CRITICAL AND SUBCRITICAL FRACTIONAL TRUDINGER-MOSER TYPE INEQUALITIES ON $\mathbb{R}$

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Abstract. In this paper, we are concerned with the critical and subcritical Trudinger-Moser type inequalities for functions in a fractional Sobolev space $H^{1/2,2}$ on the whole real line. We prove the relation between two inequalities and discuss the attainability of the suprema.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain with finite volume. Then the Sobolev embedding theorem assures that $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, however, a simple example shows that the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold. Instead, functions in $W_0^{1,N}(\Omega)$ enjoy the exponential summability:

$$W_0^{1,N}(\Omega) \hookrightarrow \{u \in L^N(\Omega) : \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \quad \text{for any } \alpha > 0\},$$

see Yudovich [29], Pohozaev [24], and Trudinger [28]. Later, Moser [18] improved the embedding above as follows, now known as the Trudinger-Moser inequality:

$$TM(\Omega, \alpha) = \sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_{L^N(\Omega)} \leq 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N, \end{cases}$$

here $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\omega_{N-1} = |S^{N-1}|$ denotes the area of the unit sphere in $\mathbb{R}^N$. On the attainability of $TM(\Omega, \alpha)$, Carleson-Chang [4], Flucher [6], and Lin [13] proved that $TM(\Omega, \alpha)$ is attained for any $0 < \alpha \leq \alpha_N$.

On domains with infinite volume, for example on the whole space $\mathbb{R}^N$, the Trudinger-Moser inequality does not hold as it is. However,

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several variants are known on the whole space. In the following, let

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}$$

denote the truncated exponential function.

First, Ogawa [20], Ogawa-Ozawa [21], Cao [3], Ozawa [23], and Adachi-Tanaka [1] proved that the following inequality holds true, which we call Adachi-Tanaka type Trudinger-Moser inequality:

$$A(N,\alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1}} \frac{1}{\|u\|^N_{L^N(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha < \alpha_N, \\ = \infty, & \alpha \ge \alpha_N. \end{cases}$$

The inequality enjoys the scale invariance under the scaling $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$. Note that the critical exponent $\alpha = \alpha_N$ is not allowed for the finiteness of the supremum. Recently, it is proved that $A(N,\alpha)$ is attained for any $\alpha \in (0, \alpha_N)$ by Ishiwata-Nakamura-Wadade [10] and Dong-Lu [5]. In this sense, Adachi-Tanaka type Trudinger-Moser inequality has a subcritical nature of the problem.

On the other hand, Ruf [26] and Li-Ruf [15] proved that the following inequality holds true:

$$B(N,\alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \\ \|u\|_{W^{1,N}(\mathbb{R}^N)} \le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha \le \alpha_N, \\ = \infty, & \alpha > \alpha_N. \end{cases}$$

Here $\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|^N_{L^N(\mathbb{R}^N)} + \|u\|^N_{L^N(\mathbb{R}^N)} \right)^{1/N}$ is the full Sobolev norm. Note that the scale invariance ($u \mapsto u_\lambda$) does not hold for this inequality. Also note that the critical exponent $\alpha = \alpha_N$ is permitted to the finiteness.

Concerning the attainability of $B(N,\alpha)$, the following facts have been proved:

- If $N \ge 3$, then $B(N,\alpha)$ is attained for $0 < \alpha \le \alpha_N$ [26].
- If $N = 2$, then there exists $\alpha_* > 0$ such that $B(2,\alpha)$ is attained for $\alpha_* < \alpha \le \alpha_2 (= 4\pi)$ [26], [9].
- If $N = 2$ and $\alpha > 0$ is sufficiently small, then $B(2,\alpha)$ is not attained. [9].

The non-attainability of $B(2,\alpha)$ for α sufficiently small is attributed to the non-compactness of "vanishing" maximizing sequences, as described in [9].

In the following, we focus our attention on the fractional Sobolev spaces.

Let $s \in (0,1)$, $p \in [1,+\infty)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. For $s > 0$, let us consider the space

$$L_s(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N) \; : \; \int_{\mathbb{R}^N} \frac{|u|}{1+|x|^{N+s}} dx < \infty \right\}.$$

For $u \in L_s(\mathbb{R}^N)$, we define the fractional Laplacian $(-\Delta)^{s/2}u$ as follows: First, for $\phi \in \mathcal{S}(\mathbb{R}^N)$, the rapidly decreasing functions on $\mathbb{R}^N$, $(-\Delta)^{s/2}\phi$ is defined via the normalized Fourier transform $\mathcal{F}$ as $(-\Delta)^{s/2}\phi(x) = \mathcal{F}^{-1}(|\xi|^s \mathcal{F}\phi(\xi))(x)$ for $x \in \mathbb{R}^N$. Then for $u \in L_s(\mathbb{R}^N)$, $(-\Delta)^{s/2}u$ is defined as the element of $\mathcal{S}'(\mathbb{R}^N)$, the tempered distributions on $\mathbb{R}^N$, by the relation

$$\langle \phi, (-\Delta)^{s/2}u \rangle = \langle (-\Delta)^{s/2}\phi, u \rangle = \int_{\mathbb{R}} (-\Delta)^{s/2}\phi \cdot u dx, \quad \phi \in \mathcal{S}(\mathbb{R}^N).$$

Note that $L^p(\mathbb{R}^N) \subset L_s(\mathbb{R}^N)$ for any $p \geq 1$. Also note that it could happen $supp((-\Delta)^{s/2}u) \not\subset \Omega$ even if $supp(u) \subset \Omega$ for some open set Ω in $\mathbb{R}^N$.

By using the above notion, we define *the Bessel potential space* $H^{s,p}(\Omega)$ for a (possibly unbounded) set $\Omega \subset \mathbb{R}^N$ as

$$H^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) \; : \; (-\Delta)^{s/2}u \in L^p(\mathbb{R}^N) \right\},$$
$$\tilde{H}^{s,p}(\Omega) = \left\{ u \in H^{s,p}(\mathbb{R}^N) \; : u \equiv 0 \quad \text{on}\, \mathbb{R}^N \setminus \Omega \right\}.$$

On the other hand, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) \; : \; [u]_{W^{s,p}(\mathbb{R}^N)} < \infty \right\},$$
$$[u]^p_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy,$$

and for a bounded domain $\Omega \subset \mathbb{R}^N$, we define

$$\tilde{W}^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}}$$

where $\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)} \right)^{1/p}$. It is known that

$$\tilde{W}^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) \; : u \equiv 0 \quad \text{on}\, \mathbb{R}^N \setminus \Omega \right\}$$

if Ω is a Lipschitz domain and $H^{s,p}(\mathbb{R}^N) = F^s_{p,2}(\mathbb{R}^N)$ (Triebel-Lizorkin space), $W^{s,p}(\mathbb{R}^N) = B^s_{p,p}(\mathbb{R}^N)$ (Besov space). Thus $H^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$, however in general, $H^{s,p}(\mathbb{R}^N) \neq W^{s,p}(\mathbb{R}^N)$ for $p \neq 2$. See [25], [11] and the references therein.

Recently, Martinazzi [17] (see also [12]) proved a fractional Trudinger-Moser type inequality on $\tilde{H}^{s,p}(\Omega)$ as follows: Let $p \in (1,\infty)$ and

$s = N/p$ for $N \in \mathbb{N}$. Then for any open $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$, it holds

$$\sup_{\substack{u \in \tilde{H}^{s,p}(\Omega) \\ \|(-\Delta)^{s/2} u\|_{L^p(\Omega)} \le 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{\frac{p}{p-1}}) dx \begin{cases} < \infty, & \alpha \le \alpha_{N,p}, \\ = \infty, & \alpha > \alpha_{N,p}. \end{cases}$$

Here $\alpha_{N,p} = \frac{N}{\omega_{N-1}} \left(\frac{\Gamma((N-s)/2)}{\Gamma(s/2) 2^s \pi^{N/2}} \right)^{-p/(p-1)}$.

We note that, differently from the classical case, the attainability of the supremum is not known even for $N = 1$ and $p = 2$.

On the Sobolev-Slobodeckij spaces $\tilde{W}^{s,p}(\Omega)$ with $sp = N$, similar fractional Trudinger-Moser inequality is also proved by Parini-Ruf [25] when $N \ge 2$ and Iula [11] when $N = 1$. In this case, the result is weaker and the inequality holds true only for $0 \le \alpha < \alpha^*_{N,p}$ for some (explicit) value $\alpha^*_{N,p}$. Also, it is not known the inequality holds or not when $\alpha = \alpha^*_{N,p}$.

In the following, we are interested in the simplest one dimensional case, that is, we put $N = 1$, $s = 1/2$ and $p = 2$. In this case, the Bessel potential space $H^{1/2,2}(\mathbb{R})$ coincides with the Sobolev-Slobodeckij space $W^{1/2,2}(\mathbb{R})$ and both seminorms are related as

$$\|(-\Delta)^{1/4} u\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} [u]^2_{W^{1/2,2}(\mathbb{R})},$$

see Proposition 3.6. in [19]. Then the fractional Trudinger-Moser inequality in [17], [12] can be read as

Proposition 1. *(A fractional Trudinger-Moser inequality on $\tilde{H}^{1/2,2}(I)$) Let $I \subset \mathbb{R}$ be an open bounded interval. Then it holds*

$$\sup_{\substack{u \in \tilde{H}^{1/2,2}(I) \\ \|(-\Delta)^{1/4} u\|_{L^2(I)} \le 1}} \frac{1}{|I|} \int_I e^{\alpha |u|^2} dx \begin{cases} < \infty, & \alpha \le \alpha_{1,2} = \pi, \\ = \infty, & \alpha > \pi \end{cases}$$

For the fractional Adachi-Tanaka type Trudinger-Moser inequality on the whole line, put

$$A(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1}} \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx. \tag{1.1}$$

Then by the precedent results by Ogawa-Ozawa [21] and Ozawa [23], it is known that $A(\alpha) < \infty$ for small exponent α.

On the other hand, it is already known a fractional Li-Ruf type Trudinger-Moser inequality on $H^{1/2,2}(\mathbb{R})$:

Proposition 2. *(Iula-Maalaoui-Martinazzi* [12]*)*

$$B(\alpha) = \sup_{\substack{u\in H^{1/2,2}(\mathbb{R})\\ \|u\|_{H^{1/2,2}(\mathbb{R})}\le 1}} \int_{\mathbb{R}} \left(e^{\alpha u^2}-1\right) dx \begin{cases} <\infty, & \alpha \le \pi, \\ =\infty, & \alpha > \pi. \end{cases} \tag{1.2}$$

Here

$$\|u\|_{H^{1/2,2}(\mathbb{R})} = \left(\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\right)^{1/2}$$

is the full Sobolev norm on $H^{1/2,2}(\mathbb{R})$.

Concerning $A(\alpha)$ in (1.1), a natural question is that to what range of the exponent α the supremum is finite. As pointed out in [8], it remained an open problem for a while. In this paper, first we prove the finiteness of supremum in the full range of values of exponent.

Theorem 1. *(Full range Adachi-Tanaka type on $H^{1/2,2}(\mathbb{R})$) We have*

$$A(\alpha) = \sup_{\substack{u\in H^{1/2,2}(\mathbb{R})\setminus\{0\}\\ \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}\le 1}} \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2}-1\right) dx. \begin{cases} <\infty, & \alpha < \pi, \\ =\infty, & \alpha \ge \pi. \end{cases}$$

Ozawa [22] proved that the Adachi-Tanaka type Trudinger-Moser inequality is equivalent to the Gagliardo-Nirenberg type inequality, and he also proved an exact relation between the best constants of both inequalities. As a result, we have the next corollary.

Corollary 1. *Set*

$$\beta_0 = \limsup_{q\to\infty} \sup_{u\in H^{1/2,2}(\mathbb{R}), u\neq 0} \frac{\|u\|_{L^q(\mathbb{R})}}{q^{1/2}\|(-\Delta)^{1/4}u\|^{1-2/q}_{L^2(\mathbb{R})}\|u\|^{2/q}_{L^2(\mathbb{R})}}.$$

Then $\beta_0 = (2\pi e)^{-1/2}$.

Furthermore, we obtain the relation between the suprema of both critical and subcritical Trudinger-Moser type inequalities along the line of Lam-Lu-Zhang [14].

Theorem 2. *(Relation) We have*

$$B(\pi) = \sup_{\alpha\in(0,\pi)} \frac{1-(\alpha/\pi)}{(\alpha/\pi)} A(\alpha).$$

Also we obtain how Adachi-Tanaka type supremum $A(\alpha)$ behaves when α tends to π.

Theorem 3. *(Asymptotic behavior) There exist* $C_1, C_2 > 0$ *such that for any* $\alpha < \pi$ *which is close enough to* π*, it holds*

$$\frac{C_1}{1-\alpha/\pi} \le A(\alpha) \le \frac{C_2}{1-\alpha/\pi}.$$

Note that the estimate from the above follows from Theorem 2 and Proposition 2. On the other hand, we will see that that the estimate from the below follows from a computation using the Moser sequence.

Concerning the existence of maximizers of Adachi-Tanaka type supremum $A(\alpha)$ in (1.1), we see

Theorem 4. *(Attainability of* $A(\alpha)$*)* $A(\alpha)$ *is attained for any* $\alpha \in (0, \pi)$*.*

On the other hand, as for $B(\alpha)$ in (1.2), we have

Theorem 5. *(Non-attainability of* $B(\alpha)$*) For* $0 < \alpha << 1$*,* $B(\alpha)$ *is not attained.*

It is plausible that there exists $\alpha_* > 0$ such that $B(\alpha)$ is attained for $\alpha_* < \alpha \le \pi$, but we do not have a proof up to now.

Finally, we improve the subcritical Adachi-Tanaka type inequality along the line of Dong-Lu [5]:

Theorem 6. *For* $\alpha > 0$*, set*

$$E(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1}} \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} u^2 e^{\alpha u^2} dx. \tag{1.3}$$

Then we have

$$E(\alpha) \begin{cases} < \infty, & \alpha < \pi, \\ = \infty, & \alpha \ge \pi. \end{cases}$$

Furthermore, $E(\alpha)$ *is attained for all* $\alpha \in (0, \pi)$*.*

Since $e^{\alpha t^2} - 1 \le \alpha t^2 e^{\alpha t^2}$ for $t \in \mathbb{R}$, Theorem 6 extends Theorem 1. In the classical case, Dong-Lu used a rearrangement technique to reduce the problem to one-dimension and obtained the similar inequality by estimating a one-dimensional integral. The method is similar to [4]. In the fractional setting $H^{1/2,2}$, we cannot follow this argument and we need a new idea.

The organization of the paper is as follows: In section 2, we prove Theorem 1, 2, and 3. In section 3, we prove Theorem 4 and 5. In section 4, we prove Theorem 6.

2. Proof of Theorem 1, 2, and 3

For the proofs of Theorem 1, 2, and 3, we prepare several lemmas.

Lemma 1. *Set*

$$\tilde{A}(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1 \\ \|u\|_{L^2(\mathbb{R})} = 1}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx. \tag{2.1}$$

Then $\tilde{A}(\alpha) = A(\alpha)$ *for any* $\alpha > 0$.

Proof. For any $u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\}$ and $\lambda > 0$, we put $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}$. Then we have

$$\begin{cases} \|(-\Delta)^{1/4} u_\lambda\|_{L^2(\mathbb{R})} = \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})}, \\ \|u_\lambda\|^2_{L^2(\mathbb{R})} = \lambda^{-1} \|u\|^2_{L^2(\mathbb{R})}, \end{cases} \tag{2.2}$$

since

$$\begin{aligned} 2\pi \|(-\Delta)^{1/4} u_\lambda\|^2_{L^2(\mathbb{R})} &= [u_\lambda]^2_{W^{1/2,2}(\mathbb{R})} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\lambda x) - u(\lambda y)|^2}{|x-y|^2} dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\lambda x) - u(\lambda y)|^2}{|\lambda x - \lambda y|^2} d(\lambda x) d(\lambda y) \\ &= [u]^2_{W^{1/2,2}(\mathbb{R})} = 2\pi \|(-\Delta)^{1/4} u\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Thus for any $u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\}$ with $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1$, if we choose $\lambda = \|u\|^2_{L^2(\mathbb{R})}$, then $u_\lambda \in H^{1/2,2}(\mathbb{R})$ satisfies

$$\|(-\Delta)^{1/4} u_\lambda\|_{L^2(\mathbb{R})} \le 1 \quad \text{and} \quad \|u_\lambda\|^2_{L^2(\mathbb{R})} = 1.$$

Thus

$$\frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx = \int_{\mathbb{R}} \left(e^{\alpha u_\lambda^2} - 1 \right) dx \le \tilde{A}(\alpha),$$

which implies $A(\alpha) \le \tilde{A}(\alpha)$. The opposite inequality is trivial. □

Lemma 2. *For any* $0 < \alpha < \pi$, *it holds*

$$A(\alpha) \le \frac{(\alpha/\pi)}{1 - (\alpha/\pi)} B(\pi).$$

Proof. Choose any $u \in H^{1/2,2}(\mathbb{R})$ with $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1$ and $\|u\|_{L^2(\mathbb{R})} = 1$. Put $v(x) = Cu(\lambda x)$ where $C^2 = \alpha/\pi \in (0,1)$ and $\lambda = \frac{C^2}{1-C^2}$. Then by scaling rules (2.2), we see

$$\begin{aligned}\|v\|^2_{H^{1/2,2}(\mathbb{R})} &= \|(-\Delta)^{1/4}v\|^2_{L^2(\mathbb{R})} + \|v\|^2_{L^2(\mathbb{R})} \\ &= C^2\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R})} + \lambda^{-1}C^2\|u\|^2_{L^2(\mathbb{R})} \\ &\le C^2 + \lambda^{-1}C^2 = 1.\end{aligned}$$

Also we have

$$\begin{aligned}\int_{\mathbb{R}} \left(e^{\pi v^2} - 1\right) dx &= \int_{\mathbb{R}} \left(e^{\pi C^2 u^2(\lambda x)} - 1\right) dx \\ &= \lambda^{-1} \int_{\mathbb{R}} \left(e^{\pi C^2 u^2(y)} - 1\right) dy \\ &= \frac{1 - C^2}{C^2} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1\right) dy \\ &= \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1\right) dy.\end{aligned}$$

Thus testing $B(\pi)$ by v, we see

$$B(\pi) \ge \int_{\mathbb{R}} \left(e^{\pi v^2} - 1\right) dx \ge \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1\right) dy.$$

By taking the supremum for $u \in H^{1/2,2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \le 1$ and $\|u\|_{L^2(\mathbb{R})} = 1$, we have

$$B(\pi) \ge \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \tilde{A}(\alpha).$$

Finally, Lemma 1 implies the result. □

Proof of Theorem 1: The assertion that $A(\alpha) < \infty$ for $\alpha < \pi$ follows from Lemma 2 and the fact $B(\pi) < \infty$ by Proposition 2.

For the proof of $A(\pi) = \infty$, we use the Moser sequence

$$u_\varepsilon = \begin{cases} (\log(1/\varepsilon))^{1/2}, & \text{if } |x| < \varepsilon, \\ \frac{\log(1/|x|)}{(\log(1/\varepsilon))^{1/2}}, & \text{if } \varepsilon < |x| < 1, \\ 0, & \text{if } 1 \le |x|, \end{cases} \tag{2.3}$$

and its estimates

$$\|(-\Delta)^{1/4}u_\varepsilon\|^2_{L^2(\mathbb{R})} = \pi + o(1), \tag{2.4}$$

$$\|(-\Delta)^{1/4}u_\varepsilon\|^2_{L^2(\mathbb{R})} \le \pi\left(1 + (C\log(1/\varepsilon))^{-1}\right), \tag{2.5}$$

$$\|u_\varepsilon\|^2_{L^2(\mathbb{R})} = O\left((\log(1/\varepsilon))^{-1}\right) \tag{2.6}$$

as $\varepsilon \to 0$ for some $C > 0$. Note $u_\varepsilon \in \tilde{W}^{1/2,2}((-1,1)) \subset W^{1/2,2}(\mathbb{R}) = H^{1/2,2}(\mathbb{R})$. For the estimate (2.4), we refer to Iula [11] Proposition 2.2. For the estimate (2.5), we refer to [11] equation (35). Actually, after a careful look of the proof of Proposition 2.2 in [11], we confirm that

$$\lim_{\varepsilon\to 0} (\log(1/\varepsilon)) \left(\|(-\Delta)^{1/4} u_\varepsilon\|^2_{L^2(\mathbb{R})} - \pi \right) \le C$$

for a positive $C > 0$, which implies (2.5). For (2.6), we compute

$$\begin{aligned}
\|u_\varepsilon\|^2_{L^2(\mathbb{R})} &= \int_{|x|\le\varepsilon} (\log(1/\varepsilon))\, dx + \int_{\varepsilon<|x|\le 1} \left(\frac{\log(1/|x|)}{(\log(1/\varepsilon))^{1/2}} \right)^2 dx \\
&= 2\varepsilon \log(1/\varepsilon) + \frac{2}{\log(1/\varepsilon)} \int_{\log(1/\varepsilon)}^0 t^2(-e^t) dx \\
&= 2\varepsilon \log(1/\varepsilon) + \frac{2}{\log(1/\varepsilon)} (\Gamma(3) + o(1))
\end{aligned}$$

as $\varepsilon \to 0$. Thus we obtain (2.6).

By testing $A(\pi)$ by $v_\varepsilon = u_\varepsilon / \|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}$, we have

$$\begin{aligned}
A(\pi) &\ge \frac{1}{\|v_\varepsilon\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\pi v_\varepsilon^2} - 1 \right) dx \\
&\ge \frac{\|(-\Delta)^{1/4} u_\varepsilon\|^2_{L^2(\mathbb{R})}}{\|u_\varepsilon\|^2_{L^2(\mathbb{R})}} \int_{|x|\le\varepsilon} \left(e^{\pi v_\varepsilon^2} - 1 \right) dx \\
&\ge \frac{\|(-\Delta)^{1/4} u_\varepsilon\|^2_{L^2(\mathbb{R})}}{\|u_\varepsilon\|^2_{L^2(\mathbb{R})}} \varepsilon \exp\left(\pi \frac{\log(1/\varepsilon)}{\|(-\Delta)^{1/4} u_\varepsilon\|^2_{L^2(\mathbb{R})}} \right) \\
&\ge \frac{\|(-\Delta)^{1/4} u_\varepsilon\|^2_{L^2(\mathbb{R})}}{\|u_\varepsilon\|^2_{L^2(\mathbb{R})}} \varepsilon \exp\left(\frac{\log(1/\varepsilon)}{1 + (C\log(1/\varepsilon))^{-1}} \right)
\end{aligned}$$

since $e^t - 1 \ge (1/2)e^t$ for t large and (2.5). Also since

$$\frac{t}{1+\frac{1}{Ct}} - t = \frac{-1/C}{1+\frac{1}{Ct}} \to -\frac{1}{C} \quad \text{as}\, t \to \infty,$$

we see $\frac{t}{1+\frac{1}{Ct}} = t - 1/C + o(1)$ as $t \to \infty$. Put $t = \log(1/\varepsilon)$, we see

$$\exp\left(\frac{\log(1/\varepsilon)}{1 + (C\log(1/\varepsilon))^{-1}} \right) = \exp\left(\log(1/\varepsilon) - 1/C + o(1)\right) = (1/\varepsilon) e^{-1/C+o(1)},$$

which leads to

$$\varepsilon \exp\left(\frac{\log(1/\varepsilon)}{1 + (C\log(1/\varepsilon))^{-1}} \right) \ge e^{-1/C+o(1)} \ge \delta > 0$$

for some $\delta > 0$ independent of $\varepsilon \to 0$. Therefore, by (2.4), (2.5), (2.6), we have for $\delta' > 0$

$$A(\pi) \geq \frac{\pi + o(1)}{(C\log(1/\varepsilon))^{-1}}\delta \geq \delta' \left(\log(1/\varepsilon)\right) \to \infty$$

as $\varepsilon \to 0$. This proves $A(\pi) = \infty$. □

Proof of Theorem 2: By Lemma 2, we have

$$B(\pi) \geq \sup_{\alpha\in(0,\pi)} \frac{1-(\alpha/\pi)}{(\alpha/\pi)} A(\alpha).$$

Let us prove the opposite inequality. Let $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$, $u_n \neq 0$, $\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})} + \|u_n\|^2_{L^2(\mathbb{R})} \leq 1$, be a maximizing sequence of $B(\pi)$. We may assume $\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})} < 1$ for any $n \in \mathbb{N}$. Put

$$\begin{cases} v_n(x) = \frac{u_n(\lambda_n x)}{\|(-\Delta)^{1/4}u_n\|_{L^2(\mathbb{R})}}, & (x \in \mathbb{R}) \\ \lambda_n = \frac{1-\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})}}{\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})}} > 0. \end{cases}$$

Thus by (2.2), we see

$$\|(-\Delta)^{1/4}v_n\|^2_{L^2(\mathbb{R})} = 1,$$

$$\|v_n\|^2_{L^2(\mathbb{R})} = \frac{\lambda_n^{-1}}{\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})}}\|u_n\|^2_{L^2(\mathbb{R})} = \frac{\|u_n\|^2_{L^2(\mathbb{R})}}{1-\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})}} \leq 1,$$

since $\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})} + \|u_n\|^2_{L^2(\mathbb{R})} \leq 1$. Thus, setting $\alpha_n = \pi\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})} < \pi$ for any $n \in \mathbb{N}$, we may test $A(\alpha_n)$ by $\{v_n\}$, which results in

$$\begin{aligned} B(\pi) + o(1) &= \int_{\mathbb{R}} \left(e^{\pi u_n^2(y)} - 1\right) dy \\ &= \lambda_n \int_{\mathbb{R}} \left(e^{\pi\|(-\Delta)^{1/4}u_n\|^2_{L^2(\mathbb{R})} v_n^2(x)} - 1\right) dx \\ &\leq \lambda_n \frac{1}{\|v_n\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha_n v_n^2(x)} - 1\right) dx \\ &\leq \lambda_n A(\alpha_n) = \frac{1-(\alpha_n/\pi)}{(\alpha_n/\pi)} A(\alpha_n) \\ &\leq \sup_{\alpha\in(0,\pi)} \frac{1-(\alpha/\pi)}{(\alpha/\pi)} A(\alpha). \end{aligned}$$

Here we have used a change of variables $y = \lambda_n x$ for the second equality, and $\|v_n\|^2_{L^2(\mathbb{R})} \leq 1$ for the first inequality. Letting $n \to \infty$, we have the desired result. □

Proof of Theorem 3:

We need to prove that there exists $C_1 > 0$ such that for any $\alpha < \pi$ which is sufficiently close to π, it holds that

$$A(\alpha) \geq \frac{C_1}{1 - \alpha/\pi}.$$

Again we use the Moser sequence (2.3) and we test $A(\alpha)$ by $v_\varepsilon = u_\varepsilon / \|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}$. As in the similar calculations in the proof of Theorem 1, we have

$$\begin{aligned}
A(\alpha) &\geq \frac{1}{\|v_\varepsilon\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha v_\varepsilon^2} - 1 \right) dx \\
&\geq \frac{(1/2)}{\|v_\varepsilon\|^2_{L^2(\mathbb{R})}} \int_{|x| \leq \varepsilon} e^{\alpha v_\varepsilon^2} dx \\
&\geq C\varepsilon \left(\log(1/\varepsilon) \right) \exp\left(\frac{\alpha}{\pi} \frac{\log(1/\varepsilon)}{1 + (C \log(1/\varepsilon))^{-1}} \right) \\
&= C\varepsilon \left(\log(1/\varepsilon) \right) \exp\left(\delta_\varepsilon \log(1/\varepsilon) \right)
\end{aligned}$$

where we put $\delta_\varepsilon = (\frac{\alpha}{\pi}) \frac{1}{1 + (C \log(1/\varepsilon))^{-1}} \in (0, 1)$.

Now, for $\alpha < \pi$ which is sufficiently close to π, we fix $\varepsilon > 0$ small such that

$$\frac{1}{1 - \alpha/\pi} \leq \log(1/\varepsilon) \leq \frac{2}{1 - \alpha/\pi}, \tag{2.7}$$

which implies

$$\exp\left(-\frac{2}{1 - \alpha/\pi} \right) \leq \varepsilon \leq \exp\left(-\frac{1}{1 - \alpha/\pi} \right).$$

With this choice of $\varepsilon > 0$, we have

$$\begin{aligned}
A(\alpha) &\geq C\varepsilon \left(\log(1/\varepsilon) \right) \exp\left(\delta_\varepsilon \log(1/\varepsilon) \right) \\
&= C\varepsilon \left(\log(1/\varepsilon) \right) (1/\varepsilon)^{\delta_\varepsilon} = C\varepsilon^{1 - \delta_\varepsilon} \left(\log(1/\varepsilon) \right).
\end{aligned} \tag{2.8}$$

Now, we estimate that

$$
\begin{aligned}
\varepsilon^{1-\delta_\varepsilon} &\geq \left(\exp\left(-\frac{2}{1-\alpha/\pi}\right)\right)^{1-\delta_\varepsilon} = \exp\left(-\frac{2}{1-\alpha/\pi}(1-\delta_\varepsilon)\right) \\
&= \exp\left(-\left(\frac{2}{1-\alpha/\pi}\right)\left\{(1-\alpha/\pi) + (\alpha/\pi)\left(1 - \frac{1}{1+(C\log 1/\varepsilon)^{-1}}\right)\right\}\right) \\
&= \exp\left(-2 - \left(\frac{2(\alpha/\pi)}{1-\alpha/\pi}\right)\left(\frac{1}{1+C\log 1/\varepsilon}\right)\right) \\
&\geq \exp\left(-2 - \left(\frac{2(\alpha/\pi)}{1-\alpha/\pi}\right)\left(\frac{1}{1+\frac{C}{1-\alpha/\pi}}\right)\right) \\
&= e^{-2} \cdot e^{-\frac{2(\alpha/\pi)}{C+1-\alpha/\pi}} = e^{-2} \cdot e^{-f(\alpha/\pi)}
\end{aligned}
$$

where $f(t) = \frac{2t}{C+1-t}$ for $t \in [0,1]$ and we have used (2.7) in the last inequality. We easily see that $f(0) = 0$, $f'(t) = \frac{2(C+1)}{(C+1-t)^2} > 0$ for $t > 0$, thus $f(t)$ is strictly increasing in t and $\max_{t\in[0,1]} f(t) = f(1) = 2/C$. Thus we have

$$\varepsilon^{1-\delta_\varepsilon} \geq e^{-2} \cdot e^{-2/C} =: C_0$$

which is independent of α. Backing to (2.8) with (2.7), we observe that

$$A(\alpha) \geq C\varepsilon^{1-\delta_\varepsilon}\left(\log(1/\varepsilon)\right) \geq CC_0\left(\log(1/\varepsilon)\right) \geq \frac{CC_0}{1-\alpha/\pi}$$

which proves the result. □

3. Proof of Theorem 4 and 5

For $u \in H^{1/2,2}(\mathbb{R})$, u^* will denote its symmetric decreasing rearrangement defined as follows: For a measurable set $A \subset \mathbb{R}$, let A^* denote an open interval $A^* = (-|A|/2, |A|/2)$. We define u^* by

$$u^*(x) = \int_0^\infty \chi_{\{y\in\mathbb{R}:|u(y)|>t\}^*}(x)dt$$

where χ_A denote the indicator function of a measurable set $A \subset \mathbb{R}$. Note that u^* is nonnegative, even, and decreasing on the positive line $\mathbb{R}_+ = [0, +\infty)$. It is known that

$$\int_{\mathbb{R}} F(u^*)dx = \int_{\mathbb{R}} F(|u|)dx \tag{3.1}$$

for any nonnegative measurable function $F : \mathbb{R}_+ \to \mathbb{R}_+$, which is the difference of two monotone increasing functions F_1, F_2 with $F_1(0) =$

$F_2(0) = 0$ such that either $F_1 \circ |u|$ or $F_2 \circ |u|$ is integrable. Also the inequality of Pólya-Szegö type

$$\int_{\mathbb{R}} |(-\Delta u^*)^{1/4}|^2 dx \leq \int_{\mathbb{R}} |(-\Delta u)^{1/4}|^2 dx$$

holds true for $u \in H^{1/2,2}(\mathbb{R})$, see for example, [2] and [16].

Remark 1. Note that Radial Compactness Lemma by Strauss [27] is violated on $\mathbb{R}$. More precisely, let

$$H^{1/2,2}_{rad}(\mathbb{R}) = \{u \in H^{1/2,2}(\mathbb{R}) \, : \, u(x) = u(-x), \, x \geq 0\},$$

then $H^{1/2,2}_{rad}(\mathbb{R})$ cannot be embedded compactly in $L^q(\mathbb{R})$ for any $q > 0$. To see this, let $\psi \neq 0$ be an even function in $C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subset (-1, 1)$ and put $u_n(x) = \psi(x - n) + \psi(x + n)$. Then we see u_n is even, compactly supported smooth function, and $u_n \rightharpoonup 0$ weakly in $H^{1/2,2}(\mathbb{R})$ as $n \to \infty$. But $\{u_n\}$ does not have any strong convergent subsequence in $L^q(\mathbb{R})$, because $\|u_n\|^q_{L^q(\mathbb{R})} = 2\|\psi\|^q_{L^q(\mathbb{R})} > 0$ for any n sufficient large.

However, for a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ with u_n even, nonnegative and decreasing on $\mathbb{R}_+$, we have the following compactness result.

Proposition 3. *Assume $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ be a sequence such that u_n is even, nonnegative and decreasing on $\mathbb{R}_+$. Let $u_n \rightharpoonup u$ weakly in $H^{1/2,2}(\mathbb{R})$. Then $u_n \to u$ strongly in $L^q(\mathbb{R})$ for any $q \in (2, +\infty)$ for a subsequence.*

Proof. Since $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ is a weakly convergent sequence, we have $\sup_{n\in\mathbb{N}} \|u_n\|_{H^{1/2,2}(\mathbb{R})} \leq C$ for some $C > 0$. We also have $u_n(x) \to u(x)$ a.e $x \in \mathbb{R}$ for a subsequence, thus u is even, nonnegative and decreasing on $\mathbb{R}_+$. Now, we use the estimate below, which is referred to a Simple Radial Lemma: If $u \in L^2(\mathbb{R})$ is even, nonnegative and decreasing on $\mathbb{R}_+$, then it holds

$$u^2(x) \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} u^2(y) dy \leq \frac{1}{2|x|} \|u\|^2_{L^2(\mathbb{R})} \quad (x \neq 0). \tag{3.2}$$

Thus $u_n^2(x) \leq \frac{C}{2|x|}$ for $x \neq 0$ by $\sup_{n\in\mathbb{N}} \|u_n\|_{H^{1/2,2}(\mathbb{R})} \leq C$ and $u^2(x) \leq \frac{C}{2|x|}$ by the pointwise convergence. Now, set $v_n = |u_n - u|^q$ for $q > 2$.

Then we see $v_n(x) \to 0$ a.e. $x \in \mathbb{R}$. Moreover,

$$\begin{aligned}\int_{|x|\geq R} |u_n - u|^q dx &= 2\int_R^\infty |u_n - u|^q dx \\ &\leq 2^q \left(\int_R^\infty |u_n|^q dx + \int_R^\infty |u|^q dx \right) \\ &\leq C \int_R^\infty \frac{dx}{|x|^{q/2}} = \frac{CR^{1-q/2}}{(q/2)-1} \to 0\end{aligned}$$

as $R \to \infty$ since $q > 2$. Thus $\{v_n\}_{n\in\mathbb{N}}$ is uniformly integrable. Also by [19] Theorem 6.9, we know that

$$H^{1/2,2}(\mathbb{R}) \subset L^{q_0}(\mathbb{R}) \quad \text{for any } q_0 \geq 2 \text{ and } \quad \|u\|_{L^{q_0}(\mathbb{R})} \leq C\|u\|_{H^{1/2,2}(\mathbb{R})}.$$

For any $q > 2$, take q_0 such that $2 < q < q_0 < \infty$. Since $u_n - u$ is uniformly bounded in $H^{1/2,2}(\mathbb{R})$, we have $\|u_n - u\|_{L^{q_0}(\mathbb{R})} \leq C$, and

$$\int_I v_n dx = \int_I |u_n - u|^q dx \leq \left(\int_I |u_n - u|^{q_0} dx \right)^{q/q_0} |I|^{1-q/q_0}$$

for any bounded measurable set $I \subset \mathbb{R}$. Therefore $\int_I v_n dx \to 0$ if $|I| \to 0$, which implies $\{v_n\}$ is uniformly absolutely continuous. Thus by Vitali's Convergence Theorem (see for example, [7] p.187), we obtain $v_n = |u_n - u|^q \to 0$ strongly in $L^1(\mathbb{R})$, which is the desired conclusion. □

Proposition 4. *Assume $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ be a sequence with $\|(-\Delta)^{1/4}u_n\|_{L^2(\mathbb{R})} \leq 1$. Let $u_n \rightharpoonup u$ weakly in $H^{1/2,2}(\mathbb{R})$ for some u and assume u_n is even, nonnegative and decreasing on $\mathbb{R}_+$. Then we have*

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 - \alpha u_n^2 \right) dx \to \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx$$

for any $\alpha \in (0, \pi)$.

Proof. The similar proposition above is already appeared, see [10] Lemma 3.1, and [5] Lemma 5.5. We prove it here for the reader's convenience.

Put $\Phi_\alpha(t) = e^{\alpha t^2} - 1$ and $\Psi_\alpha(t) = e^{\alpha t^2} - 1 - \alpha t^2$. Note that $\Phi_\alpha(t)$ is nonnegative, strictly convex and $\Psi'_\alpha(t) = 2\alpha t \Phi_\alpha(t)$. Thus by the mean value theorem, we have

$$\begin{aligned}|\Psi_\alpha(u_n) - \Psi_\alpha(u)| &\leq \Psi'_\alpha(\theta u_n + (1-\theta)u)|u_n - u| \\ &\leq 2\alpha|\theta u_n + (1-\theta)u|\Phi_\alpha(\theta u_n + (1-\theta)u)|u_n - u| \\ &\leq 2\alpha(|u_n| + |u|)\left(\theta\Phi_\alpha(u_n) + (1-\theta)\Phi_\alpha(u)\right)|u_n - u| \\ &\leq 2\alpha(|u_n| + |u|)\left(\Phi_\alpha(u_n) + \Phi_\alpha(u)\right)|u_n - u|.\end{aligned}$$

Thus we have

$$\int_{\mathbb{R}} |\Psi_\alpha(u_n) - \Psi_\alpha(u)| dx \le 2\alpha \int_{\mathbb{R}} (|u_n| + |u|) \left(\Phi_\alpha(u_n) + \Phi_\alpha(u)\right) |u_n - u| dx$$

(3.3)

$$\le 2\alpha \| |u_n| + |u| \|_{L^a(\mathbb{R})} \|\Phi_\alpha(u_n) + \Phi_\alpha(u)\|_{L^b(\mathbb{R})} \|u_n - u\|_{L^c(\mathbb{R})}$$

by Hölder's inequality, where $a, b, c > 1$ and $1/a + 1/b + 1/c = 1$ are chosen later.

First, direct calculation shows that

$$(\Phi_\alpha(t))^b < e^{b\alpha t^2} - 1 \quad (t \in \mathbb{R}) \tag{3.4}$$

for all $b > 1$. Thus if we fix $1 < b < \pi/\alpha$ so that $b\alpha < \pi$ is realized, then we have

$$\begin{aligned} &\|\Phi_\alpha(u_n) + \Phi_\alpha(u)\|_{L^b(\mathbb{R})}^b \le \left(\|\Phi_\alpha(u_n)\|_{L^b(\mathbb{R})} + \|\Phi_\alpha(u)\|_{L^b(\mathbb{R})}\right)^b \\ &\le 2^{b-1} \left(\int_{\mathbb{R}} (\Phi_\alpha(u_n))^b dx + \int_{\mathbb{R}} (\Phi_\alpha(u))^b dx \right) \\ &\le 2^{b-1} \left(\int_{\mathbb{R}} \left(e^{b\alpha u_n^2} - 1\right) dx + \int_{\mathbb{R}} \left(e^{b\alpha u^2} - 1\right) dx \right) \\ &\le 2^{b-1} A(b\alpha) \left(\|u_n\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 \right), \end{aligned}$$

here we used (3.4) for the third inequality and Theorem 1 for the last inequality, the use of which is valid since $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})} \le 1$ and $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1$ by the weak lower semicontinuity. Note that $\{u_n\}$ satisfies $\sup_{n\in\mathbb{N}} \|u_n\|_{H^{1/2,2}(\mathbb{R})} \le C$ for some $C > 0$. Thus we have obtained $\|\Phi_\alpha(u_n) + \Phi_\alpha(u)\|_{L^b(\mathbb{R})} = O(1)$ independent of n.

Next, we estimate the term $\| |u_n| + |u| \|_{L^a(\mathbb{R})}$. Since $\{u_n\}$ is a bounded sequence in $H^{1/2,2}(\mathbb{R})$, we have by [19] Theorem 6.9 that $\|u\|_{L^q(\mathbb{R})} \le C\|u_n\|_{H^{1/2,2}(\mathbb{R})}$ for any $q \ge 2$. Thus we see $\| |u_n| + |u| \|_{L^a(\mathbb{R})} \le C$ for some $C > 0$ independent of n for $a \ge 2$. Now, note that if we choose $1 < b < \pi/\alpha$ and $a > 2$ sufficiently large, then we can find $c > 2$ such that $1/a + 1/b + 1/c = 1$.

By these choices and Proposition 3, we conclude that $\|u_n - u\|_{L^c(\mathbb{R})} \to 0$ as $n \to \infty$. Backing to (3.3) with all together, we conclude that

$$\int_{\mathbb{R}} \Psi_\alpha(u_n) dx \to \int_{\mathbb{R}} \Psi_\alpha(u) dx \quad (n \to \infty),$$

which is the desired conclusion. □

Now, we prove Theorem 4. We will show that $A(\alpha)$ in (1.1) is attained for any $0 < \alpha < \pi$. Since $A(\alpha) = \tilde{A}(\alpha)$ by Lemma 1, we choose a maximizing sequence for $\tilde{A}(\alpha)$:

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1\right) dx = A(\alpha) + o(1) \quad (n \to \infty).$$

Here $\{u_n\}_{n\in\mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ satisfies $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})} \le 1$ and $\|u_n\|_{L^2(\mathbb{R})} = 1$. By appealing to the use of rearrangement, we may furthermore assume that u_n is nonnegative, even, and decreasing on $\mathbb{R}_+$. Since $\{u_n\}_{n\in\mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ is a bounded sequence, we have $u \in H^{1/2,2}(\mathbb{R})$ such that $u_n \rightharpoonup u$ in $H^{1/2,2}(\mathbb{R})$. By Proposition 4, we see

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 - \alpha u_n^2\right) dx = \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2\right) dx$$

as $n \to \infty$. Therefore, since $\|u_n\|^2_{L^2(\mathbb{R})} = 1$, we have, letting $n \to \infty$,

$$A(\alpha) = \alpha + \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2\right) dx. \tag{3.5}$$

Next, we claim that $A(\alpha) > \alpha$ for any $0 < \alpha < \pi$. Indeed, take any $u_0 \in H^{1/2,2}(\mathbb{R})$ such that $u_0 \not\equiv 0$, $\|(-\Delta)^{1/4} u_0\|_{L^2(\mathbb{R})} \le 1$ and $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then we have

$$A(\alpha) = \tilde{A}(\alpha) \ge \int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1\right) dx = \alpha + \int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1 - \alpha u_0^2\right) dx.$$

Now, since $e^{\alpha t^2} - 1 - \alpha t^2 > 0$ for any $t > 0$, we have

$$\int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1 - \alpha u_0^2\right) dx > 0$$

for $u_0 \not\equiv 0$, which results in $A(\alpha) > \alpha$, the claim.

By the claim and (3.5), we conclude that the weak limit u satisfies $u \not\equiv 0$. By the weak lower semi continuity, we have $u \not\equiv 0$ satisfies $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \le 1$ and $\|u\|_{L^2(\mathbb{R})} \le 1$. Thus by (3.5) again, we see

$$\begin{aligned}
A(\alpha) &= \alpha + \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2\right) dx \\
&\le \alpha + \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2\right) dx \\
&= \alpha + \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right) dx - \alpha \frac{\|u\|^2_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}} \\
&= \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right) dx.
\end{aligned}$$

Thus we have shown that $u \in H^{1/2,2}(\mathbb{R})$ maximizes $A(\alpha)$. $\square$

Next, we prove Theorem 5. We follow Ishiwata's argument in [9]. Let

$$M = \left\{ u \in H^{1/2,2}(\mathbb{R}) \,:\, \|u\|_{H^{1/2,2}(\mathbb{R})} = 1 \right\},$$
$$J_\alpha : M \to \mathbb{R}, \quad J_\alpha(u) = \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx.$$

Actually, we will show a stronger claim that J_α has no critical point on M for sufficiently small $\alpha > 0$. Assume the contrary that there exists a critical point $v \in M$ of J_α for small $\alpha > 0$. Then we define an orbit on M through v as

$$v_\tau(x) = \sqrt{\tau} v(\tau x) \quad \tau \in (0, \infty), \quad w_\tau = \frac{v_\tau}{\|v_\tau\|_{H^{1/2}}} \in M.$$

Note that $w_1 = v$ thus it must be $\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) = 0$. By scaling rules (2.2), we see for any $p \geq 2$,

$$\|v_\tau\|^p_{L^p(\mathbb{R})} = \tau^{p/2-1} \|v\|^p_{L^p(\mathbb{R})} \quad \text{and} \quad \|(-\Delta)^{1/4} v_\tau\|_{L^2(\mathbb{R})} = \tau \|(-\Delta)^{1/4} v\|_{L^2(\mathbb{R})}.$$

Now, we see

$$\begin{aligned}
J_\alpha(w_\tau) &= \int_{\mathbb{R}} \left(e^{\alpha w_\tau^2} - 1 \right) dx = \int_{\mathbb{R}} \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{v_\tau^{2j}(x)}{\left(\|v_\tau\|_2^2 + \|(-\Delta)^{1/4} v_\tau\|_2^2 \right)^j} \\
&= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{\|v_\tau\|_{2j}^{2j}}{\left(\|v_\tau\|_2^2 + \|(-\Delta)^{1/4} v_\tau\|_2^2 \right)^j} = \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{\tau^{j-1} \|v\|_{2j}^{2j}}{\left(\|v\|_2^2 + \tau \|(-\Delta)^{1/4} v\|_2^2 \right)^j} \\
&= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} f_j(\tau)
\end{aligned}$$

where $f_j(\tau) = \frac{\tau^{j-1} c}{(b+\tau a)^j}$ with $a = \|(-\Delta)^{1/4} v\|_2^2$, $b = \|v\|_2^2$ and $c = \|v\|_{2j}^{2j}$. Since

$$f_j'(\tau) = \frac{\tau^{j-2} c}{(b + \tau a)^{j+1}} \left\{ -\tau a + (j-1) b \right\}$$

and $\|(-\Delta)^{1/4}v\|_2^2 + \|v\|_2^2 = 1$, we calculate

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau)$$
$$= \sum_{j=1}^{\infty}\left[\frac{\alpha^j}{j!}\frac{\tau^{j-2}\|v\|_{2j}^{2j}}{(\|v\|_2^2+\tau\|(-\Delta)^{1/4}v\|_2^2)^{j+1}}\left\{-\tau\|(-\Delta)^{1/4}v\|_2^2+(j-1)\|v\|_2^2\right\}\right]_{\tau=1}$$
$$\le -\alpha\|(-\Delta)^{1/4}v\|_2^2\|v\|_2^2 + \sum_{j=2}^{\infty}\frac{\alpha^j}{(j-1)!}\|v\|_{2j}^{2j}$$
$$= \alpha\|(-\Delta)^{1/4}v\|_2^2\|v\|_2^2\left\{-1+\sum_{j=2}^{\infty}\frac{\alpha^{j-1}}{(j-1)!}\frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2\|v\|_2^2}\right\}.$$

Here, we need the following lemma:

Lemma 3. *(Ogawa-Ozawa [21]) There exists $C > 0$ such that for any $u \in H^{1/2,2}(\mathbb{R})$ and $p \ge 2$, it holds*

$$\|u\|_{L^p(\mathbb{R})}^p \le Cp^{p/2}\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^{p-2}\|u\|_{L^2(\mathbb{R})}^2.$$

For $p = 2j$, Lemma 3 implies

$$\frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2\|v\|_2^2} \le C(2j)^j \underbrace{\|(-\Delta)^{1/4}v\|_2^{2j-4}}_{\le 1\,(j\ge 2)} \le C(2j)^j.$$

Thus for $0 < \alpha << 1$ sufficiently small (it would be enough that $\alpha < 1/(2e)$), Stirling's formula $j! \sim j^j e^{-j}\sqrt{2\pi j}$ implies that

$$\sum_{j=2}^{\infty}\frac{\alpha^{j-1}}{(j-1)!}\frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2\|v\|_2^2} \le \sum_{j=2}^{\infty}\frac{\alpha^{j-1}}{(j-1)!}(2j)^j \le \alpha C$$

for some $C > 0$ independent of α. Therefore we have $\frac{d}{d\tau}J_\alpha(w_\tau)\Big|_{\tau=1} < 0$ for small α, which is a desired contradiction. □

4. Proof of Theorem 6.

In order to prove Theorem 6, first we set

$$F(\beta) = \sup_{\substack{u\in H^{1/2,2}(\mathbb{R})\\ \|u\|_{H^{1/2,2}(\mathbb{R})}\le 1}} \int_{\mathbb{R}} u^2 e^{\beta u^2} dx \tag{4.1}$$

for $\beta > 0$. Then we have

Proposition 5. *We have $F(\beta) < \infty$ for $\beta < \pi$*

Proof. We follow the proof of Theorem 1.5 in [12]. Take any $u \in H^{1/2,2}(\mathbb{R})$ with $\|u\|_{H^{1/2,2}(\mathbb{R})} \le 1$ in the admissible sets for $F(\beta)$ in (4.1). By appealing to the rearrangement, we may assume that u is even, nonnegative and decreasing on $\mathbb{R}_+$. We divide the integral

$$\int_{\mathbb{R}} u^2 e^{\beta u^2} dx = \int_{\mathbb{R}\setminus I} u^2 e^{\beta u^2} dx + \int_I u^2 e^{\beta u^2} dx = (I) + (II),$$

where $I = (-1/2, 1/2)$.

First, we estimate (I). By the Radial Lemma (3.2), we see for any $k \in \mathbb{N}$, $k \ge 2$,

$$u^{2k}(x) \le \left(\frac{\|u\|^2_{L^2(\mathbb{R})}}{2|x|} \right)^k = \frac{\|u\|^{2k}_{L^2(\mathbb{R})}}{2^k} \frac{1}{|x|^k} \quad \text{for} \quad x \ne 0.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}\setminus I} u^{2k}(x) dx &\le \frac{\|u\|^{2k}_{L^2(\mathbb{R})}}{2^k} \int_{\mathbb{R}\setminus I} \frac{dx}{|x|^k} \\ &= \frac{\|u\|^{2k}_{L^2(\mathbb{R})}}{2^{k-1}} \int_{1/2}^\infty \frac{dx}{x^k} = \frac{\|u\|^{2k}_{L^2(\mathbb{R})}}{k-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(I) &= \int_{\mathbb{R}\setminus I} u^2 e^{\beta u^2} dx = \int_{\mathbb{R}\setminus I} u^2 \left(1 + \sum_{k=1}^\infty \frac{\beta^k u^{2k}}{k!} \right) dx \\ &= \int_{\mathbb{R}\setminus I} u^2 dx + \sum_{k=2}^\infty \frac{\beta^{k-1}}{(k-1)!} \int_{\mathbb{R}\setminus I} u^{2k} dx \\ &\le \|u\|^2_{L^2(\mathbb{R})} + \sum_{k=2}^\infty \frac{\beta^{k-1}}{(k-1)!} \frac{\|u\|^{2k}_{L^2(\mathbb{R})}}{k-1} \\ &= \|u\|^2_{L^2(\mathbb{R})} \left(1 + \sum_{k=2}^\infty \frac{\beta^{k-1}}{(k-1)(k-1)!} \|u\|^{2(k-1)}_{L^2(\mathbb{R})} \right).\end{aligned}$$

Now by the constraint $\|u\|_{H^{1/2,2}(\mathbb{R})} \le 1$, we have $\|u\|_{L^2(\mathbb{R})} \le 1$. Also if we put $a_k = \frac{\beta^{k-1}}{(k-1)(k-1)!}$, then $\sum_{k=2}^\infty a_k$ converges since $a_{k+1}/a_k = \beta \frac{k-1}{k^2} \to 0$ as $k \to \infty$. Thus we obtain

$$(I) \le 1 + \sum_{k=2}^\infty \frac{\beta^{k-1}}{(k-1)(k-1)!} \le C$$

where $C > 0$ is independent of $u \in H^{1/2,2}(\mathbb{R})$ with $\|u\|_{H^{1/2,2}(\mathbb{R})} \le 1$.

Next, we estimate (II). Set

$$v(x) = \begin{cases} u(x) - u(1/2), & |x| \le 1/2, \\ 0, & |x| > 1/2. \end{cases}$$

Then by the argument of [12], we know that

$$\|(-\Delta)^{1/4} v\|^2_{L^2(\mathbb{R})} \le \|(-\Delta)^{1/4} u\|^2_{L^2(\mathbb{R})},$$
$$u^2(x) \le v^2(x) \left(1 + \|u\|^2_{L^2(\mathbb{R})}\right) + 2$$

for $x \in I$. Put $w = v\sqrt{1 + \|u\|^2_{L^2(\mathbb{R})}}$. Then we have $w \in \tilde{H}^{1/2,2}(I)$ since $v \equiv 0$ on $\mathbb{R} \setminus I$, and

$$\|(-\Delta)^{1/4} w\|^2_{L^2(\mathbb{R})} = \left(1 + \|u\|^2_{L^2(\mathbb{R})}\right) \|(-\Delta)^{1/4} v\|^2_{L^2(\mathbb{R})}$$
$$\le \left(1 + \|u\|^2_{L^2(\mathbb{R})}\right) \left(1 - \|u\|^2_{L^2(\mathbb{R})}\right) \le 1.$$

Thus we may use the fractional Trudinger-Moser inequality (Proposition 1) to w to obtain

$$\int_I e^{\pi w^2} dx \le C$$

for some $C > 0$ independent of u. By $u^2 \le w^2 + 2$ on I, we conclude that

$$\int_I e^{\pi u^2} dx \le \int_I e^{\pi(w^2+2)} dx = e^{2\pi} \int_I e^{\pi w^2} dx \le C'.$$

Now, since $\beta < \pi$, there is an absolute constant C_0 such that $t^2 e^{\beta t^2} \le C_0 e^{\pi t^2}$ for any $t \in \mathbb{R}$. Finally, we obtain

$$(II) = \int_I u^2 e^{\beta u^2} dx \le C_0 \int_I e^{\pi u^2} dx \le C_0 C'.$$

Proposition 5 follows from the estimates (I) and (II). □

By using Proposition 5 and arguing as in the proof of Theorem 1 (after establishing the similar claims as in Lemma 1 and Lemma 2), it is easy to obtain the following Proposition:

Proposition 6. *For any* $0 < \alpha < \beta < \pi$, *we have*

$$E(\alpha) \le \left(\frac{1}{1 - \alpha/\beta}\right) F(\beta).$$

Since $F(\beta) < \infty$ for any $\beta < \pi$, this proves the first part of Theorem 6. For the attainability of $E(\alpha)$ for $\alpha \in (0, \pi)$, it is enough to argue as in the proof of Theorem 4. We omit the details. □

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