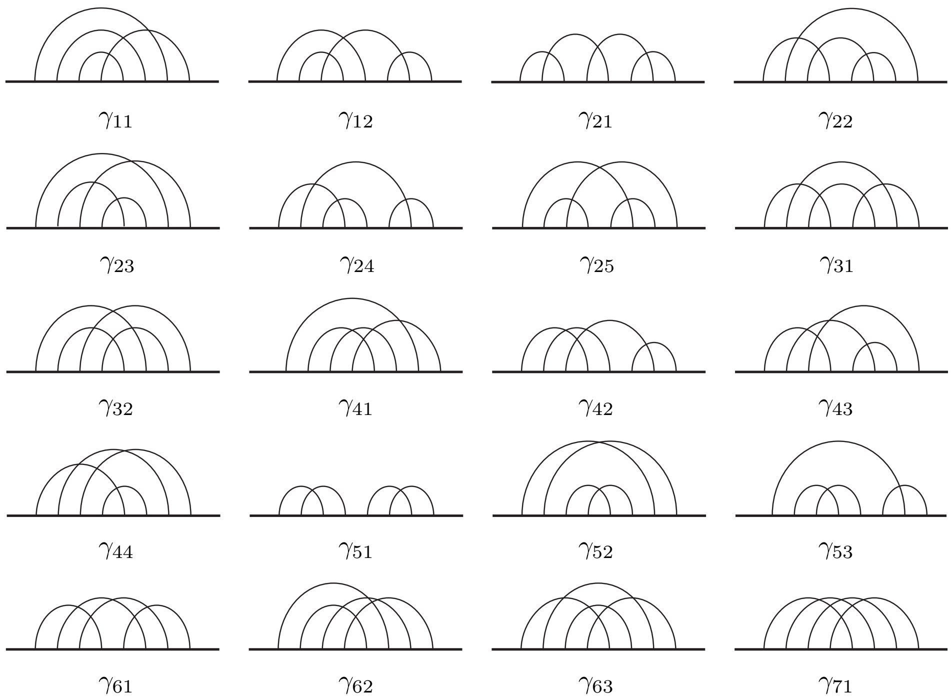


Fig. 37. Linear Gauss diagrams γ_{ij} obtained from γ_i, $1 \leq i \leq 7$.

$$\gamma_{531}(p,q,r,s) \overset{\mathrm{r}}{\sim} \Gamma^4_8(r,p,q,s) \qquad \text{(by the move } \Omega_7\text{)}, \tag{6.6}$$

$$\gamma_{423}(p,q,r,s) \overset{\mathrm{r}}{\sim} \gamma_{243}(p,q,r,s) \qquad \text{(by the move } \Omega_7\text{)}, \tag{6.7}$$

$$\gamma_{622}(p,q,r,s) \overset{\mathrm{r}}{\sim} \gamma_{113}(p,q,s,r) \qquad \text{(by the move } \Omega_8\text{)}, \tag{6.8}$$

$$\gamma_{713}(p,q,r,s) \overset{\mathrm{r}}{\sim} \gamma_{442}(q,p,r,s) \qquad \text{(by the move } \Omega_7\text{)}. \tag{6.9}$$

Thus, each linear Gauss diagram $\gamma_{ijk}(p,q,r,s)$ is r-equivalent to either a diagram with up to 3 chords or $\Gamma^4_l(p',q',r',s')$ for some l and some signs p', q', r', s'. This completes the proof. □

7. Ribbon 2-Knots Obtained from the Linear Gauss Diagrams Γ^4_l

In this section, we prove Theorem 4.1 for a linear Gauss diagram with 4 chords using Lemma 6.1; we examine the associated ribbon 2-knot for each $\Gamma^4_l(p,q,r,s)$, $1 \leq l \leq 13$.

7.1. *Ribbon 2-knots obtained from* Γ^4_1

By the move $\mathrm{A_{II}}$ if either $q=-r$ or $r=-s$, then $\Gamma^4_1(p,q,r,s)$ is r-equivalent to a diagram with 2 chords. So, we consider the case $q=r=s$. Then we obtain the

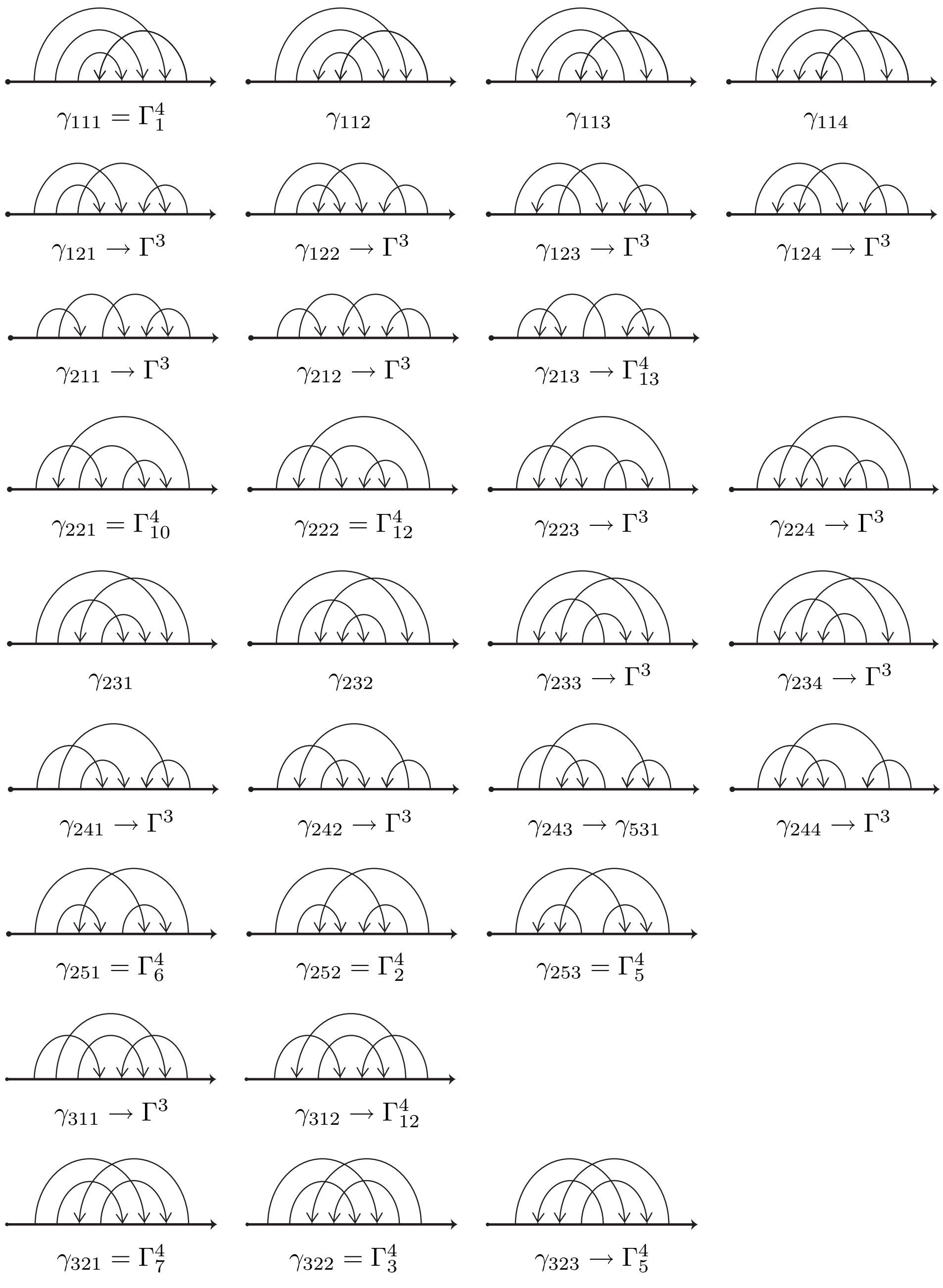


Fig. 38. Oriented linear Gauss diagrams γ_{ijk} obtained from γ_{ij}.

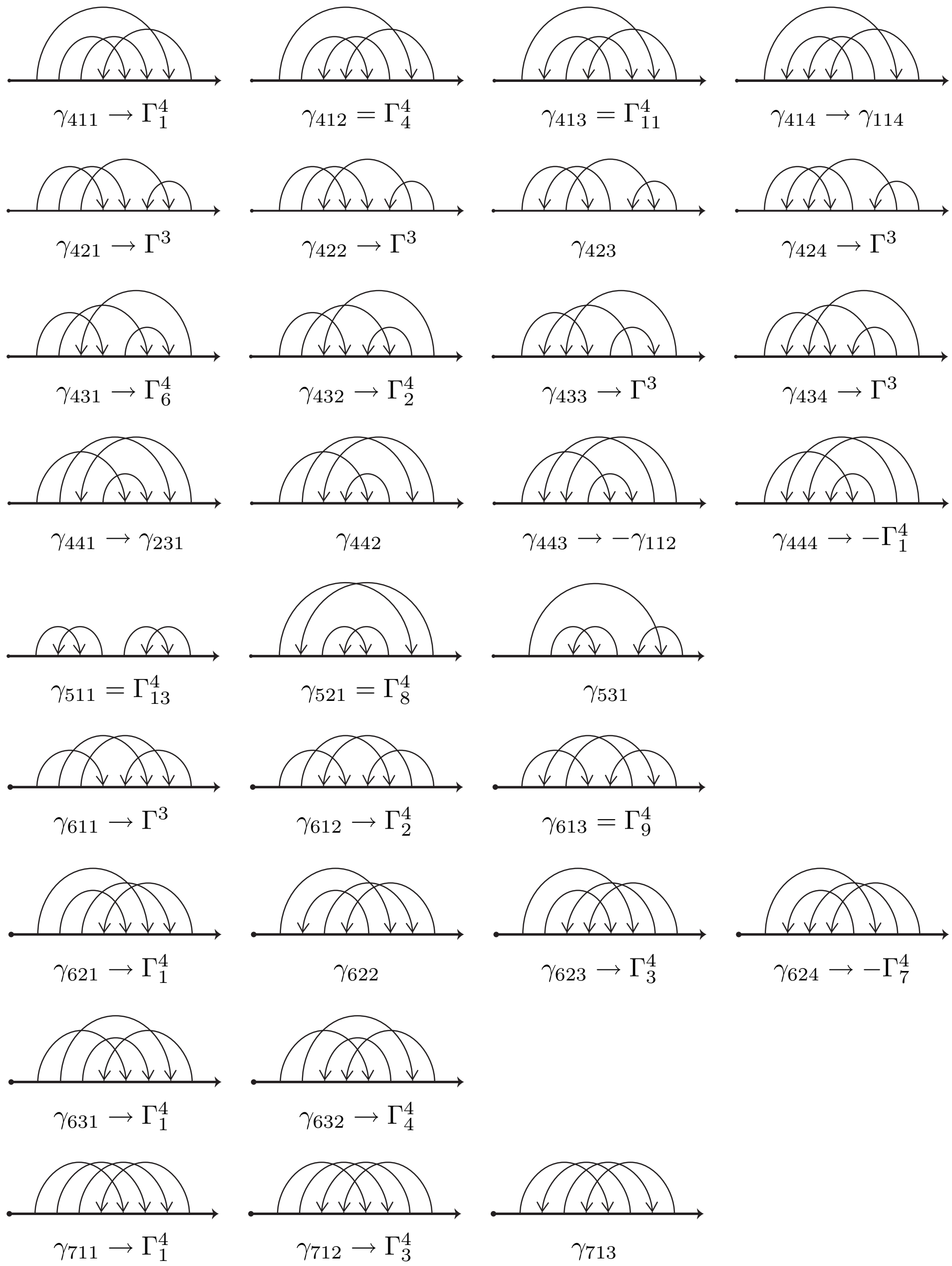


Fig. 38. Cont'd.

ribbon 2-knots $R^4_{1,j}$, $R^4_{1,j}!$, $j = 1, 2$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma_1^4(p,q,r,s) = G(p, q+r+s). \tag{7.1}$$

7.2. *Ribbon 2-knots obtained from* Γ_2^4

By the move D the diagram $\Gamma_2^4(p,q,r,s)$ is deformed into $\gamma_{612}(p,q,r,s)$ given in Fig. 38. If either $s = -q$ or $r = -p$, then by the move $\mathrm{A_{III}}$ the diagram $\gamma_{612}(p,q,r,s)$ is deformed into a diagram with a single separated chord; see Fig. 39. So, we consider the case $s = q$ and $r = p$. Note that $\Gamma_2^4(++++) \overset{\mathrm{r}}{\sim} \Gamma_2^4(----)$ by the reversing move. Then we obtain the ribbon 2-knots $R_{2,1}^4$, $R_{2,2}^4$, $R_{2,2}^4!$ as listed in Table 3. The group is presented as follows:

$$\pi\Gamma_2^4(p,q,r,s) = G(p,q,r,s). \tag{7.2}$$

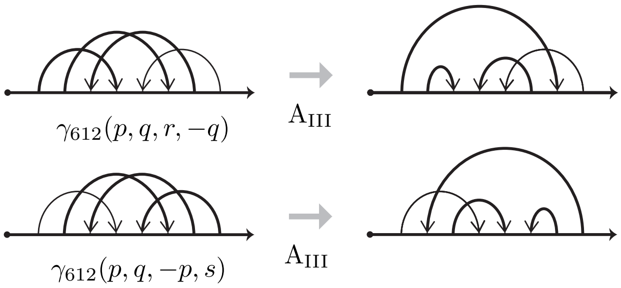


Fig. 39. The diagrams $\gamma_{612}(p,q,r,-q)$ and $\gamma_{612}(p,q,-p,s)$ are deformed into diagrams with a single separated chord.

7.3. *Ribbon 2-knots obtained from* Γ_3^4

If either $p = -q$ or $q = -r$, then by the move $\mathrm{A_{II}}$ the diagram $\Gamma_3^4(p,q,r,s)$ is r-equivalent to a diagram with 2 chords. So, we consider the case $p = q$ and $r = s$. Note that $\Gamma_3^4(++++) \overset{\mathrm{r}}{\sim} \Gamma_3^4(----)$ by the reversing move. Then we obtain the ribbon 2-knots $R_{3,1}^4$, $R_{3,2}^4$, $R_{3,2}^4!$ as listed in Table 3. The group is presented as follows:

$$\pi\Gamma_3^4(p,q,r,s) = G(p+q, r+s). \tag{7.3}$$

7.4. *Ribbon 2-knots obtained from* Γ_4^4

By the move D the diagram $\Gamma_4^4(p,q,r,s)$ is deformed into $\gamma_{632}(p,q,r,s)$ given in Fig. 38. Then if $s = -r$, by the move $\mathrm{A_{III}}$ the diagram $\gamma_{632}(p,q,r,s)$ is deformed into a Gauss diagram with a single separated chord; see Fig. 40. So, we consider the cases $r = s$. Then we obtain the ribbon 2-knots $R^4_{4,j}$, $R^4_{4,j}!$, $j = 1, 2, 3, 4$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_4(p, q, r, s) = G(p, -s, q, r + s). \tag{7.4}$$

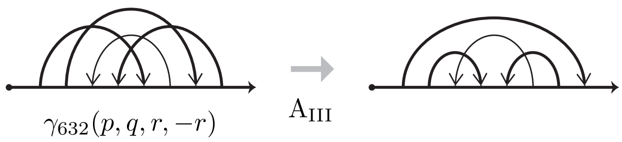


Fig. 40. $\gamma_{632}(p, q, r, -r)$ is r-equivalent to a diagram with a single separated chord.

7.5. *Ribbon 2-knots obtained from* Γ^4_5

By the move Ω_5 the diagram $\Gamma^4_5(p, q, r, s)$ is r-equivalent to $\gamma_{114}(-s, p, q, r)$ given in Fig. 38; see Fig. 41. If $p = s$, then by the move A_{II} the diagram $\gamma_{114}(-s, p, q, r)$ is r-equivalent to a diagram with 2 chords. So, we consider the case $p = -s$. Note that by the reversing move we have $\Gamma^4_5(+---) \overset{r}{\sim} \Gamma^4_5(+++-)$ and $\Gamma^4_5(-+++) \overset{r}{\sim} \Gamma^4_5(---+)$.

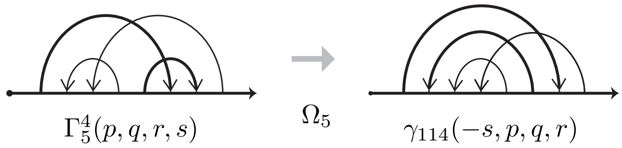


Fig. 41. The diagram $\Gamma^4_6(p, q, r, s)$ is deformed into $\gamma_{114}(-s, p, q, r)$.

Furthermore, we have the following r-equivalent pairs.

Lemma 7.1.

$$\Gamma^4_4(+-++) \overset{r}{\sim} \Gamma^4_5(-+-+); \tag{7.5}$$
$$\Gamma^4_4(-+--) \overset{r}{\sim} \Gamma^4_5(+-+-); \tag{7.6}$$
$$\Gamma^4_4(-+++) \overset{r}{\sim} \Gamma^4_5(---+); \tag{7.7}$$
$$\Gamma^4_4(+---) \overset{r}{\sim} \Gamma^4_5(+++-). \tag{7.8}$$

Proof. Figure 42 shows the diagrams $\Gamma^4_4(p, -p, +, +)$ and $\Gamma^4_5(-, -, p, +)$ are r-equivalent to the same diagram, proving Eqs. (7.5) and (7.7); Eqs. (7.6) and (7.8) are their mirror images. □

Then the remaining diagrams are $\Gamma^4_5(--++)$ and $\Gamma^4_5(++--)$, which present the ribbon 2-knots $R^4_{5,1}$ and $R^4_{5,1}!$, respectively, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_5(p,q,r,s) = G(-r,-q,r,s-p). \tag{7.9}$$

7.6. *Ribbon 2-knots obtained from* $\mathbf{\Gamma^4_6}$

First we show the following lemma.

Lemma 7.2.

$$\Gamma^4_6(p,q,r,-q) \overset{\mathrm{r}}{\sim} \Gamma^4_9(p,r,p,q); \tag{7.10}$$

Proof. See Fig. 43. □

Now we show the following r-equivalence.

Lemma 7.3.

$$\Gamma^4_6(+++-) \overset{\mathrm{r}}{\sim} \Gamma^4_6(---+); \tag{7.11}$$

$$\Gamma^4_6(--+-) \overset{\mathrm{r}}{\sim} \Gamma^4_6(++--) \overset{\mathrm{r}}{\sim} \Gamma^4_6(+-++); \tag{7.12}$$

$$\Gamma^4_6(++-+) \overset{\mathrm{r}}{\sim} \Gamma^4_6(--++) \overset{\mathrm{r}}{\sim} \Gamma^4_6(-+--). \tag{7.13}$$

Proof. First, we show Eq. (7.11). By Lemma 7.2 we have $\Gamma^4_6(+++-) \overset{\mathrm{r}}{\sim} \Gamma^4_9(++++)$ and $\Gamma^4_6(---+) \overset{\mathrm{r}}{\sim} \Gamma^4_9(----)$. Since $\Gamma^4_9(++++) \overset{\mathrm{r}}{\sim} \Gamma^4_9(----)$ by the reversing move, we obtain Eq. (7.11).

Next, we show Eq. (7.13); Eq. (7.12) is its mirror image. We start with the ribbon handlebody H_0 as shown in Fig. 44, which is consisting of four 0-handles and three 1-handles. We deform this into three ribbon handlebodies H_1, H_2, H_3 as shown in Fig. 44 by the stably equivalent move introduced by Marumoto [16, Fig. 2-2], and so these four ribbon handlebodies represent the same ribbon 2-knot. Then H_1, H_2, H_3 are represented by the virtual arcs A_1, A_2, A_3, and their diagrams are δ_1, δ_2, δ_3, respectively, as shown in Fig. 45. Note that $\delta_1 = \gamma_{441}(+-++)$, $\delta_2 = -\gamma_{441}(+--+)$, $\delta_3 = \gamma_{441}(---+)$. Then δ_1 is r-equivalent to $\Gamma^4_6(++-+)$ by the move Ω_8; δ_2 is r-equivalent to $\gamma_{441}(-++-)$ by the reversing move, which is r-equivalent to $\Gamma^4_6(--++)$ by the move Ω_8; δ_3 is r-equivalent to $\Gamma^4_6(-+--)$ by the move Ω_8; see Eq. (6.4). Thus we obtain Eq. (7.13). This completes the proof.□

Therefore, we obtain the ribbon 2-knots $R^4_{6,j}$ $R^4_{6,j}!$, $j = 1, 3, 4, 5, 6$, and $R^4_{6,2}$ as listed in Table 3. For the group $\pi\Gamma^4_6(p,q,r,s)$, see Example 4.3.

Remark 7.4. In [24, Example 4.1] Yasuda has shown the ribbon 2-knot represented by H_0 has three different ribbon handlebodies consisting of two 0-handles and one 1-handle. The ribbon handlebodies H_0 and H_1 in Fig. 44 are given in [24, Fig. 7].

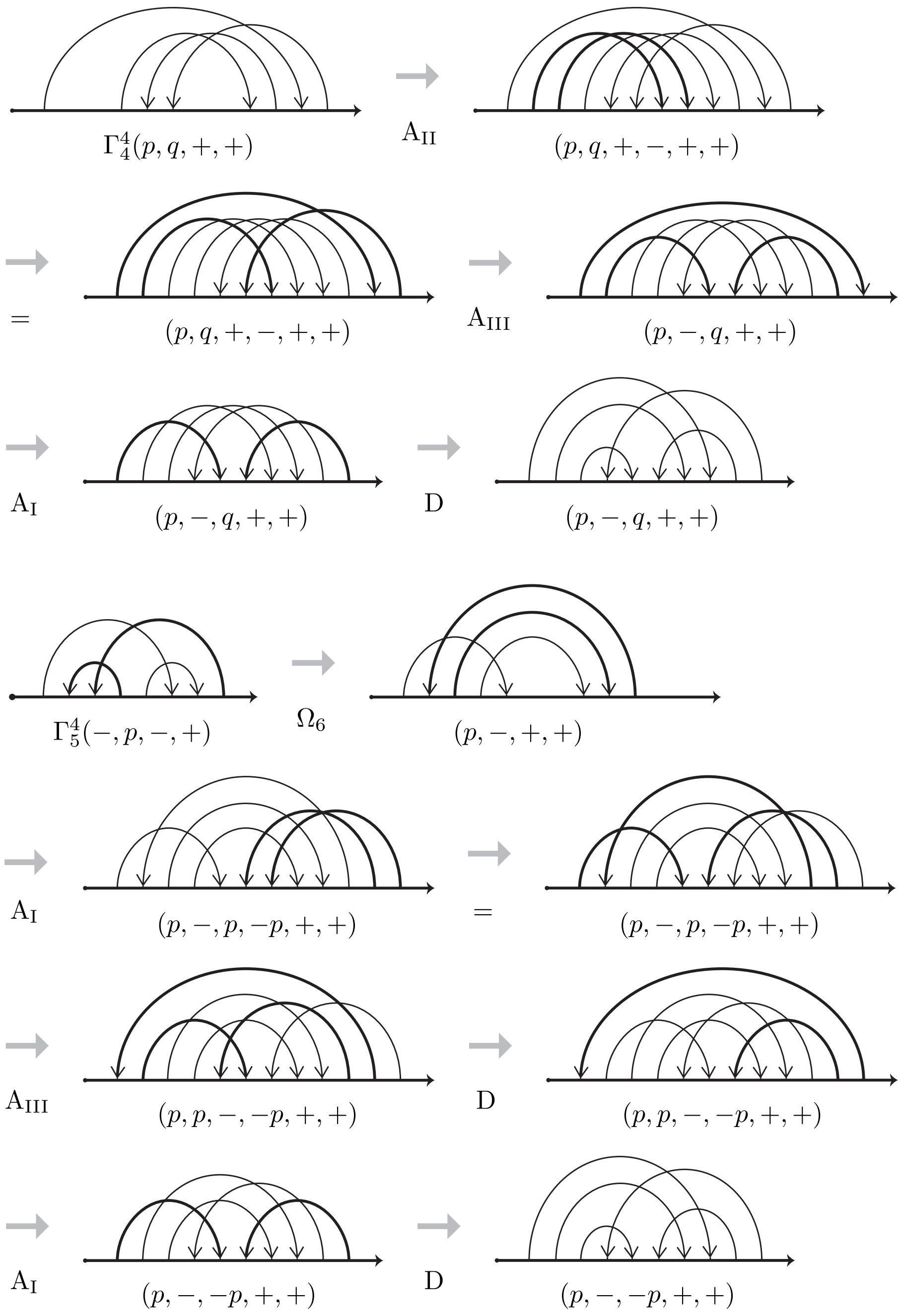


Fig. 42. Proof of Lemma 7.1.

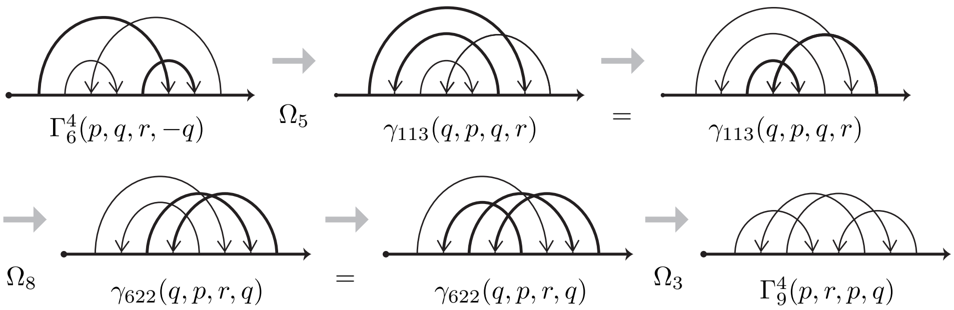


Fig. 43. Proof of Lemma 7.2.

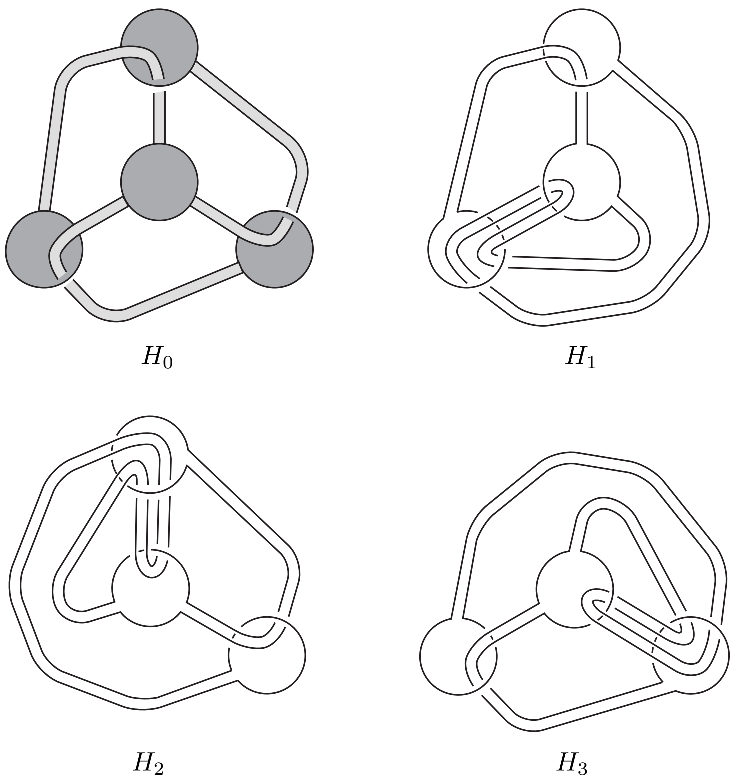


Fig. 44. Ribbon handlebodies.

7.7. *Ribbon 2-knots obtained from* $\mathbf{\Gamma_7^4}$

If $q = -r$, then the diagram $\Gamma_7^4(p,q,r,s)$ is r-equivalent to a diagram with 2 chords by the move $\mathrm{A_{II}}$. So, we consider the case $q = r$. Then we obtain the ribbon 2-knots

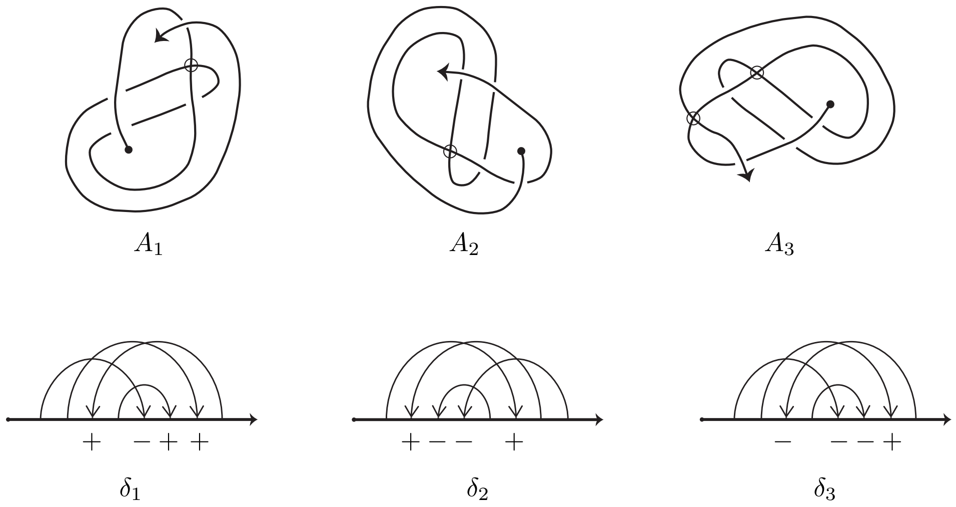


Fig. 45. Virtual arcs A_1, A_2, A_3 representing ribbon handlebodies H_1, H_2, H_3, and their diagrams δ_1, δ_2, δ_3.

$R^4_{7,j}$, $R^4_{7,j}!$, $j = 1, 2, 3, 4$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_7(p,q,r,s) = G(-s,p,-q-r,-p). \tag{7.14}$$

7.8. *Ribbon 2-knots obtained from* $\mathbf{\Gamma^4_8}$

Note that $\Gamma^4_8(p,q,r,s) \overset{\mathrm{r}}{\sim} \Gamma^4_8(-s,-r,-q,-p)$ by the reversing move. Then we obtain the ribbon 2-knots $R^4_{8,1}$, $R^4_{8,6}$, and $R^4_{8,j}$, $R^4_{8,j}!$, $j = 2, 3, 4, 5$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_8(p,q,r,s) = G(p,s,q,-s,-p,r,p,s). \tag{7.15}$$

7.9. *Ribbon 2-knots obtained from* $\mathbf{\Gamma^4_9}$

If either $p = r$ or $q = s$, then $\Gamma^4_9(p,q,r,s)$ is r-equivalent to $\Gamma^4_6(p',q',r',s')$ for some signs p', q', r', s'. In fact, if $p = r$, then by Lemma 7.2 we have $\Gamma^4_9(p,q,p,s) \overset{\mathrm{r}}{\sim} \Gamma^4_7(p,s,q,-s)$. Next, suppose $q = s$. By the reversing move we have $\Gamma^4_9(p,q,r,q) \overset{\mathrm{r}}{\sim} \Gamma^4_9(-q,-r,-q,-p)$, which is r-equivalent to $\Gamma^4_6(-q,-p,-r,p)$ by Lemma 7.2. Then we obtain the ribbon 2-knots $R^4_{9,1}$, $R^4_{9,1}!$, and $R^4_{9,2}$ as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_9(p,q,r,s) = G(p,q,-p,-s,r,s). \tag{7.16}$$

7.10. *Ribbon 2-knots obtained from* $\boldsymbol{\Gamma^4_{10}}$

From the diagram Γ^4_{10} we obtain the ribbon 2-knots $R^4_{10,j}$, $R^4_{10,j}!$, $j = 1, \dots, 8$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_{10}(p, q, r, s) = G(p, q, -p, r, p, -q, -p, s, p, q). \tag{7.17}$$

7.11. *Ribbon 2-knots obtained from* $\boldsymbol{\Gamma^4_{11}}$

We have the following r-equivalent pairs.

Lemma 7.5.

$$\Gamma^4_{11}(++-+) \overset{\mathrm{r}}{\sim} \Gamma^4_{11}(+-+-); \tag{7.18}$$

$$\Gamma^4_{11}(--+-) \overset{\mathrm{r}}{\sim} \Gamma^4_{11}(-+-+). \tag{7.19}$$

Proof. We show Eq. (7.18). The diagram $\Gamma^4_{11}(++-+)$ represents the virtual arc diagram A_4 as shown in Fig. 46, which is r-equivalent to $-A_4^\dagger$. Then it is deformed into A'_4 by the moves $\mathrm{A_{II}}$, $\mathrm{A_{III}}$, D, $\mathrm{E_{II}}$ as shown in Fig. 46, which has the diagram δ_4 as shown in Fig. 48.

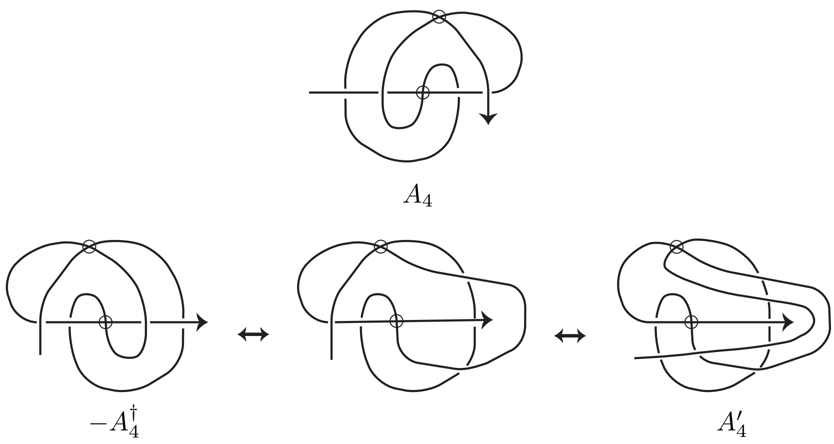


Fig. 46. Virtual arc diagram A_4 having diagram $\Gamma^4_{11}(++-+)$, and r-equivalent virtual arcs.

The diagram $\Gamma^4_{11}(+-+-)$ represents the virtual arc diagram A_5 as shown in Fig. 47. It is deformed into A'_5 by the moves $\mathrm{A_{II}}$, $\mathrm{A_{III}}$, $\mathrm{B_{II}}$, $\mathrm{B_{III}}$, D, $\mathrm{E_{I}}$, $\mathrm{E_{II}}$ as shown in Fig. 47, which has the Gauss diagram δ_5 as shown in Fig. 48. Both the Gauss diagrams γ_4 and γ_5 are deformed into γ_6 as shown in Fig. 48 by the move D, which completes the proof. □

Then we obtain the ribbon 2-knots $R^4_{11,j}$, $R^4_{11,j}!$, $j = 1, \dots, 7$, as listed in Table 3. For the group $\pi\Gamma^4_{11}(p, q, r, s)$, see Example 4.4

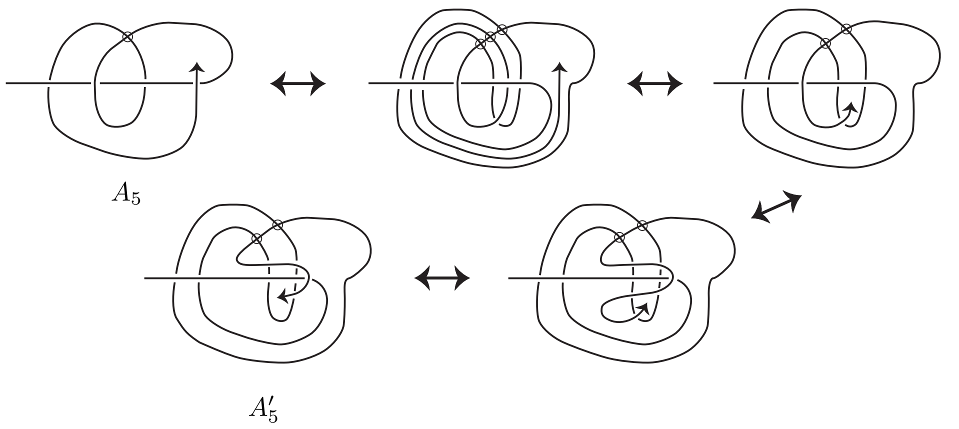


Fig. 47. Virtual arc diagram A_5 having diagram $\Gamma^4_{11}(+-+-)$, and r-equivalent virtual arcs.

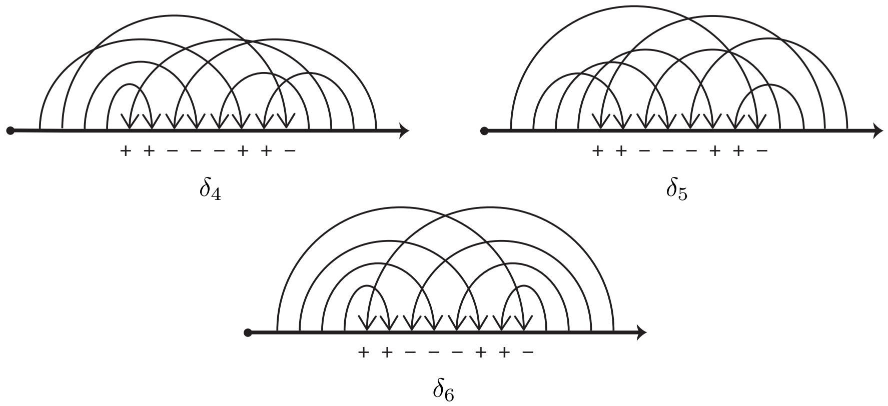


Fig. 48. Gauss linear diagrams for the virtual arcs A'_4 and A'_5.

7.12. *Ribbon 2-knots obtained from* $\mathbf{\Gamma^4_{12}}$

If $p = r$, then the diagram $\Gamma^4_{12}(p, q, r, s)$ is r-equivalent to a linear Gauss diagram with less than 4 chords. In fact, by the move $\mathrm{A_{III}}$ the diagram $\Gamma^4_{12}(p, q, p, s)$ is r-equivalent to a linear Gauss diagram with a separated chord; see Fig. 49.

Moreover, if $s = -q$, then we have $\Gamma^4_{12}(p, q, r, -q) \overset{\mathrm{r}}{\sim} -\Gamma^4_7(p, r, -p, -q)$ as shown in Fig. 50. By the reversing move $-\Gamma^4_7(p, r, -p, -q) \overset{\mathrm{r}}{\sim} \Gamma^4_7(q, p, -r, -p)$. So, we consider $\Gamma^4_{12}(p, q, r, s)$ with $p = -r$ and $q = s$. Then we obtain the ribbon 2-knots $R^4_{12,j}$, $R^4_{12,j}!$, $j = 1, 2$, as listed in Table 3. The group is presented as follows:

$$\pi\Gamma^4_{12}(p, q, r, s) = G(p, q, r - p, s) \tag{7.20}$$

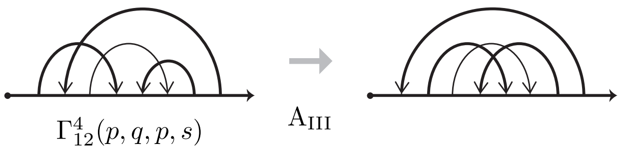


Fig. 49. $\Gamma^4_{12}(p,q,p,s)$ is deformed into a diagram with a single separated chord.

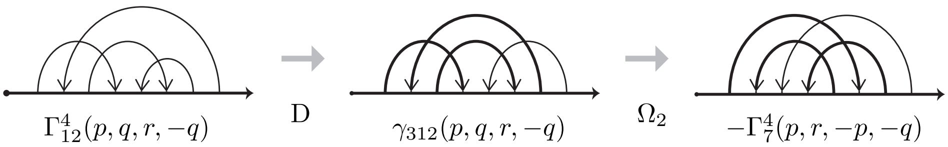


Fig. 50. $\Gamma^4_{12}(p,q,r,-q)$ is r-equivalent to $-\Gamma^4_7(p,r,-p,-q)$.

7.13. *Ribbon 2-knots obtained from* $\boldsymbol{\Gamma^4_{13}}$

Since the diagram $\Gamma^4_{13}(p,q,r,s)$ is the product $\Gamma^2_1(p,q) \cdot \Gamma^2_1(r,s)$ and $\Gamma^2_1(++) \overset{\mathrm{r}}{\sim} \Gamma^2_1(--)$, by Eq. (3.1) we have:

$$\Gamma^4_{13}(++++) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(----) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(++--) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(--++), \tag{7.21}$$

$$\Gamma^4_{13}(+++-) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(+-++) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(--+-) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(+---), \tag{7.22}$$

$$\Gamma^4_{13}(++-+) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(-+++) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(---+) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(-+--), \tag{7.23}$$

$$\Gamma^4_{13}(+--+) \overset{\mathrm{r}}{\sim} \Gamma^4_{13}(-++-), \tag{7.24}$$

which present the composite ribbon 2-knots $R^2_1\#R^2_1$, $R^2_1\#R^2_2$, $R^2_1\#R^2_2!$, $R^2_2\#R^2_2!$, respectively. Additionally, we have $R^2_2\#R^2_2$ and $R^2_2!\#R^2_2!$, which are presented by $\Gamma^4_{13}(+-+-)$ and $\Gamma^4_{13}(-+-+)$, respectively. For the composition of two ribbon 2-knots K and K', $K\#K'$, we have $\Delta_{K\#K'}(t) = \Delta_K(t)\Delta_{K'}(t)$, and so we obtain Table 4.

8. Final Remarks

(i) In Tables 2, 3 and 4 there are several pairs and triples sharing the same Alexander polynomials. Except for the pair $(R^4_{8,1}, R^4_{8,6})$ we can distinguish using the knot group and the twisted Alexander polynomial; see the forthcoming paper [11] for the detail. The groups of this pair are actually isomorphic.

(ii) The following ribbon 2-knots are missing in Yasuda's table [22,23,25,26,27]. In fact, in Yasuda's table there do not exist ribbon knots having the Alexander polynomials of these knots.

$$R^4_{11,4},\ R^4_{11,4}!,\ R^4_{11,5},\ R^4_{11,5}!,\ R^4_{11,6},\ R^4_{11,6}!. \tag{8.1}$$

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