SCALING INVARIANT HARDY TYPE INEQUALITIES WITH NON-STANDARD REMAINDER TERMS

MEGUMI SANO

Abstract. We consider the Hardy inequality on $\mathbb{R}^N$, the critical Hardy inequality on a ball, and the Rellich inequality on $\mathbb{R}^N$. These three Hardy type inequalities can be refined by adding remainder terms. Our remainder terms are expressed by a distance from the families of "virtual" extremals. A key ingredient is the critical Hardy inequality on $\mathbb{R}^N$ which was proved by Machihara, Ozawa, and Wadade [20].

1. Introduction

Let $N \geq 2$ and $1 < p < N$. The Hardy inequality

$$\int_{\mathbb{R}^N} \left| \nabla u \cdot \frac{x}{|x|} \right|^p dx \geq \left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \tag{1.1}$$

holds for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, where $\mathcal{D}^{1,p}(\mathbb{R}^N))$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla \cdot \|_{L^p(\mathbb{R}^N)}$. The inequality (1.1) is also called the Uncertainty Principle, and has many applications for the elliptic and the parabolic equations with the singular potential (see [6], [4] etc.). In the higher-order generalization of (1.1), for $2 \leq k < kp < N$, the inequality

$$|u|^p_{k,p} \geq C^p_{k,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}} dx \tag{1.2}$$

holds for all $u \in \mathcal{D}^{k,p}(\mathbb{R}^N)$ (see [24], [9], [22]). Here we set

$$|u|^p_{k,p} = \begin{cases} \int_{\mathbb{R}^N} |\Delta^m u|^p \, dx & \text{if } k = 2m, \\ \int_{\mathbb{R}^N} |\nabla(\Delta^m u)|^p \, dx & \text{if } k = 2m+1, \end{cases}$$

$$C_{k,p} = \begin{cases} p^{-2m} \prod_{j=1}^m \{N - 2pj\}\{N(p-1) + 2p(j-1)\} & \text{if } k = 2m, \\ \frac{(N-p)}{p^{2(m+1)}} \prod_{j=1}^m (N - (2j+1)p) \{N(p-1) + (2j-1)p\} & \text{if } k = 2m+1, \end{cases}$$

for $k, m \in \mathbb{N}, m \geq 1$. For the sake of simplicity, we define $C_{0,p} = 1$ and $C_{1,p} = \frac{N-p}{p}$.

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For (1.1), it is known that the constant $(\frac{N-p}{p})^p$ is optimal, and equality of (1.1) is not achieved unless $u \equiv 0$. Furthermore, Cianchi-Ferone [8] provided a remainder term of (1.1) as follows: Let $p^* = \frac{Np}{N-p}$, $v_a(x) = a|x|^{-\frac{N-p}{p}}$ for $x \in \mathbb{R}^N$, $a \in \mathbb{R}$ and $L^{\rho,\sigma}(\mathbb{R}^N)$ $(0 < \rho \le \infty, 1 \le \sigma \le \infty)$ is the Lorentz space. Then there exists a constant $C = C(p, N)$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \left(1 + C \left(\inf_{a\in\mathbb{R}} \frac{\|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^N)}}{\|u\|_{L^{p^*,p}(\mathbb{R}^N)}}\right)^{2p^*}\right) \tag{1.3}$$

holds for every real-valued weakly differentiable function u in $\mathbb{R}^N$ decaying to zero at infinity with $|\nabla u| \in L^p(\mathbb{R}^N)$.

This type remainder term expresses not only the absence of extremal of (1.1), but also the cause of that. Indeed, the improved Hardy inequality (1.3) says that if there exists a extremal $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ of (1.1), then $u = v_a$ for some $a \in \mathbb{R}$. However $v_a \notin \mathcal{D}^{1,p}(\mathbb{R}^N)$ (note that $v_a \in L^{p^*,\infty}(\mathbb{R}^N)$) which yields a contradiction. Here we call v_a "virtual" extremal of (1.1). In the paper [8], the proof of (1.3) is based on the rearrangement theory which is well suited for the Hardy inequality (1.1). On the other hand, it is not suited for the Rellich inequality (1.2). Therefore it seems difficult to obtain remainder term of (1.2) by using same way as [8]. One of our aims is to provide a remainder term of (1.2). Our method is quite different from theirs in [8]. In our proofs, there are two key ingredients. One is the magical computation via the transformation (3.1) (resp. (4.2)) using a virtual extremal of the inequality (1.1) (resp. (1.2)). This idea was implicitly used in [21] or [6]. The other is the critical Hardy inequality on the whole space which was proved by Machihara, Ozawa, and Wadade :

$$\int_{\mathbb{R}^N} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|x|^N |\log \frac{R}{|x|}|^\alpha} dx \le \left(\frac{\beta}{\alpha - 1}\right)^\beta \int_{\mathbb{R}^N} \frac{\left|\nabla f(x) \cdot \frac{x}{|x|}\right|^\beta}{|x|^{N-\beta} \left|\log \frac{R}{|x|}\right|^{\alpha-\beta}} dx. \tag{1.4}$$

We call (1.4) Machihara-Ozawa-Wadade's inequality (briefly, MOW inequality) in this paper. The crucial point of MOW inequality is taking away its boundary value $f(R\frac{x}{|x|})$ from $f(x)$ to weaken the singularity of the logarithmic term $|\log \frac{R}{|x|}|^{-\alpha}$. Actually, its boundary value of MOW inequality plays the role of virtual extremal for improvements of Hardy type inequalities. Our main results are as follows:

Theorem 1. *(Improved Hardy inequality on $\mathbb{R}^N$) Let $2 \le p < N$. Set $v_a(x) := a|x|^{-\frac{N-p}{p}}$ for $a \in \mathbb{R}, x \in \mathbb{R}^N$ and*

$$d_H(u; R) := \left(\int_{\mathbb{R}^N} \frac{|u(x) - v_a(x)|^p}{|x|^p |\log \frac{R}{|x|}|^p} dx\right)^{\frac{1}{p}}.$$

Then there exists a constant $C > 0$ such that the inequality

$$\int_{\mathbb{R}^N} \left| \nabla u \cdot \frac{x}{|x|} \right|^p dx - \left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \geq C \sup_{R>0} d_H(u;R)^p. \tag{1.5}$$

holds for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, where $a = a(x,u,R) = R^{\frac{N-p}{p}} u\left(R\frac{x}{|x|}\right)$.

Concerning improvement on a bounded domain, see [1], [6], [7], [11], [14], to name a few.

Remark 2. We can check that the distance $d_H(u;R)$ in Theorem 1 is well-defined. Indeed, set

$$d_H(u;R)^p = \int_{B_{R/2}(0)\cup B^c_{2R}(0)} + \int_{B_{2R}(0)\setminus B_{R/2}(0)} =: I_1 + I_2.$$

We see that $u(x)\,|x|^{\frac{N-p}{p}} \leq C$ near $|x| = 0, \infty$, since $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and the Sobolev embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-p}}(\mathbb{R}^N)$ holds. Thus we obtain

$$I_1 = \int_{B_{R/2}(0)\cup B^c_{2R}(0)} \frac{\left| u(x)|x|^{\frac{N-p}{p}} - a \right|^p}{|x|^N |\log \frac{R}{|x|}|^p} dx \leq \int_{B_{R/2}(0)\cup B^c_{2R}(0)} \frac{C}{|x|^N |\log \frac{R}{|x|}|^p} dx < \infty.$$

On the other hand, by the elementary inequality $\log x \geq \frac{x-1}{x}$ for $x \in [1,+\infty)$ and the mean value property we have

$$\begin{aligned} I_2 &\leq \left(\frac{2}{R}\right)^N \int_{B_{2R}(0)\setminus B_{R/2}(0)} \frac{R^p \left| u(x)|x|^{\frac{N-p}{p}} - u(R\frac{x}{|x|})R^{\frac{N-p}{p}} \right|^p}{|\,|x| - R\,|^p} dx \\ &\leq C \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty. \end{aligned}$$

Therefore the distance $d_H(u;R)$ is well-defined for $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$.

For derivative term $\int_{\mathbb{R}^N} \left| \nabla u \cdot \frac{x}{|x|} \right|^p dx$, we do not know whether to apply rearrangement theory (actually, the Pólya-Szegö inequality) or not. Therefore one of good points of (1.5) is that we can obtain a remainder term of the Hardy inequality with its derivative term.

For (1.2), it is known that the constant $C^p_{k,p}$ is optimal. We also have the following.

Theorem 3. *(Improved Rellich inequality on $\mathbb{R}^N$) Let $N, k \in \mathbb{N}$ satisfy $N, k \geq 2$ and $k < kp < N$. Set $w_a(x) := a|x|^{-\frac{N-kp}{2}}$ and*

$$d_{RE}(u;R) := \left(\int_{\mathbb{R}^N} \frac{\left| |u(x)|^{\frac{p-2}{2}} u(x) - w_a(x) \right|^2}{|x|^{kp} |\log \frac{R}{|x|}|^2} dx \right)^{\frac{1}{2}}.$$

Then there exists a constant $C > 0$ such that the inequality

$$|u|_{k,p} - C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx \geq C \sup_{R>0} d_{RE}(u;R)^2 \tag{1.6}$$

holds for all radial functions $u \in \mathcal{D}^{k,p}(\mathbb{R}^N)$, where $a = R^{\frac{N-kp}{2}} |u(R)|^{\frac{p-2}{2}} u(R)$.

We can obtain the non-radially symmetric case for $k = 2$.

corollary 4. *Let $k = 2$. If $u \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ is a non-radial function, then it holds*

$$\int_{\mathbb{R}^N} |\Delta u|^p\, dx - C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\, dx \geq C \sup_{R>0} d_{RE}(\tilde{u};R)^2 \tag{1.7}$$

where $\tilde{u}(x) = \int_{\mathbb{R}^N} \frac{(-\Delta u)^\#(y)}{|x-y|^{N-2}}\, dy \,(\in \mathcal{D}^{2,p}(\mathbb{R}^N))$, where $a = R^{\frac{N-kp}{2}} |\tilde{u}(R\frac{x}{|x|})|^{\frac{p-2}{2}} \tilde{u}(R\frac{x}{|x|})$.

Especially, if there exists a extremal $u \in \mathcal{D}^{2,p}(\mathbb{R}^N)$, then $\tilde{u}(x) = a|x|^{-\frac{N-2p}{p}} \notin \mathcal{D}^{2,p}(\mathbb{R}^N)$. Therefore the equality of (1.2) is not achieved unless $u \equiv 0$.

Concerning improvement on a bounded domain, see [2], [3], [5], [10], [12], [15], [29] and the references therein. And also for the another type improvement on $\mathbb{R}^N$, see [25].

This paper is organized as follows: In §2, we state preliminaries to show our results. In §3, we give the proof of Theorem 1. In §4, we prove Theorem 3 and Corollary 4. In §5, we discuss the critical Hardy inequality (5.1) with a remainder term (Theorem 10).

We fix several notations: $B_R(0)$ is a ball centered 0 with radius R in $\mathbb{R}^N$. ω_N is the area of a unit sphere in $\mathbb{R}^N$. $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$. The Schwarz symmetrization $u^\# : \mathbb{R}^N \to [0,\infty]$ is given by $u^\#(|x|) = \inf \left\{ \tau > 0 \,:\, |\{x \in \mathbb{R}^N \,:\, |u(x)| > \tau\}| \leq |B_{|x|}(0)| \right\}$. $\mathcal{D}^{k,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $|\cdot|_{k,p}$. Throughout the paper, if u is a radial function in $\mathbb{R}^N$, then we can write as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$ in $\mathbb{R}_+$. Then we write $u(x) = u(|x|)$ with admitting some ambiguity. We hope no confusion occurs by this abbreviation. And also, we use C as a general constant.

2. Preliminaries

The critical Hardy inequalities on $\mathbb{R}^N$ were proved by Machihara, Ozawa, and Wadade [20] by using only integration by parts and Hölder's inequality.

Theorem 5. *([20] Theorem 1.1.) Let $N \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha-1\} < \beta < \infty$. Then for any $R > 0$, the inequalities*

$$\int_{\mathbb{R}^N} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|x|^N |\log\frac{R}{|x|}|^\alpha}\, dx \leq \left(\frac{\beta}{\alpha-1}\right)^\beta \int_{\mathbb{R}^N} \frac{\left|\nabla f(x) \cdot \frac{x}{|x|}\right|^\beta}{|x|^{N-\beta} \left|\log\frac{R}{|x|}\right|^{\alpha-\beta}}\, dx \tag{2.1}$$

hold for all $f \in W^1 L_{N,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$*, where the embedding constant* $\left(\frac{\beta}{\alpha-1}\right)^\beta$ *in (2.1) is best-possible.*

Here, $L_{p,q,\lambda}(\mathbb{R}^N)$ is the Lorentz-Zygmund space and $W^1 L_{p,q,\lambda}(\mathbb{R}^N)$ is the Sobolev-Lorentz-Zygmund space. On their definitions, see [20]. In [20], they proved (2.1) for $f \in W^1 L_{N,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$. However, we need (2.1) for $f \in C_0(\mathbb{R}^N) \cup C^1(\mathbb{R}^N \setminus \{0\})$ with only $\nabla f \in L_{N,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$ to prove our theorems. In fact, we can obtain Lemma 6 by the minor change in their proof.

Lemma 6. *(Machiha-Ozawa-Wadade's inequality) Let* $N \in \mathbb{N}$*,* $1 < \alpha < \infty$ *and* $\max\{1, \alpha - 1\} < \beta < \infty$*. Then for any* $R > 0$*, the inequalities (2.1) hold for all* $f \in C_0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ *with* $\nabla f \in L_{N,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$*.*

To show our Theorems, We provide two point-wise estimates of $|a - b|^p$ from below. First, we prepare the following point-wise estimate for $p \geq 1$. We omit the proof.

Lemma 7. *Let* $p \geq 1$ *and* $a, b \in \mathbb{R}$*. Then the inequality*

$$|a - b|^p - |a|^p \geq -p|a|^{p-2}ab \tag{2.2}$$

holds true.

In $p \geq 2$ case, it is known a better estimate (2.3) than former one (2.2). Here, we provide the simple proof of (2.3).

Lemma 8. *Let* $p \geq 2$*. Then there exists a constant* $C = C(p) > 0$ *such that*

$$|a - b|^p - |a|^p \geq -p|a|^{p-2}ab + C|b|^p \tag{2.3}$$

holds true for $a, b \in \mathbb{R}$*.*

Proof of Lemma 8. Set

$$f(t) = |1 - t|^p - |t|^p + p|t|^{p-2}t \quad (t \in \mathbb{R}).$$

It is enough to show

$$f(t) \geq C > 0, \tag{2.4}$$

since (2.3) follows from (2.4), on taking $t = \frac{a}{b}$. When $t \geq 1$, the mean value theorem for the function x^{p-2} which is defined for $x \geq 0$ yields that

$$f'(t) = p\,[(t-1)^{p-1} - t^{p-1}] + p(p-1)t^{p-2} = p(p-1)[t^{p-2} - s^{p-2}] \geq 0,$$

for $p \geq 2$, where $s \geq 0$ satisfies $t - 1 \leq s \leq t$. Hence we obtain

$$f(t) \geq f(1) = p - 1 \quad \text{for all} \quad t \geq 1. \tag{2.5}$$

In the same manner as above, we also obtain

$$f(t) \geq f(0) = 1 \quad \text{for all} \quad t \leq 0. \tag{2.6}$$

When $0 \le t \le 1$, we define $C_p = \min_{0\le t\le 1} f(t) \left(= \min_{0\le t\le 1}((1-t)^p - t^p + pt^{p-1})\right)$. Let $0 \le a \le 1$ satisfy $C_p = f(a)$. From Lemma 7, we observe that $C_p \ge 0$. If $C_p = 0$, then the following equalities hold

$$0 = f(a) = (1-a)^p - a^p + pa^{p-1} \quad \text{and}$$
$$0 = \frac{a-1}{p} f'(a) = (1-a)^p - a^{p-1}(a-1) + (p-1)a^{p-2}(1-a)$$

which implies $a = 0$. However this contradicts to $f(0) = 1$. We can also derive a contradiction when $p = 2$. Thus we obtain

$$f(t) \ge C_p > 0 \quad \text{for all} \quad 0 \le t \le 1. \tag{2.7}$$

Consequently, from (2.5), (2.6), and (2.7), we obtain Lemma 8. □

3. The Hardy inequality

In this section, We prove Theorem 1.

Proof of Theorem 1. [Step 1] Let $x = r\omega (r > 0, \omega \in S^{N-1})$. First, we show that the inequality (1.5) holds for a smooth function $u = u(r\omega) \in C_0^\infty(\mathbb{R}^N)$. We consider the following transformation:

$$v(r\omega) = r^{\frac{N-p}{p}} u(r\omega), \quad \text{where } r \in [0,\infty), \omega \in S^{N-1}. \tag{3.1}$$

Note that $v(0) = \lim_{r\to\infty} v(r\omega) = 0$ for $\omega \in S^{N-1}$ since the support of u is compact. Now, direct calculation shows that

$$\begin{aligned}
I &:= \int_{\mathbb{R}^N} \left|\nabla u \cdot \frac{x}{|x|}\right|^p dx - \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \\
&= \int_{S^{N-1}} \int_0^\infty \left|-\frac{\partial}{\partial r} u(r\omega)\right|^p r^{N-1} - \left(\frac{N-p}{p}\right)^p |u(r\omega)|^p r^{N-p-1}\, dr dS_\omega \\
&= \int_{S^{N-1}} \int_0^\infty \left|\frac{N-p}{p} r^{-\frac{N}{p}} v(r\omega) - r^{-\frac{N-p}{p}} \frac{\partial}{\partial r} v(r\omega)\right|^p r^{N-1} - \left(\frac{N-p}{p}\right)^p |v(r\omega)|^p r^{-1}\, dr dS_\omega.
\end{aligned}$$

Applying Lemma 8 with the choice $a = \dfrac{N-p}{p} r^{-\frac{N}{p}} v(r\omega)$ and $b = r^{-\frac{N-p}{p}} \dfrac{\partial}{\partial r} v(r\omega)$, and using the fact $\int_0^\infty |v|^{p-2} v \left(\frac{\partial}{\partial r} v\right) dr = 0$, we have

$$\begin{aligned}
I &\ge \int_{S^{N-1}} \int_0^\infty -p\left(\frac{N-p}{p}\right)^{p-1} |v(r\omega)|^{p-2} v(r\omega) \frac{\partial}{\partial r} v(r\omega) + C \left|\frac{\partial}{\partial r} v(r\omega)\right|^p r^{p-1}\, dr dS_\omega \\
&= C \int_{\mathbb{R}^N} |x|^{p-N} \left|\nabla v \cdot \frac{x}{|x|}\right|^p dx.
\end{aligned} \tag{3.2}$$

Now, we apply Lemma 6 for $f = v \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C_0(\mathbb{R}^N)$ in $\alpha = \beta = p$ case, and combine this with (3.2), we obtain

$$I \geq C \int_{\mathbb{R}^N} \frac{|v(x) - v(R\frac{x}{|x|})|^p}{|x|^N |\log \frac{R}{|x|}|^p}\, dx = C \int_{S^{N-1}} \int_0^\infty \frac{|v(r\omega) - v(R\omega)|^p}{r |\log \frac{R}{r}|^p}\, dr dS_\omega$$

(3.3)

$$= C \int_{S^{N-1}} \int_0^\infty \frac{|u(r\omega) - R^{\frac{N-p}{p}} u(R\omega) r^{-\frac{N-p}{p}}|^p}{r^{1+p-N} |\log \frac{R}{r}|^p}\, dr dS_\omega$$

for any $R > 0$. Therefore we have (1.5) for $u \in C_0^\infty(\mathbb{R}^N)$.

[Step 2] In this step, we prove (1.5) for $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ by using same argument as it in [20]. Let $\{u_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ be a sequence such that $u_m \to u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as $m \to \infty$. Then there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ such that

$$\begin{aligned} u_m &\to u \quad \text{in } L^{\frac{Np}{N-p}}(\mathbb{R}^N), \\ u_m &\to u \quad \text{in } L^p(\mathbb{R}^N; |x|^{-p} dx), \\ u_{m_j} &\to u \quad \text{a.e. in } \mathbb{R}^N \end{aligned}$$

by Sobolev embedding and Hardy inequality (1.1). Here, we define

$$\tilde{u}_R(x) := \frac{u(x) - a|x|^{-\frac{N-p}{p}}}{|x| |\log \frac{R}{|x|}|}$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $R > 0$, where $a = R^{\frac{N-p}{p}} u(R\frac{x}{|x|})$. Since the inequality (3.3) holds for $u_m - u_j \in C_0^\infty(\mathbb{R}^N)$, we can observe that $\{\widetilde{(u_{m_j})}_R\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Hence there exists a function $f \in L^p(\mathbb{R}^N)$ such that $\widetilde{(u_{m_j})}_R \to f$ in $L^p(\mathbb{R}^N)$ as $m \to \infty$. Since $u_{m_j} \to u$ a.e. in $\mathbb{R}^N$, we can see that $\tilde{u}_R = f$. Therefore the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \geq C \int_{\mathbb{R}^N} \frac{|u(x) - a|x|^{-\frac{N-p}{p}}|^p}{|x|^p |\log \frac{R}{|x|}|^p} dx$$

holds for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $R > 0$. □

Remark 9. In the case $1 < p < 2$, instead of the point-wise estimate (2.3), the following inequality is known

$$|a - b|^p - |a|^p \geq -p|a|^{p-2}ab + C\frac{|b|^2}{(|a-b| + |a|)^{2-p}} \tag{3.4}$$

for $a, b \in \mathbb{R}$ (see e.g., [19]). If we use the point-wise estimate (3.4), we also have the case $1 < p < 2$ of Theorem 1, but we omit here.

4. The Rellich inequality

In this section, we discuss the improvement of the Rellich inequality (1.2). And also, by using this improvement, we show the non-existence of extremals of (1.2). In the proof of Theorem 3, it is enough to use the weaken estimate (2.2) than (2.3).

Proof of Theorem 3. From (i) Step 2 in the proof of Theorem 1 and higher order Sobolev embedding $\mathcal{D}^{k,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-kp}}(\mathbb{R}^N)$ (see [13]), it is enough to prove the inequality (1.5) for radial functions $u = u(r) \in C_0^\infty(\mathbb{R}^N)$ where $r = |x|$.

First, note that the inequality

$$|u|^p_{k,p} = |\Delta u|^p_{k-2,p} \geq C^p_{k-2,p} \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \tag{4.1}$$

holds from Rellich's inequality (1.2). Actually when $k = 2$, this is the equality. Here, we consider the following transformation:

$$v(r) = r^{\frac{N-kp}{p}} u(r), \quad \text{where } r \in [0, \infty). \tag{4.2}$$

Note that $v(0) = 0$ and also $v(+\infty) = 0$ since the support of u is compact. For $k \geq 2, k \in \mathbb{N}$ and $k < kp < N$, put

$$\theta_k = \theta(k, N, p) = 2k + \frac{N(p-2)}{p} \quad \text{and} \quad \Delta_{\theta_k} f = f''(r) + \frac{\theta_k - 1}{r} f'(r)$$

for smooth radial functions $f = f(r)$. Define

$$A_{k,p} = \frac{(N-kp)[(k-2)p + (p-1)N]}{p^2}.$$

Then direct calculation shows that $-\Delta u = r^{k-2-\frac{N}{p}} \left(A_{k,p} v(r) - r^2 \Delta_{\theta_k} v(r)\right)$. Now applying Lemma 7 with the choice $a = A_{k,p} v(r)$ and $b = r^2 \Delta_{\theta_k} v(r)$, and using the fact $\int_0^\infty |v|^{p-2}vv' dr = 0$ since $v(0) = v(+\infty) = 0$, we have

$$
\begin{aligned}
J &:= \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\,dx - A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx \\
&= \omega_N \int_0^\infty |-\Delta u(r)|^p\, r^{N-1-(k-2)p}\,dr - A_{k,p}^p \omega_N \int_0^\infty |u(r)|^p r^{N-kp-1}\,dr \\
&= \omega_N \int_0^\infty \left(\left| A_{k,p} v(r) - r^2 \Delta_{\theta_k} v(r) \right|^p - (A_{k,p} v(r))^p \right) r^{-1}\,dr \\
&\geq -p\omega_N A_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v \Delta_{\theta_k} v\, r\,dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v \left(v'' + \frac{\theta_k - 1}{r} v' \right) r\,dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |v|^{p-2} v v'' r\,dr. \qquad (4.3)
\end{aligned}
$$

Moreover we observe that

$$
\begin{aligned}
-\int_0^\infty |v|^{p-2} v v'' r\,dr &= (p-1) \int_0^\infty |v|^{p-2} (v')^2 r\,dr + \int_0^\infty |v|^{p-2} v v'\,dr \\
&= \frac{4(p-1)}{p^2} \int_0^\infty |(\,|v|^{\frac{p-2}{2}} v)'|^2 r\,dr \\
&= \frac{4(p-1)}{p^2 \omega_2} \int_{\mathbb{R}^2} \left| \nabla(\,|v|^{\frac{p-2}{2}} v) \cdot \frac{x}{|x|} \right|^2 dx. \qquad (4.4)
\end{aligned}
$$

Now, we apply Lemma 2.1 for $|v|^{\frac{p-2}{2}} v \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C_0(\mathbb{R}^N)$ in $\alpha = \beta = N = 2$ case, and combine this to (4.3) and (4.4), we obtain

$$
\begin{aligned}
J &\geq C \int_{\mathbb{R}^2} \frac{\left| |v(x)|^{\frac{p-2}{2}} v(x) - |v(R\frac{x}{|x|})|^{\frac{p-2}{2}} v(R\frac{x}{|x|}) \right|^2}{|x|^2 |\log \frac{R}{|x|}|^2}\,dx \\
&= C \int_0^\infty \frac{\left| |v(r)|^{\frac{p-2}{2}} v(r) - |v(R)|^{\frac{p-2}{2}} v(R) \right|^2}{r |\log \frac{R}{r}|^2}\,dr \\
&= C \int_0^\infty \frac{\left| |u(r)|^{\frac{p-2}{2}} u(r) - R^{\frac{N-kp}{2}} |u(R)|^{\frac{p-2}{2}} u(R) r^{-\frac{N-kp}{2}} \right|^2}{r^{1-N+kp} |\log \frac{R}{r}|^2}\,dr \qquad (4.5)
\end{aligned}
$$

for any $R > 0$. Consequently, from (4.1), (4.5) and $C_{k-2,p}A_{k,p} = C_{k,p}$, we obtain

$$\begin{aligned}|u|_{k,p} &\geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \\ &= C_{k-2,p}^p \left(J + A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx \right) \\ &\geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx + C \sup_{R>0} \int_{\mathbb{R}^N} \frac{\left| |u(x)|^{\frac{p-2}{2}} u(x) - a|x|^{-\frac{N-kp}{2}} \right|^2}{|x|^{kp} | \log \frac{R}{|x|} |^2}\, dx,\end{aligned}$$

where $a = R^{\frac{N-kp}{2}} |u(R)|^{\frac{p-2}{2}} u(R)$. The proof of Theorem 3 is now complete. □

For the Rellich inequality (1.2), we can not apply the rearrangement techniques, namely, Hardy-Littlewood inequality and Pólya-Szegö inequality directly. However, thanks to Talenti's comparison principle [28], we can prove Corollary 4 by using Theorem 3.

Proof of Corollary 4. Let $u \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ be a non-radial function. Set $f := -\Delta u \in L^p(\mathbb{R}^N)$ and $\tilde{u}(x) := \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}}\, dy$. Note that $\tilde{u}$ is a radial function, since $\tilde{u}(Ox) = \tilde{u}(x)$ for any $O \in O(N)$, where $O(N)$ is the group of orthogonal matrices in $\mathbb{R}^N$. By $f^\# \in L^p(\mathbb{R}^N)$ and the Calderon-Zygmund inequality (see [16] Theorem 9.9.), we obtain that $\tilde{u} \in \mathcal{D}^{2,p}(\mathbb{R}^N)$, and $\tilde{u}$ satisfies $-\Delta \tilde{u} = f^\#$ a.e. in $\mathbb{R}^N$. Therefore we have

$$||\Delta \tilde{u}||_p = ||\Delta u||_p. \tag{4.6}$$

By Talenti's comparison principle [28], we know that $\tilde{u} \geq u^\# \geq 0$. Hence we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\tilde{u}|^\beta |x|^\gamma\, dx &\geq \int_{\mathbb{R}^N} |u^\#|^\beta |x|^\gamma\, dx \quad \text{if } \beta \geq 0 \\ &\geq \int_{\mathbb{R}^N} |u|^\beta |x|^\gamma\, dx \quad \text{if } \beta \geq 0 \text{ and } \gamma \leq 0.\end{aligned} \tag{4.7}$$

where second inequality comes from the Hardy-Littlewood inequality (see e.g., [23]). From (4.6) and (4.7), we obtain

$$\int_{\mathbb{R}^N} |\Delta u|^p\, dx - C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\, dx \geq \int_{\mathbb{R}^N} |\Delta \tilde{u}|^p\, dx - C_{2,p}^p \int_{\mathbb{R}^N} \frac{|\tilde{u}|^p}{|x|^{2p}}\, dx.$$

Thus, by using theorem 3 for the radial function $\tilde{u}$, we obtain (1.7).

□

5. The critical Hardy inequality

On the critical case $p = N$, the Hardy inequality (1.1) fails for every constant and instead of (1.1) the critical Hardy inequality

$$\int_{B_R(0)} \left| \nabla u(x) \cdot \frac{x}{|x|} \right|^N dx \geq \left(\frac{N-1}{N} \right)^N \int_{B_R(0)} \frac{|u(x)|^N}{|x|^N (\log \frac{R}{|x|})^N} dx \tag{5.1}$$

holds for all $u \in W_0^{1,N}(B_R(0))$ (see [17], [27] etc.). Note that the inequality (5.1) is not invariant under the standard scaling $u_\lambda(x) = u(\lambda x)$ due to the logarithmic term. However the following scaling is introduced

$$u_\lambda(x) = \lambda^{-\frac{N-1}{N}} u \left(\left(\frac{|x|}{R} \right)^{\lambda - 1} x \right) \tag{5.2}$$

for $\lambda > 0$. Under this scaling (5.2), the critical Hardy inequality (5.1) has the scale invariance (see [17]). Furthermore it is known that the optimal constant $(\frac{N-1}{N})^N$ is not attained unless $u \equiv 0$ and the function $\left(\log \frac{R}{|x|}\right)^{\frac{N-1}{N}}$ is virtual extremal of (5.1) (see [17]).

In this section, we refine the critical Hardy inequality (5.1) on $B_R(0)$ by adding the non-standard remainder term.

Theorem 10. *(Improved critical Hardy inequality on $B_R(0)$) Let $N \geq 2$. Set $x_a(x) := a(\log \frac{R}{|x|})^{\frac{N-1}{N}}$ and*

$$d_{CH}(u; T) := \left(\int_{B_R(0)} \frac{|u(x) - x_a(x)|^N}{|x|^N |\log \frac{R}{|x|}|^N |\log(T \log \frac{R}{|x|})|^N} dx \right)^{\frac{1}{N}}.$$

Then there exists a constant C such that the inequality

$$\int_{B_R(0)} \left| \nabla u(x) \cdot \frac{x}{|x|} \right|^N dx - \left(\frac{N-1}{N} \right)^N \int_{B_R(0)} \frac{|u(x)|^N}{|x|^N (\log \frac{R}{|x|})^N} dx \geq C \sup_{T>0} d_{CH}(u; T)^N \tag{5.3}$$

holds for all $u \in W_0^{1,N}(B_R(0))$, where $a = T^{\frac{N-1}{N}} u(Re^{-\frac{1}{T}} \frac{x}{|x|})$.

For the another type remainder term to (5.1) on a ball, see [18], [26].

It is difficult to show Theorem 10 by applying the transformation used only the virtual extremal and Lemma 6. One of the reasons is that the MOW inequality (2.1) can not treat the term $\int \left| \nabla v \cdot \frac{x}{|x|} \right|^N \left(\log \frac{R}{|x|} \right)^{N-1} dx$. Therefore we introduce the transformation (5.4) to change the remainder term from $\int \left| \nabla v \cdot \frac{x}{|x|} \right|^N \left(\log \frac{R}{|x|} \right)^{N-1} dx$ to $\int \left| \nabla v \cdot \frac{x}{|x|} \right|^N dx$.

Proof of Theorem 10. From (i) Step 2 in the proof of Theorem 1 and the Poincaré inequality $W_0^{1,N}(B_R(0)) \hookrightarrow L^N(B_R(0))$, it is enough to prove the inequality (5.3) for $u = u(r\omega) \in C_0^\infty(B_R(0))$ where $r = |x|$. We consider the following transformation

$$(5.4)\quad v(s\omega) = \left(\log \frac{R}{r}\right)^{-\frac{N-1}{N}} u(r\omega), \text{ where } s = s(r) = \left(\log \frac{R}{r}\right)^{-1}, \omega \in S^{N-1}.$$

Note that $v(0) = 0$ and v has a compact support since $u \in C_0^\infty(B_R(0))$. Then direct calculation shows that

$$\frac{\partial}{\partial r}u(r) = -\left(\frac{N-1}{N}\right)\left(\log \frac{R}{r}\right)^{-\frac{1}{N}} \frac{v(s\omega)}{r} + \left(\log \frac{R}{r}\right)^{\frac{N-1}{N}} \frac{\partial}{\partial s}v(s\omega)s'(r).$$

Set

$$(5.5)\qquad K := \int_{B_R(0)} \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx - \left(\frac{N-1}{N}\right)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N(\log \frac{R}{|x|})^N} dx.$$

Then we have

$$\begin{aligned} K &= \int_{S^{N-1}} \int_0^R \left|\frac{\partial}{\partial r}u(r\omega)\right|^N r^{N-1} - \left(\frac{N-1}{N}\right)^N \frac{|u(r\omega)|^N}{r(\log \frac{R}{r})^N} dr dS_\omega \\ &= \int_{S^{N-1}} \int_0^R \left|\frac{N-1}{N}\left(r\log \frac{R}{r}\right)^{-\frac{1}{N}} v(s\omega) - \left(r\log \frac{R}{r}\right)^{\frac{N-1}{N}} \frac{\partial}{\partial s}v(s\omega)s'(r)\right|^N \\ &\qquad\qquad - \left(\frac{N-1}{N}\right)^N \frac{|v(s\omega)|^N}{r\log \frac{R}{r}} dr dS_\omega. \end{aligned}$$

Here, we can apply Lemma 8 with the choice

$$a = \frac{N-1}{N}\left(r\log \frac{R}{r}\right)^{-\frac{1}{N}} v(s\omega) \quad \text{and} \quad b = \left(r\log \frac{R}{r}\right)^{\frac{N-1}{N}} \frac{\partial}{\partial s}v(s\omega)s'(r).$$

By using the boundary conditions $v(0) = \lim_{r\to\infty} v(r\omega) = 0$ and (5.5), we obtain

$$\begin{aligned} K &\geq \int_{S^{N-1}} \int_0^R -N\left(\frac{N-1}{N}\right)^{N-1} |v(s\omega)|^{N-2}v(s\omega)\frac{\partial}{\partial s}v(s\omega)s'(r) \\ &\qquad + C\left|\frac{\partial}{\partial s}v(s\omega)\right|^N (s'(r))^N \left(r\log \frac{R}{r}\right)^{N-1} dr dS_\omega \\ &= \int_{S^{N-1}} \int_0^\infty -N\left(\frac{N-1}{N}\right)^{N-1} |v(s\omega)|^{N-2}v(s\omega)\frac{\partial}{\partial s}v(s)\, ds + C\left|\frac{\partial}{\partial s}v(s\omega)\right|^N s^{N-1}\, ds dS_\omega \end{aligned}$$

(5.6)

$$= C\int_{\mathbb{R}^N} \left|\nabla v \cdot \frac{x}{|x|}\right|^N dx.$$

Now, we apply Lemma 6 for $v \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C_0(\mathbb{R}^N)$ in $\alpha = \beta = N$ case, and combine this with (5.6), we obtain

$$
\begin{aligned}
K &\geq C \int_{\mathbb{R}^N} \frac{|v(x) - v(T\frac{x}{|x|})|^N}{|x|^N |\log \frac{T}{|x|}|^N} dx = C \int_{S^{N-1}} \int_0^\infty \frac{|v(s\omega) - v(T\omega)|^N}{s|\log \frac{T}{s}|^N} dsdS_\omega \\
&= C \int_{S^{N-1}} \int_0^R \frac{\left| \left(\log \frac{R}{r}\right)^{-\frac{N-1}{N}} u(r\omega) - T^{\frac{N-1}{N}} u(Re^{-\frac{1}{T}}\omega) \right|^N}{r(\log \frac{R}{r}) |\log \left(T \log \frac{R}{r}\right)|^N} drdS_\omega \\
&= C \int_{S^{N-1}} \int_0^R \frac{\left| u(r\omega) - T^{\frac{N-1}{N}} u(Re^{-\frac{1}{T}}\omega)(\log \frac{R}{r})^{\frac{N-1}{N}} \right|^N}{r(\log \frac{R}{r})^N |\log \left(T \log \frac{R}{r}\right)|^N} drdS_\omega.
\end{aligned}
$$

Therefore the inequality

$$
K \geq C \int_{B_R(0)} \frac{|u(x) - a(\log \frac{R}{|x|})^{\frac{N-1}{N}}|^N}{|x|^N |\log \frac{R}{|x|}|^N |\log(T \log \frac{R}{|x|})|^N} dx
$$

holds for any $T > 0$. The proof of Theorem 10 is now complete. □

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Department of Mathematics, Graduate School of Science, Osaka City University, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail address: `megumisano0609@st.osaka-cu.ac.jp`