ON A WEIGHTED TRUDINGER-MOSER TYPE INEQUALITY ON THE WHOLE SPACE AND ITS (NON-)EXISTENCE OF MAXIMIZERS

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Abstract. In this paper, we establish a weighted Trudinger-Moser type inequality with the full Sobolev norm constraint on the whole Euclidean space. The radial weight is allowed to increase in the radial direction, therefore we cannot use a rearrangement argument directly. Also we discuss the non-attainability of the supremum related to the inequality when the exponent is sufficiently small.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain with finite volume. Then the Sobolev embedding theorem assures that $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, however, as the function $\log(\log(e/|x|)) \in W_0^{1,N}(B)$, B the unit ball in $\mathbb{R}^N$, shows, the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold. Instead, functions in $W_0^{1,N}(\Omega)$ enjoy the exponential summability:

$$W_0^{1,N}(\Omega) \hookrightarrow \{u \in L^N(\Omega) : \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \quad \text{for any } \alpha > 0\},$$

see Yudovich [31], Pohozaev [26], and Trudinger [30]. Moser [22] improved the above embedding as follows, now known as the Trudinger-Moser inequality: Define

$$TM(N, \Omega, \alpha) = \sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_{L^N(\Omega)} \leq 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) dx.$$

Then we have

$$TM(N, \Omega, \alpha) \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N, \end{cases}$$

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here and henceforth $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} denotes the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. On the attainability of the supremum, Carleson-Chang [5], Flucher [12], and Lin [17] proved that $TM(N,\Omega,\alpha)$ is attained on any bounded domain for all $0 < \alpha \le \alpha_N$.

Later, Adimurthi-Sandeep [2] established a weighted (singular) Trudinger-Moser inequality as follows: Let $0 \le \beta < N$ and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$. Define

$$\widetilde{TM}(N,\Omega,\alpha,\beta) = \sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_{L^N(\Omega)} \le 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}.$$

Then it is proved that

$$\widetilde{TM}(N,\Omega,\alpha,\beta) \begin{cases} < \infty, & \alpha \le \alpha_{N,\beta}, \\ = \infty, & \alpha > \alpha_{N,\beta}. \end{cases}$$

On the attainability of the supremum, recently Csató-Roy [9], [10] proved that $\widetilde{TM}(2,\Omega,\alpha,\beta)$ is attained for $0 < \alpha \le \alpha_{2,\beta} = 2\pi(2-\beta)$ for any bounded domain $\Omega \subset \mathbb{R}^2$. For other types of weighted Trudinger-Moser inequalities, see for example, [6], [7], [8], [13], [18], [28], [29], [32], to name a few.

On domains with infinite volume, for example on the whole space $\mathbb{R}^N$, the Trudinger-Moser inequality does not hold as it is. However, several variants are known on the whole space. In the following, let

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}$$

denote the truncated exponential function.

First, Ogawa [23], Ogawa-Ozawa [24], Cao [4], Ozawa [25], and Adachi-Tanaka [1] proved that the following inequality holds true, which we call Adachi-Tanaka type Trudinger-Moser inequality: Define

$$A(N,\alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1}} \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx. \tag{1.1}$$

Then

$$A(N,\alpha) \begin{cases} < \infty, & \alpha < \alpha_N, \\ = \infty, & \alpha \ge \alpha_N. \end{cases} \tag{1.2}$$

The functional in (1.1)

$$F(u) = \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx$$

enjoys the scale invariance under the scaling $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$, i.e., $F(u_\lambda) = F(u)$ for any $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$. Note that the critical exponent $\alpha = \alpha_N$ is not allowed for the finiteness of the supremum. Recently, Ishiwata-Nakamura-Wadade [16] and Dong-Lu [11] proved that $A(N, \alpha)$ is attained for any $\alpha \in (0, \alpha_N)$. In this sense, Adachi-Tanaka type Trudinger-Moser inequality has a subcritical nature of the problem.

On the other hand, Ruf [27] and Li-Ruf [20] proved that the following inequality holds true: Define

$$B(N, \alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \\ \|u\|_{W^{1,N}(\mathbb{R}^N)} \le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx. \tag{1.3}$$

Then

$$B(N, \alpha) \begin{cases} < \infty, & \alpha \le \alpha_N, \\ = \infty, & \alpha > \alpha_N. \end{cases} \tag{1.4}$$

Here $\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|^N_{L^N(\mathbb{R}^N)} + \|u\|^N_{L^N(\mathbb{R}^N)} \right)^{1/N}$ is the full Sobolev norm. Note that the scale invariance ($u \mapsto u_\lambda$) does not hold for this inequality. Also the critical exponent $\alpha = \alpha_N$ is permitted to the finiteness of (1.3). Concerning the attainability of $B(N, \alpha)$, it is known that $B(N, \alpha)$ is attained for $0 < \alpha \le \alpha_N$ if $N \ge 3$ [27]. On the other hand when $N = 2$, there exists an explicit constant $\alpha_* > 0$ related to the Gagliardo-Nirenberg inequality in $\mathbb{R}^2$ such that $B(2, \alpha)$ is attained for $\alpha_* < \alpha \le \alpha_2 (= 4\pi)$ [27], [15]. However, if $\alpha > 0$ is sufficiently small, then $B(2, \alpha)$ is not attained [15]. The non-attainability of $B(2, \alpha)$ for α sufficiently small is attributed to the non-compactness of "vanishing" maximizing sequences, as described in [15].

Intuitively, the different nature of both inequalities may be explained as follows: For the Adachi-Tanaka type Trudinger-Moser inequality (1.2), the constraint $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$ is weak, thus it holds only for $\alpha < \alpha_N$ and the limiting case $\alpha = \alpha_N$ is excluded. On the other hand, for the Li-Ruf type Trudinger-Moser inequality (1.4), the constraint $\|u\|_{W^{1,N}(\mathbb{R}^N)} \le 1$ is strong, thus it holds even for $\alpha = \alpha_N$. From this point of view, a natural question is what kind of Trudinger-Moser type inequality would hold even for $\alpha = \alpha_N$ under the weaker constraint

$\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$. Obviously, it is necessary to weaken the (exponential) growth of the integrand somehow. Recently, Ibrahim-Masmoudi-Nakanishi [14] and Masmoudi-Sani [21] answered the question as follows: Define

$$C(N,\alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1}} \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}^N} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha |u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}} dx. \tag{1.5}$$

Then

$$C(N,\alpha) \begin{cases} < \infty, & \alpha \le \alpha_N, \\ = \infty, & \alpha > \alpha_N. \end{cases} \tag{1.6}$$

If we replace the functional in (1.5) by

$$\frac{1}{\|u\|_{L^N(\mathbb{R}^N)}^N} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha |u|^{\frac{N}{N-1}})}{(1+|u|)^p} dx$$

for $p < N/(N-1)$, then we easily check that the corresponding supremum is $+\infty$ when $\alpha = \alpha_N$. In this sense, the inequality (1.6) is called as the "exact growth" Trudinger-Moser type inequality. Note that the scale invariance under $u \mapsto u_\lambda$ holds for the inequality. Also it is known that the exact growth Trudinger-Moser inequality (1.6) yields Adachi-Tanaka type and Li-Ruf type Trudinger-Moser inequalities.

In the following, we are interested in the weighted version of the Trudinger-Moser inequalities on the whole space. Let $N \ge 2$, $-\infty < \gamma < N$ and define the weighted Sobolev space

$$\begin{aligned} X_\gamma^{1,N}(\mathbb{R}^N) &= \dot{W}^{1,N}(\mathbb{R}^N) \cap L^N(\mathbb{R}^N, |x|^{-\gamma} dx) \\ &= \{u \in L^1_{loc}(\mathbb{R}^N) \,:\, \|\nabla u\|_{X_\gamma^{1,N}(\mathbb{R}^N)} < \infty\}, \\ \|u\|_{X_\gamma^{1,N}(\mathbb{R}^N)} &= \left(\|\nabla u\|_N^N + \|u\|_{N,\gamma}^N\right)^{1/N}, \text{ here} \\ \|u\|_{N,\gamma} &= \|u\|_{L^N(\mathbb{R}^N;|x|^{-\gamma}dx)} = \left(\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\gamma} dx\right)^{1/N}, \\ \|u\|_N &= \|u\|_{N,0}. \end{aligned}$$

We note that a special form of the Caffarelli-Kohn-Nirenberg inequality in [3]:

$$\|u\|_{N,\beta} \le C \|u\|_{N,\gamma}^{\frac{N-\beta}{N-\gamma}} \|\nabla u\|_N^{1-\frac{N-\beta}{N-\gamma}} \tag{1.7}$$

implies that $X_\gamma^{1,N}(\mathbb{R}^N) \subset X_\beta^{1,N}(\mathbb{R}^N)$ when $\gamma \le \beta$. From now on, we assume

$$N \ge 2, \quad -\infty < \gamma \le \beta < N \tag{1.8}$$

and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$. Recently, Ishiwata-Nakamura-Wadade [16] (in the radial case) and Dong-Lu [11] (in the general case) proved that the following weighted Adachi-Tanaka type Trudinger-Moser inequality holds true: Define

$$(1.9)\quad \tilde{A}(N,\alpha,\beta,\gamma) = \sup_{\substack{u\in X^{1,N}_{\gamma}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1}} \frac{1}{\|u\|_{N,\gamma}^{N(\frac{N-\beta}{N-\gamma})}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}.$$

Then for β,γ satisfying (1.8), we have

$$(1.10)\qquad \tilde{A}(N,\alpha,\beta,\gamma)\begin{cases} <\infty, & \alpha<\alpha_{N,\beta}, \\ =\infty, & \alpha\ge\alpha_{N,\beta}. \end{cases}$$

In particular, if we take $\gamma=\beta$ and put

$$(1.11)\qquad \tilde{A}(N,\alpha,\beta) = \sup_{\substack{u\in X^{1,N}_{\beta}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1}} \frac{1}{\|u\|_{N,\beta}^{N}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta},$$

then we have $\tilde{A}(N,\alpha,\beta)<\infty$ when $\alpha<\alpha_{N,\beta}$, and $\tilde{A}(N,\alpha,\beta)=\infty$ when $\alpha\ge\alpha_{N,\beta}$. Attainability of the best constant (1.9) is also considered in [16] and [11]: $\tilde{A}(N,\alpha,\beta,\gamma)$ is attained for any $0<\alpha<\alpha_{N,\beta}$.

First purpose of this note is to establish the weighted Li-Ruf type Trudinger-Moser inequality on the weighted Sobolev space $X^{1,N}_{\beta}(\mathbb{R}^N)$, where the space dimension N and the weight β satisfies

$$(1.12)\qquad N\ge 2, \quad \text{and} \quad -\infty<\beta<N.$$

Theorem 1. *(Weighted Li-Ruf type inequality) Assume (1.12) and put* $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$. *Define*

$$(1.13)\qquad \tilde{B}(N,\alpha,\beta) = \sup_{\substack{u\in X^{1,N}_{\beta}(\mathbb{R}^N) \\ \|u\|_{X^{1,N}_{\beta}}\le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}.$$

Then

$$(1.14)\qquad \tilde{B}(N,\alpha,\beta)\begin{cases} <\infty, & \alpha\le\alpha_{N,\beta}, \\ =\infty, & \alpha>\alpha_{N,\beta}. \end{cases}$$

Here $\|u\|_{X^{1,N}_{\beta}} = \left(\|\nabla u\|_N^N + \|u\|_{N,\beta}^N\right)^{1/N}$ *is the full Sobolev norm of the space* $X^{1,N}_{\beta}(\mathbb{R}^N)$.

As a former result, de Souza-de O [29] proved that

$$\sup_{\substack{u\in W^{1,N}(\mathbb{R}^N)\\ \|u\|_{W^{1,N}(\mathbb{R}^N)}\le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \begin{cases} <\infty, & \alpha \le \alpha_{N,\beta},\\ =\infty, & \alpha > \alpha_{N,\beta} \end{cases}$$

for $N \ge 2$ and $0 \le \beta < N$. Note that $W^{1,N} = X_0^{1,N} \subset X_\beta^{1,N}$ when $0 < \beta$. In [29], the rearrangement technique is used, and for this reason, the authors in [29] need to assume $\beta \ge 0$ for the weight $\frac{1}{|x|^\beta}$.

In this paper, we cannot use the rearrangement directly since the weight β in (1.12) may be negative. Instead, we use the following inequality to prove Theorem 1.

Theorem 2. *(Weighted exact growth type) Assume (1.8). Then*

$$\tilde{C}(N,\alpha,\beta,\gamma) = \sup_{\substack{u\in X_\gamma^{1,N}\setminus\{0\}\\ \|\nabla u\|_N\le 1}} \frac{1}{\|u\|_{N,\gamma}^{N(\frac{N-\beta}{N-\gamma})}} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}$$
$$\begin{cases} <\infty, & \alpha \le \alpha_{N,\beta},\\ =\infty, & \alpha > \alpha_{N,\beta}. \end{cases}$$

It is easy to see that the weighted exact growth Trudinger-Moser inequality in Theorem 2 yields the weighted Adachi-Tanaka type inequality (1.10). Also Theorem 2 derives the weighted Li-Ruf type Trudinger-Moser inequality Theorem 1, as shown later.

Next, we obtain the relation between the suprema of Adachi-Tanaka type and Li-Ruf type weighted Trudinger-Moser inequalities, along the line of Lam-Lu-Zhang [19]. Set $\tilde{B}(N,\beta) = \tilde{B}(N,\alpha_{N,\beta},\beta)$ in (1.13), i.e.,

$$\tilde{B}(N,\beta) = \sup_{\substack{u\in X_\beta^{1,N}(\mathbb{R}^N)\\ \|u\|_{X_\beta^{1,N}}\le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}. \tag{1.15}$$

Then $\tilde{B}(N,\beta) < \infty$ by Theorem 1.

Theorem 3. *(Relation) Assume (1.12). Then we have*

$$\tilde{B}(N,\beta) = \sup_{\alpha\in(0,\alpha_{N,\beta})} \frac{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \tilde{A}(N,\alpha,\beta).$$

Furthermore, we prove how $\tilde{A}(N,\alpha,\beta)$ behaves as α approaches to $\alpha_{N,\beta}$ from the below:

Theorem 4. *(Asymptotic behavior of $\tilde{A}(N,\alpha,\beta)$) Assume (1.12). Then there exist positive constants C_1, C_2 (depending on N and β) such that for α close enough to $\alpha_{N,\beta}$, the estimate*

$$\frac{C_1}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \leq \tilde{A}(N,\alpha,\beta) \leq \frac{C_2}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}$$

holds.

Note that the estimate from the above follows from Theorem 3. On the other hand, we will see that the estimate from the below follows from a computation using the Moser sequence.

Lastly, we prove the following non-attainability result:

Theorem 5. *(Non-attainability of the best constant) Let $N = 2$, $\beta < 2$ and $\alpha > 0$ is sufficiently small. Then $\tilde{B}(2,\alpha,\beta)$ in (1.13) is not attained.*

According to the results by [27], [20], and [15], we may conjecture that

- When $N \geq 3$, $\tilde{B}(N,\alpha,\beta)$ is attained for $0 < \alpha \leq \alpha_{N,\beta}$.
- When $N = 2$, there exists $\alpha_* > 0$ such that $\tilde{B}(2,\alpha,\beta)$ is attained for $\alpha_* < \alpha \leq \alpha_{2,\beta}$.

But we do not have a proof up to now.

The organization of the paper is as follows: In section 2, first we prove Theorem 2. Main tools are a transformation which eliminates the weights and the (unweighted) exact growth Trudinger-Moser type inequality (1.6). Next, we prove Theorem 1 by using Theorem 2 and an argument by [14], [21]. In section 3, we prove Theorem 3 and Theorem 4. Finally in section 4, we prove Theorem 5. The letter C will denote various positive constant which varies from line to line, but is independent of functions under consideration.

2. Proof of Theorem 1.

In this section, first we prove Theorem 2 and then Theorem 1 by the use of Theorem 2. For the proof of Theorem 2, it is enough to prove its special case:

Proposition 1. *(Special case of the weighted exact growth type) Assume (1.12). Then it holds that*

$$\sup_{\substack{u \in X^{1,N}_{\beta} \setminus \{0\} \\ \|\nabla u\|_N \leq 1}} \frac{1}{\|u\|^N_{N,\beta}} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha |u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}} \frac{dx}{|x|^{\beta}} \begin{cases} < \infty, & \alpha \leq \alpha_{N,\beta}, \\ = \infty, & \alpha > \alpha_{N,\beta}. \end{cases}$$

Once this proposition is established, then the Caffarelli-Korn-Nirenberg inequality (1.7):

$$\|u\|_{N,\beta}^N \le C\|u\|_{N,\gamma}^{N(\frac{N-\beta}{N-\gamma})}\|\nabla u\|_N^{N(1-\frac{N-\beta}{N-\gamma})}$$

with the assumption $\|\nabla u\|_N \le 1$ yields the weighted exact growth Trudinger-Moser inequality in Theorem 2 easily.

Proof of Proposition 1. By abuse of the notation, we write $u(y) = u(s,\omega)$ for $y = s\omega \in \mathbb{R}^N$, $s = |y|$ and $\omega \in S^{N-1}$. Let $\lambda > 0$. We use a change of variables which eliminates the weight

$$\begin{cases} U_\lambda(x) = U_\lambda(r,\omega) = \lambda^{-\frac{N-1}{N}} u(y), \\ x = r\omega \in \mathbb{R}^N, r = |x|, \quad y = s\omega \in \mathbb{R}^N, s = |y|, \\ s = r^\lambda, \quad ds = \lambda r^{\lambda-1} dr. \end{cases}$$

Then by a direct calculation, we see

$$|\frac{\partial}{\partial r} U_\lambda(r,\omega)|^N r^{N-1} dr = |\frac{\partial}{\partial s} u(s,\omega)|^N s^{N-1} ds.$$

Integrating both sides by $\int_{S^{N-1}}\int_0^\infty (\cdots) d(\cdot) dS_\omega$ implies

$$\int_{\mathbb{R}^N} |\nabla U_\lambda(x)|^N dx = \int_{\mathbb{R}^N} |\nabla u(y)|^N dy.$$

On the other hand, we have

$$\int_{\mathbb{R}^N} F(U_\lambda(x)) dx = \lambda^{-1} \int_{\mathbb{R}^N} F\left(\lambda^{-\frac{N-1}{N}} u(y)\right) |y|^{N(1/\lambda - 1)} dy$$

for any $F = F(t) \in C(\mathbb{R})$. In particular, by choosing $F(t) = \frac{\Phi_N(\alpha|t|^{N(N-1)})}{(1+|t|)^{N/(N-1)}}$ for $\alpha > 0$ and $\lambda = \frac{N}{N-\beta} > 0$ so that $N(1/\lambda - 1) = -\beta$, we see

$$\int_{\mathbb{R}^N} \frac{\Phi_N(\alpha|U_\lambda|^{\frac{N}{N-1}})}{(1+|U_\lambda|)^{\frac{N}{N-1}}} dx = \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha(\frac{N-\beta}{N})|u|^{\frac{N}{N-1}})}{((\frac{N}{N-\beta})^{\frac{N-1}{N}} + |u|)^{\frac{N}{N-1}}} \frac{dy}{|y|^\beta}$$

$$\simeq \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha(\frac{N-\beta}{N})|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}} \frac{dy}{|y|^\beta},$$

where $A \simeq B$ means $c_1 B \le A \le c_2 B$ for some $c_1, c_2 > 0$. Similarly, we have

$$\int_{\mathbb{R}^N} |U_\lambda(x)|^N dx = \left(\frac{N-\beta}{N}\right)^N \int_{\mathbb{R}^N} |u(y)|^N \frac{dy}{|y|^\beta}$$

and thus $u \in X_\beta^{1,N}$ implies that $U_\lambda \in W^{1,N}(\mathbb{R}^N)$. Therefore, we may apply the unweighted exact growth Trudinger-Moser inequality (1.6) by [14], [21] to $U_\lambda \in W^{1,N}(\mathbb{R}^N)$, which results in Proposition 1. □

Proof of Theorem 1:

Here we follow the argument by Masmoudi and Sani (see [21] Section 6). Assume $N \geq 2$, $-\infty < \beta < N$. We will prove that there exists $C > 0$ such that for any $u \in X^{1,N}_\beta$ with $\|u\|_{X^{1,N}_\beta} = \left(\|\nabla u\|_N^N + \|u\|_{N,\beta}^N\right)^{1/N} \leq 1$, it holds

$$\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \leq C. \tag{2.1}$$

We take $\theta \in (0,1)$ such that $\|u\|_{N,\beta}^N = \theta$ and $\|\nabla u\|_N^N \leq 1-\theta$. We divide the proof into two cases:

Case 1: $\theta \geq \frac{N-1}{N}$.

In this case, we put $\tilde{u} = N^{1/N}u$. Then

$$\|\tilde{u}\|_{N,\beta}^N = N\theta, \quad \|\nabla \tilde{u}\|_N^N \leq N(1-\theta) \leq 1$$

since $\theta \geq \frac{N-1}{N}$. Take $\alpha \in (0, \alpha_{N,\beta})$ so that $\alpha N^{1/(N-1)} = \alpha_{N,\beta}$ and apply the weighted Adachi-Tanaka type Trudinger-Moser inequality (1.10) with $\beta = \gamma$ to $\tilde{u} \in X^{1,N}_\beta$. Then we have $C > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_N(\alpha|\tilde{u}|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \leq C \int_{\mathbb{R}^N} \frac{|\tilde{u}|^N}{|x|^\beta} dx \leq CN\theta.$$

Since the left hand side coincides with

$$\int_{\mathbb{R}^N} \Phi_N(\alpha N^{\frac{1}{N-1}}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} = \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta},$$

thus we have (2.1) for some $C > 0$.

Case 2: $\theta < \frac{N-1}{N}$. Put

$$A = \{x \in \mathbb{R}^N : |u(x)| \geq 1\}$$

First, we derive

$$\int_{\mathbb{R}^N \setminus A} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \leq C \tag{2.2}$$

for some $C > 0$. Since $|u| < 1$ on $\mathbb{R}^N \setminus A$, and $\Phi_N(t) \leq C_N t^{N-1}$ for some $C_N > 0$ for all $t \in [0, \alpha_{N,\beta}]$, we have

$$\int_{\mathbb{R}^N \setminus A} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \leq C_N \alpha_{N,\beta}^{N-1} \int_{\mathbb{R}^N \setminus A} \frac{|u|^N}{|x|^\beta} dx \leq C\|u\|_{N,\beta}^N \leq C.$$

Next, we prove

$$\int_A \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \leq C. \tag{2.3}$$

By a direct calculation, we observe that $\Phi_N(t)^p \le \Phi_N(pt)$ for $p \ge 1$; see also [16] Lemma A.2. Thus by Hölder's inequality,

$$
\begin{aligned}
&\int_A \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&\le \left(\int_A \frac{\Phi_N(p\alpha_{N,\beta}|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}\right)^{1/p}\left(\int_A \frac{(1+|u|)^{\frac{N}{(N-1)(p-1)}}}{|x|^\beta}dx\right)^{(p-1)/p} \\
&\le 2^{N/(N-1)}\left(\int_A \frac{\Phi_N(p\alpha_{N,\beta}|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}\right)^{1/p}\left(\int_A \frac{|u|^{\frac{N}{(N-1)(p-1)}}}{|x|^\beta}dx\right)^{(p-1)/p} \\
&\le 2^{N/(N-1)}\left(\int_A \frac{\Phi_N(p\alpha_{N,\beta}|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}\right)^{1/p}\||u|^{\frac{N}{N-1}}\|^{1/p}_{\frac{1}{p-1},\beta}.
\end{aligned}
\tag{2.4}
$$

Now, take $p = \frac{N-1}{N-1-\theta} > 1$. Note that $\frac{1}{p-1} > N-1$ holds in this case. Then by expanding the exponential function into the power series in the weighted Adachi-Tanaka inequality (1.10) (with $\gamma = \beta$) and by the Stirling formula, we have $C > 0$ such that

$$\||u|^{\frac{N}{N-1}}\|_{q,\beta} \le Cq\|u\|^{N/q}_{N,\beta}$$

for any $u \in X^{1,N}_\beta$ with $\|\nabla u\|_N \le 1$ and for any $q \ge N-1$. Thus in particular, putting $q = \frac{1}{p-1}$, we have

$$\||u|^{\frac{N}{N-1}}\|^{1/p}_{\frac{1}{p-1},\beta} \le C\left(\frac{1}{p-1}\right)^{1/p}\|u\|^{N(p-1)/p}_{N,\beta}. \tag{2.5}$$

On the other hand, if we put $\tilde{u} = p^{\frac{N-1}{N}}u$, then we see $\|\nabla \tilde{u}\|^N_N = p^{N-1}\|\nabla u\|^N_N \le 1$. Now, applying the weighted exact growth Trudinger-Moser inequality in Theorem 2 to $\tilde{u}$, we have

$$
\begin{aligned}
\left(\int_A \frac{\Phi_N(p\alpha_{N,\beta}|u|^{\frac{N}{N-1}})}{(1+|u|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}\right)^{1/p} &\le \left(p\int_A \frac{\Phi_N(\alpha_{N,\beta}|\tilde{u}|^{\frac{N}{N-1}})}{(1+|\tilde{u}|)^{\frac{N}{N-1}}}\frac{dx}{|x|^\beta}\right)^{1/p} \\
&\le C\left(p\|\tilde{u}\|^N_{N,\beta}\right)^{1/p} = Cp^{N/p}\|u\|^{N/p}_{N,\beta}.
\end{aligned}
\tag{2.6}
$$

Thus backing to (2.4) with (2.5) and (2.6), we see

$$\begin{aligned}\int_A \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} &\le C\left(\frac{1}{p-1}\right)^{1/p} p^{N/p}\|u\|_{N,\beta}^{N(\frac{p-1}{p})+\frac{N}{p}} \\ &= C\left(\frac{p}{p-1}\right)^{1/p} p^{(N-1)/p}\|u\|_{N,\beta}^{N} \\ &= C\left(\frac{N-1}{\theta}\right)^{\frac{N-1-\theta}{N-1}}\left(\frac{N-1}{N-1-\theta}\right)^{N-1-\theta}\theta \\ &\le C(N-1)\theta^{\frac{\theta}{N-1}}\frac{1}{(1-\frac{\theta}{N-1})^{N-1-\theta}}.\end{aligned}$$

Now, we see $\theta^{\frac{\theta}{N-1}} < 1$ and $\left(1-\frac{\theta}{N-1}\right)^{N-1-\theta} \ge (1-\frac{1}{N})^{N-1}$ for $0<\theta<(N-1)/N$. Hence the last expression is bounded by a constant which depends only on N and (2.3) is proved. By (2.2) and (2.3), we have (2.1) so the first part of Theorem 1 is obtained.

For the proof of $B(N,\alpha,\beta)=\infty$ when $\alpha>\alpha_{N,\beta}$, we use the weighted Moser sequence as in [16], [19]: Let $-\infty<\gamma\le\beta<N$ and for $n\in\mathbb{N}$ set

$$A_n = \left(\frac{1}{\omega_{N-1}}\right)^{1/N}\left(\frac{n}{N-\beta}\right)^{-1/N}, \quad b_n = \frac{n}{N-\beta},$$

so that $(A_n b_n)^{\frac{N}{N-1}} = n/\alpha_{N,\beta}$. Put

$$u_n = \begin{cases} A_n b_n, & \text{if } |x| < e^{-b_n}, \\ A_n \log(1/|x|), & \text{if } e^{-b_n} < |x| < 1, \\ 0, & \text{if } 1 \le |x|. \end{cases} \tag{2.7}$$

Then direct calculation shows that

$$\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1, \tag{2.8}$$

$$\|u_n\|_{N,\gamma}^N = \frac{N-\beta}{(N-\gamma)^{N+1}}\Gamma(N+1)(1/n) + o(1/n) \tag{2.9}$$

as $n\to\infty$. Note $u_n \in X_\gamma^{1,N}(\mathbb{R}^N)$. In fact for (2.9), we compute

$$\begin{aligned}\|u_n\|_{N,\gamma}^N &= \omega_{N-1}\int_0^{e^{-b_n}}(A_n b_n)^N r^{N-1-\gamma}dr + \omega_{N-1}\int_{e^{-b_n}}^1 A_n^N(\log(1/r))^N r^{N-1-\gamma}dr \\ &= I + II.\end{aligned}$$

We see

$$I = \omega_{N-1}(A_n b_n)^N\left[\frac{r^{N-\gamma}}{N-\gamma}\right]_{r=0}^{r=e^{-b_n}} = \omega_{N-1}\left(\frac{n}{\alpha_{N,\beta}}\right)^{N-1}\frac{e^{-(\frac{N-\gamma}{N-\beta})n}}{N-\gamma} = o(1/n)$$

as $n \to \infty$. Also

$$
\begin{aligned}
II &= \left(\frac{N-\beta}{n}\right) \int_{e^{-b_n}}^{1} (\log(1/r))^N r^{N-1-\gamma} dr \\
&= \left(\frac{N-\beta}{n}\right) \int_0^{b_n} \rho^N e^{-(N-\gamma)\rho} d\rho = \frac{N-\beta}{(N-\gamma)^{N+1}} (1/n) \int_0^{(N-\gamma)b_n} \rho^N e^{-\rho} d\rho \\
&= \frac{N-\beta}{(N-\gamma)^{N+1}} (1/n)\Gamma(N+1) + o(1/n).
\end{aligned}
$$

Thus we obtain (2.9).

Now, put $v_n(x) = \lambda_n u_n(x)$ where u_n is the weighted Moser sequence in (2.7) and $\lambda_n > 0$ is chosen so that $\lambda_n^N + \lambda_n^N \|u_n\|_{N,\beta}^N = 1$. Thus we have $\|\nabla v_n\|_{L^N}^N + \|v_n\|_{N,\beta}^N = 1$ for any $n \in \mathbb{N}$. By (2.9) with $\beta = \gamma$, we see that $\lambda_n^N = 1 - O(1/n)$ as $n \to \infty$. For $\alpha > \alpha_{N,\beta}$, we calculate

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Phi_N(\alpha |v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \ge \int_{\{0 \le |x| \le e^{-b_n}\}} \Phi_N(\alpha |v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \\
&= \int_{\{0 \le |x| \le e^{-b_n}\}} \left(e^{\alpha |v_n|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!} |v_n|^{\frac{Nj}{N-1}} \right) \frac{dx}{|x|^\beta} \\
&\ge \left\{ \exp\left(\frac{n\alpha}{\alpha_{N,\beta}} \lambda_n^{\frac{N}{N-1}} \right) - O(n^{N-1}) \right\} \int_{\{0 \le |x| \le e^{-b_n}\}} \frac{dx}{|x|^\beta} \\
&\ge \left\{ \exp\left(\frac{n\alpha}{\alpha_{N,\beta}} \left(1 - O\left(\frac{1}{n^{\frac{1}{N-1}}} \right) \right) \right) - O(n^{N-1}) \right\} \left(\frac{\omega_{N-1}}{N-\beta} \right) e^{-n} \to +\infty
\end{aligned}
$$

as $n \to \infty$. Here we have used that for $0 \le |x| \le e^{-b_n}$,

$$
\alpha |v_n|^{\frac{N}{N-1}} = \alpha \lambda_n^{\frac{N}{N-1}} (A_n b_n)^{\frac{N}{N-1}} = \frac{n\alpha}{\alpha_{N,\beta}} \lambda_n^{\frac{N}{N-1}}
$$

by definition of A_n and b_n. Also we used that for $0 \le |x| \le e^{-b_n}$,

$$
|v_n|^{\frac{Nj}{N-1}} = \lambda_n^{\frac{Nj}{N-1}} (A_n b_n)^{\frac{Nj}{N-1}} \le C n^j \le C n^{N-1}
$$

for $0 \le j \le N-2$ and n is large. This proves Theorem 1 completely. □

3. Proof of Theorem 3 and 4.

In this section, we prove Theorem 3 and Theorem 4. As stated in the Introduction, we follow the argument by Lam-Lu-Zhang [19]. First, we prepare several lemmata.

Lemma 1. *Assume (1.8) and set*

$$\widehat{A}(N,\alpha,\beta) = \sup_{\substack{u\in X^{1,N}_\beta(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1\\ \|u\|_{N,\beta}=1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}. \tag{3.1}$$

Let $\tilde{A}(N,\alpha,\beta)$ be defined as in (1.11). Then $\tilde{A}(N,\alpha,\beta) = \widehat{A}(N,\alpha,\beta)$ for any $\alpha > 0$.

Proof. For any $u \in X^{1,N}_\beta(\mathbb{R}^N) \setminus \{0\}$ and $\lambda > 0$, we put $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}^N$. Then it is easy to see that

$$\begin{cases} \|\nabla u_\lambda\|^N_{L^N(\mathbb{R}^N)} = \|\nabla u\|^N_{L^N(\mathbb{R}^N)}, \\ \|u_\lambda\|^N_{N,\beta} = \lambda^{-(N-\beta)}\|u\|^N_{N,\beta}. \end{cases} \tag{3.2}$$

Thus for any $u \in X^{1,N}_\beta(\mathbb{R}^N) \setminus \{0\}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$, if we choose $\lambda = \|u\|^{N/(N-\beta)}_{N,\beta}$, then $u_\lambda \in X^{1,N}_\beta(\mathbb{R}^N)$ satisfies

$$\|\nabla u_\lambda\|_{L^N(\mathbb{R}^N)} \le 1 \quad \text{and} \quad \|u_\lambda\|^N_{N,\beta} = 1.$$

Thus

$$\widehat{A}(N,\alpha,\beta) \ge \int_{\mathbb{R}^N} \Phi_N(\alpha|u_\lambda|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} = \frac{1}{\|u\|^N_{N,\beta}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}$$

which implies $\widehat{A}(N,\alpha,\beta) \ge \tilde{A}(N,\alpha,\beta)$. The opposite inequality is trivial. □

Lemma 2. *Assume (1.8) and set $\tilde{B}(N,\beta)$ as in (1.15). Then we have*

$$\tilde{A}(N,\alpha,\beta) \le \frac{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}\tilde{B}(N,\beta)$$

for any $0 < \alpha < \alpha_{N,\beta}$.

Proof. Choose any $u \in X^{1,N}_\beta$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$ and $\|u\|_{N,\beta} = 1$. Put $v(x) = Cu(\lambda x)$ where $C \in (0,1)$ and $\lambda > 0$ are defined as

$$C = \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{\frac{N-1}{N}} \quad \text{and} \quad \lambda = \left(\frac{C^N}{1-C^N}\right)^{1/(N-\beta)}.$$

Then by scaling rules (3.2), we see

$$\begin{aligned} \|v\|^N_{X^{1,N}_\beta} &= \|\nabla v\|^N_N + \|v\|^N_{N,\beta} = C^N\|\nabla u\|^N_N + \lambda^{-(N-\beta)}C^N\|u\|^N_{N,\beta} \\ &\le C^N + \lambda^{-(N-\beta)}C^N = 1. \end{aligned}$$

Also we have

$$\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|v|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} = \lambda^{-(N-\beta)} \int_{\mathbb{R}^N} \Phi_N\left(\alpha_{N,\beta} C^{\frac{N}{N-1}} |u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}$$
$$= \lambda^{-(N-\beta)} \int_{\mathbb{R}^N} \Phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}.$$

Thus testing $\tilde{B}(N,\beta)$ by v, we see

$$\tilde{B}(N,\beta) \geq \left(\frac{1-C^N}{C^N}\right) \int_{\mathbb{R}^N} \Phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}.$$

By taking the supremum for $u \in X^{1,N}_\beta$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1$ and $\|u\|_{N,\beta} = 1$, we have

$$\tilde{B}(N,\beta) \geq \frac{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \widehat{A}(N,\alpha,\beta).$$

Finally, Lemma 1 implies the result. □

Proof of Theorem 3: The assertion that

$$\tilde{B}(N,\beta) \geq \sup_{\alpha \in (0,\alpha_{N,\beta})} \frac{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \tilde{A}(N,\alpha,\beta)$$

follows from Lemma 2. Note that $\tilde{B}(N,\beta) < \infty$ by Theorem 1.

Let us prove the opposite inequality. Let $\{u_n\} \subset X^{1,N}_\beta(\mathbb{R}^N)$, $u_n \neq 0$, $\|\nabla u_n\|^N_{L^N} + \|u_n\|^N_{N,\beta} \leq 1$, be a maximizing sequence of $\tilde{B}(N,\beta)$:

$$\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} = \tilde{B}(N,\beta) + o(1)$$

as $n \to \infty$. We may assume $\|\nabla u_n\|^N_{L^N(\mathbb{R}^N)} < 1$ for any $n \in \mathbb{N}$. Define

$$\begin{cases} v_n(x) = \frac{u_n(\lambda_n x)}{\|\nabla u_n\|_N}, & (x \in \mathbb{R}^N) \\ \lambda_n = \left(\frac{1-\|\nabla u_n\|^N_N}{\|\nabla u_n\|^N_N}\right)^{1/(N-\beta)} > 0. \end{cases}$$

Thus by (3.2), we see

$$\|\nabla v_n\|^N_{L^N(\mathbb{R}^N)} = 1,$$
$$\|v_n\|^N_{N,\beta} = \frac{\lambda_n^{-(N-\beta)}}{\|\nabla u_n\|^N_N} \|u_n\|^N_{N,\beta} = \frac{\|u_n\|^N_{N,\beta}}{1 - \|\nabla u_n\|^N_N} \leq 1,$$

since $\|\nabla u_n\|_N^N + \|u_n\|_{N,\beta}^N \le 1$. Thus, setting

$$\alpha_n = \alpha_{N,\beta}\|\nabla u_n\|_N^{\frac{N}{N-1}} < \alpha_{N,\beta}$$

for any $n \in \mathbb{N}$, we may test $\tilde{A}(N, \alpha_n, \beta)$ by $\{v_n\}$, which results in

$$\begin{aligned}
\tilde{B}(N,\beta) + o(1) &= \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u_n(y)|^{\frac{N}{N-1}})\frac{dy}{|y|^\beta} \\
&= \lambda_n^{N-\beta}\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}\|\nabla u_n\|_N^{\frac{N}{N-1}}|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&= \lambda_n^{N-\beta}\int_{\mathbb{R}^N} \Phi_N(\alpha_n|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&\le \lambda_n^{N-\beta}\frac{1}{\|v_n\|_{N,\beta}^N}\int_{\mathbb{R}^N} \Phi_N(\alpha_n|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&\le \lambda_n^{N-\beta}\tilde{A}(N,\alpha_n,\beta) = \left(\frac{1-\|\nabla u_n\|_N^N}{\|\nabla u_n\|_N^N}\right)\tilde{A}(N,\alpha_n,\beta) \\
&= \frac{1-\left(\frac{\alpha_n}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha_n}{\alpha_{N,\beta}}\right)^{N-1}}\tilde{A}(N,\alpha_n,\beta) \\
&\le \sup_{\alpha\in(0,\alpha_{N,\beta})}\frac{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}\tilde{A}(N,\alpha,\beta).
\end{aligned}$$

Here we have used a change of variables $y = \lambda_n x$ for the second equality, and $\|v_n\|_{N,\beta}^N \le 1$ for the first inequality. Letting $n \to \infty$, we have the desired result. □

Proof of Theorem 4: The assertion that

$$\tilde{A}(N,\alpha,\beta) \le \frac{C_2}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}$$

follows form Theorem 3 and the fact that $\tilde{B}(N,\beta) < \infty$.

For the rest, we need to prove that there exists $C > 0$ such that for any $\alpha < \alpha_{N,\beta}$ sufficiently close to $\alpha_{N,\beta}$, it holds that

$$\frac{C}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \le \tilde{A}(N,\alpha,\beta). \tag{3.3}$$

For that purpose, we use the weighted Moser sequence (2.7) again. By (2.9) with $\gamma = \beta$, we have $N_1 \in \mathbb{N}$ such that if $n \in \mathbb{N}$ satisfies $n \ge N_1$, then it holds

$$\|u_n\|_{N,\beta}^N \le \frac{2\Gamma(N+1)}{(N-\beta)^N}(1/n). \tag{3.4}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N} \Phi_N(\alpha |u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} &\ge \omega_{N-1} \int_0^{e^{-b_n}} \Phi_N\left(\alpha (A_n b_n)^{N/(N-1)}\right) r^{N-1-\beta} dr \\ &= \frac{\omega_{N-1}}{N-\beta} \Phi_N\left((\alpha/\alpha_{N,\beta})n\right) \left[r^{N-\beta}\right]_{r=0}^{r=e^{-b_n}} \\ &= \frac{\omega_{N-1}}{N-\beta} \Phi_N\left((\alpha/\alpha_{N,\beta})n\right) e^{-n}.\end{aligned}$$

Note that there exists $N_2 \in \mathbb{N}$ such that if $n \ge N_2$ then $\Phi_N\left((\alpha/\alpha_{N,\beta})n\right) \ge \frac{1}{2} e^{(\alpha/\alpha_{N,\beta})n}$. Thus we have

$$\int_{\mathbb{R}^N} \Phi_N(\alpha |u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \ge \frac{1}{2}\left(\frac{\omega_{N-1}}{N-\beta}\right) e^{-(1-\frac{\alpha}{\alpha_{N,\beta}})n}. \tag{3.5}$$

Combining (3.4) and (3.5), we have $C_1(N,\beta) > 0$ such that

$$\frac{1}{\|u_n\|_{N,\beta}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha |u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \ge C_1(N,\beta) n e^{-(1-\frac{\alpha}{\alpha_{N,\beta}})n} \tag{3.6}$$

holds when $n \ge \max\{N_1, N_2\}$.

Note that $\lim_{x\to 1}\left(\frac{1-x^{N-1}}{1-x}\right) = N-1$, thus

$$\frac{1-(\alpha/\alpha_{N,\beta})^{N-1}}{1-(\alpha/\alpha_{N,\beta})} \ge \frac{N-1}{2}$$

if $\alpha/\alpha_{N,\beta} < 1$ is very close to 1. Now, for any $\alpha > 0$ sufficiently close to $\alpha_{N,\beta}$ so that

$$\begin{cases} \max\{N_1, N_2\} < \left(\frac{2}{1-\alpha/\alpha_{N,\beta}}\right), \\ \frac{1-(\alpha/\alpha_{N,\beta})^{N-1}}{1-(\alpha/\alpha_{N,\beta})} \ge \frac{N-1}{2}, \end{cases} \tag{3.7}$$

we can find $n \in \mathbb{N}$ such that

$$\begin{cases} \max\{N_1, N_2\} \le n \le \left(\frac{2}{1-\alpha/\alpha_{N,\beta}}\right), \\ \left(\frac{1}{1-\alpha/\alpha_{N,\beta}}\right) \le n. \end{cases} \tag{3.8}$$

We fix $n \in \mathbb{N}$ satisfying (3.8). Then by $1 \le n(1-\alpha/\alpha_{N,\beta}) \le 2$, (3.6) and (3.7), we have

$$\begin{aligned}
&\frac{1}{\|u_n\|_{N,\beta}^N}\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)})\frac{dx}{|x|^\beta} \ge C_1(N,\beta)ne^{-2} \\
&\ge C_2(N,\beta)\frac{1}{1-(\alpha/\alpha_{N,\beta})} \ge \frac{N-1}{2}C_2(N,\beta)\frac{1}{1-(\alpha/\alpha_{N,\beta})^{N-1}} \\
&= C_3(N,\beta)\frac{1}{1-(\alpha/\alpha_{N,\beta})^{N-1}},
\end{aligned}$$

where $C_2(N,\beta) = e^{-2}C_1(N,\beta)$ and $C_3(N,\beta) = \frac{N-1}{2}C_2(N,\beta)$. Thus we have (3.3) for some $C > 0$ independent of α which is sufficiently close to $\alpha_{N,\beta}$. □

4. Proof of Theorem 5.

In this section, we prove Theorem 5. We follow Ishiwata's argument in [15].

Assume $-\infty < \beta < 2$ and $0 < \alpha \le \alpha_{2,\beta} = 2\pi(2-\beta)$ and define

$$\tilde{B}(2,\alpha,\beta) = \sup_{\substack{u \in X_\beta^{1,2}(\mathbb{R}^2) \\ \|u\|_{X_\beta^{1,2}(\mathbb{R}^2)} \le 1}} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)\frac{dx}{|x|^\beta}.$$

We will show that $\tilde{B}(2,\alpha,\beta)$ is not attained if $\alpha > 0$ sufficiently small. Set

$$M = \left\{u \in X_\beta^{1,2}(\mathbb{R}^2) \,:\, \|u\|_{X_\beta^{1,2}} = \left(\|\nabla u\|_2^2 + \|u\|_{2,\beta}^2\right)^{1/2} = 1\right\}$$

be the unit sphere in the Hilbert space $X_\beta^{1,2}(\mathbb{R}^2)$ and

$$J_\alpha : M \to \mathbb{R}, \quad J_\alpha(u) = \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)\frac{dx}{|x|^\beta}$$

be the corresponding functional defined on M. Actually, we will prove the stronger claim that J_α has no critical point on M when $\alpha > 0$ is sufficiently small.

Assume the contrary that there existed $v \in M$ such that v is a critical point of J_α on M. Define an orbit on M through v as

$$v_\tau(x) = \sqrt{\tau}v(\sqrt{\tau}x) \quad \tau \in (0,\infty), \quad w_\tau = \frac{v_\tau}{\|v_\tau\|_{X_\beta^{1,2}}} \in M.$$

Since $w_\tau|_{\tau=1} = v$, we must have

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) = 0. \tag{4.1}$$

Note that

$$\|\nabla v_\tau\|_{L^2(\mathbb{R}^2)}^2 = \tau \|\nabla v\|_{L^2(\mathbb{R}^2)}^2, \quad \|v_\tau\|_{p,\beta}^p = \tau^{\frac{p+\beta-2}{2}} \|v\|_{p,\beta}^p$$

for $p > 1$. Thus,

$$J_\alpha(w_\tau) = \int_{\mathbb{R}^2} \left(e^{\alpha w_\tau^2} - 1\right) \frac{dx}{|x|^\beta} = \int_{\mathbb{R}^2} \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{v_\tau^{2j}(x)}{\|v_\tau\|_{X_\beta^{1,2}}^{2j}} \frac{dx}{|x|^\beta}$$

$$= \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{\|v_\tau\|_{2j,\beta}^{2j}}{\left(\|\nabla v_\tau\|_2^2 + \|v_\tau\|_{2,\beta}^2\right)^j} = \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{\tau^{j-1+\frac{\beta}{2}} \|v\|_{2j,\beta}^{2j}}{\left(\tau \|\nabla v\|_2^2 + \tau^{\frac{\beta}{2}} \|v\|_{2,\beta}^2\right)^j}.$$

By using an elementary computation

$$f(\tau) = \frac{\tau^{j-1+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^j}, \quad a = \|\nabla v\|_2^2,\ b = \|v\|_{2,\beta}^2,\ c = \|v\|_{2j,\beta}^{2j},$$

$$f'(\tau) = (1 - \frac{\beta}{2}) \frac{\tau^{j-2+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^{j+1}} \left\{-\tau a + (j-1) b\right\},$$

we estimate $\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau)$:

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau)$$

$$= \sum_{j=1}^\infty \left[\frac{\alpha^j}{j!}(1-\frac{\beta}{2}) \frac{\tau^{j-2+\beta/2} \|v\|_{2j,\beta}^{2j}}{\left(\tau \|\nabla v\|_2^2 + \tau^{\beta/2} \|v\|_{2,\beta}^2\right)^{j+1}} \left\{-\tau \|\nabla v\|_2^2 + (j-1)\|v\|_{2,\beta}^2\right\}\right]_{\tau=1}$$

$$= -\alpha(1-\frac{\beta}{2}) \|\nabla v\|_2^2 \|v\|_{2,\beta}^2 + \sum_{j=2}^\infty \frac{\alpha^j}{j!}(1-\frac{\beta}{2}) \|v\|_{2j,\beta}^{2j} \left\{-\|\nabla v\|_2^2 + (j-1)\|v\|_{2,\beta}^2\right\}$$

(4.2)

$$\le \alpha(1-\frac{\beta}{2}) \|\nabla v\|_2^2 \|v\|_{2,\beta}^2 \left\{-1 + \sum_{j=2}^\infty \frac{\alpha^{j-1}}{(j-1)!} \frac{\|v\|_{2j,\beta}^{2j}}{\|\nabla v\|_2^2 \|v\|_{2,\beta}^2}\right\},$$

since $-\|\nabla v\|_2^2 + (j-1)\|v\|_{2,\beta}^2 \le j$.

Now, we state a lemma. Unweighted version of the next lemma is proved in [15]:Lemma 3.1, and the proof of the next is a simple modification of the one given there using the weighted Adachi-Tanaka type Trudinger-Moser inequality (1.10) (with $\gamma = \beta$) and the expansion of the exponential function.

Lemma 3. *For any $\alpha \in (0, \alpha_{2,\beta})$, there exists $C_\alpha > 0$ such that*

$$\|u\|_{2j,\beta}^{2j} \le C_\alpha \frac{j!}{\alpha^j} \|\nabla u\|_2^{2j-2} \|u\|_{2,\beta}^2$$

holds for any $u \in X^{1,2}_\beta(\mathbb{R}^2)$ *and* $j \in \mathbb{N}$, $j \geq 2$.

By this lemma, if we take $\alpha < \tilde{\alpha} < \alpha_{2,\beta}$ and put $C = C_{\tilde{\alpha}}$, we see

$$\frac{\|v\|^{2j}_{2j,\beta}}{\|\nabla v\|_2^2 \|v\|^2_{2,\beta}} \leq C\frac{j!}{\tilde{\alpha}^j}\|\nabla v\|_{2j}^{2j-4} \leq C\frac{j!}{\tilde{\alpha}^j}$$

for $j \geq 2$ since $v \in M$. Thus we have

$$\sum_{j=2}^\infty \frac{\alpha^{j-1}}{(j-1)!}\frac{\|v\|^{2j}_{2j,\beta}}{\|\nabla v\|_2^2 \|v\|^2_{2,\beta}} \leq \sum_{j=2}^\infty \frac{C\alpha^{j-1}}{(j-1)!}\frac{j!}{\tilde{\alpha}^j} = (\frac{C\alpha}{\tilde{\alpha}^2})\sum_{j=2}^\infty \left(\frac{\alpha}{\tilde{\alpha}}\right)^{j-2} j \leq \alpha C'$$

for some $C' > 0$. Inserting this into the former estimate (4.2), we obtain

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) \leq (1-\frac{\beta}{2})\alpha\|\nabla v\|_2^2\|v\|^2_{2,\beta}(-1 + C'\alpha) < 0$$

when $\alpha > 0$ is sufficiently small. This contradicts to (4.1). □

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