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# On the generalized critical Hardy inequalities with the optimal constant

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**Abstract** On a bounded domain $\Omega$, we consider the minimization problem associated with the optimal constant of generalized critical Hardy inequalities in the boundary singularity case and other cases. Especially, in the boundary singularity case, we show that the validity of the inequality depends on the sharpness of the corner of the domain $\Omega$. We also reveal the explicit optimal constant and its minimizers of the inequalities on balls.

**Keywords** Hardy inequality · Limiting case · Minimization problem · Boundary singularity

**Mathematics Subject Classification (2000)** 35A23 · 35B33 · 35J20 · 35J75.

## 1 Introduction and main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$. In the subcritical case $1 < p < N$, the classical Hardy inequality holds for all $u \in W_0^{1,p}(\Omega)$ as follows:

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \leq \int_\Omega \left|\nabla u \cdot \frac{x}{|x|}\right|^p dx, \tag{1}$$

Here $W_0^{1,p}(\Omega)$ is a completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_{L^p(\Omega)}$. We refer the celebrated work [29] in 1920. For physical background of (1), see e.g. [38]. For (1), it is known that the optimal constant $(\frac{N-p}{p})^p$ is not attained for any bounded domain $\Omega$. Therefore we can expect the existence of remainder terms of (1). Indeed there exist several remainder terms of (1), see [7], [17], [4], [5], [18], [21], [12]. And also there are applications of remainder terms to PDE, see [58], [3], [8], [1], [2].

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In this paper, we focus on the critical case $p = N$. In this case, the classical one loses its meaning. However, instead of (1), the critical Hardy inequality

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{aR}{|x|})^N} dx \le \int_\Omega \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx \tag{2}$$

holds for all $u \in W_0^{1,N}(\Omega)$ and $a \ge 1$, where $R = \sup_{x\in\Omega} |x|$ and $N \ge 2$ (see e.g. [36], [20] Corollary 9.1.2., [4], [5], [44], [56]). It is known that the optimal constant of (2) with $a \ge e$ is $(\frac{N-1}{N})^N$ and is not attained for any bounded domain $\Omega$ with $0 \in \Omega$ by using rearrangement technique and a improvement of (2) (see [3], [2], [54] etc.). On the other hand, in the case $1 \le a < e$, the optimal constant $(\frac{N-1}{N})^N$ is also not attained in spite of luck of rearrangement technique (see [32], [21] Theorem 9.1.4., Theorem 1(ii) in §1, Corollary 2 in §2). For a generalization of (2), the following inequality

$$C \left( \int_\Omega \frac{|u(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}} \le \int_\Omega |\nabla u|^N dx \tag{3}$$

holds for all $u \in W_0^{1,N}(\Omega)$ and $a > 1$, where appropriate $q$ and $\beta$, see [43], Corollary 1 in §2. We define $G$ as the optimal constant of the inequality (3) as follows:

$$G = G(\Omega; a, q, \beta) := \inf_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N \, dx}{\left( \int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}}} \tag{4}$$

for $a \ge 1$ and $q, \beta > 1$. Our aim of this paper is to study positivity and attainability of $G$ for a general bounded domain $\Omega$ with $0 \in \Omega$. Note that some results are obtained by [31] only for balls.

Our minimization problem (4) is related to the following $N-$Laplace equation with the singular potential:

$$\begin{cases} -\mathrm{div}\,(\,|\nabla u|^{N-2} \nabla u\,) = \frac{|u|^{q-2} u}{|x|^N (\log \frac{aR}{|x|})^\beta} & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The minimizers for $G$ are grand state solutions of the Euler-Lagrange equation (5). One virtue of our problem is that the phenomenon occurring on $G$ undergoes a drastic change with respect to the exponents $a, \beta$ and the boundary $\partial\Omega$ since $G$ is affected by the shape of the potential function $f_{a,\beta}(x) := |x|^{-N} (\log \frac{aR}{|x|})^{-\beta}$. Especially the structure of the singularity of the potential function $f_{a,\beta}$ is quite different between $a > 1$ and $a = 1$. Indeed, the singularity of it with $a > 1$ is only at the origin. In contrast, the singularity of it with $a = 1$ is not only at the origin but also on a portion of the boundary, that is $\partial B_R(0) \cap \partial\Omega$. In this paper, we call $a = 1$ case as "sharp" one and $a > 1$ case as "non-sharp" one. First result is concerning non-sharp case.

**Theorem 1** *(Non-sharp case : $a > 1$) Let $a > 1$, $\Omega$ be a bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, $N \ge 2$, $R = \sup_{x\in\Omega} |x|$, $q, \beta > 1$ satisfy*

$$\begin{cases} \beta > \frac{N-1}{N} q + 1 & \text{if} \quad 1 < q < N, \\ \beta \ge \frac{N-1}{N} q + 1 & \text{if} \quad N \le q. \end{cases} \tag{6}$$

*G is defined by (4). Then the following statements (i)∼(iii) hold.*
*(i) If* $\beta > \frac{N-1}{N}q + 1$*, then G is attained.*
*(ii) If* $\beta = q = N$*, then* $G = (\frac{N-1}{N})^N$ *independent of* $\Omega$ *and a, and G is not attained.*
*(iii) If* $\beta = \frac{N-1}{N}q + 1, q > N$*, and* $a \geq e^{\frac{\beta}{N}}$*, then G is independent of* $\Omega$ *and is not attained.*

*Remark 1* In Theorem 1 (iii), we assume the condition $a \geq e^{\frac{\beta}{N}}$. In this case, the potential function $f_{a,\beta}$ is radially decreasing in the domain $\Omega$. Therefore, we can apply the rearrangement technique to our minimization problem (4). We do not know whether Theorem 1 (iii) holds for any $a > 1$.

In sharp case $a = 1$, the positivity of $G$ heavily depends on geometry of $\partial\Omega$ near $\partial B_R(0)$ and the exponent $\beta$ which expresses the strength of the boundary singularity of the potential function $f_{1,\beta}$. Roughly speaking, if $\beta$ is small, then the singularity of $f_{1,\beta}$ at the origin is too strong. On the other hand, if $\beta$ is large, then the boundary singularity of $f_{1,\beta}$ is too strong. Moreover the sharpness of the corner of $\Omega$, which touches $\partial B_R(0)$, plays the role of weakening the boundary singularity of $f_{1,\beta}$. We can observe that the critical Hardy inequality has a good balance concerning these singularities. Second result is as follows.

**Theorem 2** *(Sharp case :* $a = 1$*) Let* $N \geq 2$*,* $\beta, q > 1$ *satisfy (6),* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with* $0 \in \Omega$*, and* $R = \sup_{x\in\Omega}|x|$*. G is defined by (4). Then the following statements (i), (ii) hold.*
*(i) If there exists a neighborhood* $\Gamma \subset \partial\Omega \cap \partial B_R(0)$ *in* $\partial B_R(0)$*, then* $G > 0$ *if and only if* $q = \beta = N$*.*
*(ii) Let* $\partial\Omega \cap \partial B_R(0) = \{(0, \cdots, 0, -R)\}$*. There exist a function* $\phi : \mathbb{R}^{N-1} \to [-R, \infty)$ *and a small* $\delta > 0$ *such that* $\partial\Omega$ *is represented by the graph :* $x_N = \phi(x')$ *for* $|x'| \leq \delta$*. Furthermore there exist positive constants* $C_1, C_2$ *and* $0 < \alpha \leq 1$ *such that* $C_1|x'|^\alpha \leq \phi(x') + R \leq C_2|x'|^\alpha$ *for* $|x'| \leq \delta$*. Then there exists* $\beta^* = \beta^*(\alpha, q)$ *such that* $\frac{N-1}{\alpha} + 1 \leq \beta^* \leq \frac{N}{\alpha}$*, and the following statements (ii)', (ii)" hold.*

*(ii)' If* $\beta < \beta^*$*, then* $G > 0$*. Moreover, if* $\frac{N-1}{N}q + 1 < \beta < \beta^*$*, then G is attained.*
*(ii)" If* $\beta > \beta^*$*, then* $G = 0$*.*

Third result is concerning the explicit optimal constant and its minimizers of $G$ for radial functions.

**Theorem 3** *(Optimal constant and its minimizers) Set*

$$G_{rad} = G_{rad}(B_R(0); a, q, \beta) = \inf_{u \in W^{1,N}_{0,rad}(B_R(0))\setminus\{0\}} \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left( \int_{B_R(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}}}. \tag{7}$$

*Then the following statements (i)∼(iii) hold.*
*(i)* $G_{rad}$ *is independent of* $a \geq 1$ *if* $\beta = \frac{N-1}{N}q + 1$*.*
*(ii)* $G_{rad}(1, q, \beta) > 0$ *if and only if* $q \geq N$ *and* $\beta = \frac{N-1}{N}q + 1$*. In the case* $\beta = \frac{N-1}{N}q + 1$*,* $G_{rad}(a, q, \beta)$ *is attained if and only if* $a = 1$ *and* $q > N$*.*

*(iii) When $q > N$ and $\beta = \frac{N-1}{N}q + 1$, the optimal constant $G_{rad}(1, q, \beta)$ is given by*

$$G_{rad} = \omega_N^{1-\frac{N}{q}}(N-1)\left(\frac{N}{q}\right)^{1-\frac{2N}{q}}\left(1-\frac{N}{q}\right)^{-2+\frac{2N}{q}}\left(\frac{\Gamma\left(\frac{q(N-1)}{q-N}\right)\Gamma\left(\frac{N}{q-N}\right)}{\Gamma\left(\frac{qN}{q-N}\right)}\right)^{1-\frac{N}{q}},$$

*where $\Gamma(\cdot)$ is the gamma function, and the minimizers are given by the family of functions*

$$U(y) = C\lambda^{-\frac{N-1}{N}}\left(1+\left(\lambda\log\frac{R}{|y|}\right)^{-\frac{q-N}{N}}\right)^{-\frac{N}{q-N}}, \ C \in \mathbb{R}\setminus\{0\}, \lambda > 0.$$

Only for two dimensional case $N = 2$, the form of minimizer in Theorem 3 (iii) was already found by [59].

*Remark 2* In the case $q = N, a = 1$ in Theorem 3 (ii), Ioku-Ishiwata [32] showed not only the non-existence of the minimizers of $G_{\rm rad}(1, N, N) = (\frac{N-1}{N})^N$ but also the existence of "virtual" minimizer $(\log\frac{R}{|x|})^{\frac{N-1}{N}}$. To be more precise, if there exists a minimizer $u$ of $G_{\rm rad}(1, N, N)$, then $u(x) = (\log\frac{R}{|x|})^{\frac{N-1}{N}}$. However, since $(\log\frac{R}{|x|})^{\frac{N-1}{N}} \notin W_0^{1,N}(B(R))$, the radial minimizer does not exist. This phenomenon is also observed in other Hardy type inequalities (see [12], [33], [52]).

A few comments are in order.

Scale invariance often plays a important role in consideration of minimization problems. It is well-known that the Hardy-Sobolev type inequality (1) has the scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-p}{p}}u\left(\frac{x}{\lambda}\right) \tag{8}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. When $\Omega \neq \mathbb{R}^N$, the inequality (1) is also invariant under (8) excluding variation of the domain. For simplicity, we call this invariance as "quasi-scale" invariance. On the other hand, the critical Hardy inequality (2) is not invariant under the scaling $u_\lambda(x) = u(\lambda x)$ due to the logarithmic term. However, under the non-standard scaling

$$u_\lambda(x) = \lambda^{-\frac{N-1}{N}}u\left(\left(\frac{|x|}{aR}\right)^{\lambda-1}x\right) \quad (\lambda > 0), \tag{9}$$

the inequality (2) has the scale invariance in the case $a = 1$ and has the quasi-scale invariance in the case $a > 1$ when $\Omega = B_R(0)$ (see [32], Proposition 3 in §5). Unfortunately, the generalized critical Hardy inequality (3) does not have even the quasi-scale invariance under the scaling (9) due to the derivative term $\int_\Omega |\nabla u|^N dx$ of (3) (see Proposition 3 in §5). The lack of the scale invariance makes it difficult to study our minimization problem (4).

In the subcritical case $p < N$ (especially $p = 2$ case), there is an enormous number of researches concerning the following type minimization problem associated with the optimal constant of the Hardy-Sobolev inequality:

$$\inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left( \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx \right)^{\frac{p}{p^*(s)}}}, \tag{10}$$

where $p^*(s) := \frac{p(N-s)}{N-p}$. On the whole space $\Omega = \mathbb{R}^N$, the explicit optimal constant and its minimizers were revealed (see [57], [39] [13], [26], [30] and the references therein). On the other hand, if $\Omega$ is a bounded domain and $0 \in \partial\Omega$, it is known that the existence of minimizers heavily depend on the mean curvature of $\partial\Omega$ at 0 (see [19], [22], [25], [23], [11], [41]). We refer the survey paper [24] and the book [20].
In the critical case $p = N$, there are several researches of maximization problem associated with the Trudinger-Moser inequality which expresses the embedding of the the critical Sobolev space $W^{1,N}$ to the Orlicz space (see e.g. [10], [55], [16], [42], [51], [37], [34], [35] and so on). On the other hand, our problem (4) is a minimization problem associated with the embedding of it to the weighted Lebesgue space like the subcritical one. In this view, we can consider our minimization problem (4) as a limiting case of (10) (see also §4).

This paper is organized as follows: In §2, we show Theorem 1. Again, note that the generalized critical Hardy inequality (3) does not have even the quasi-scale invariance in general. However, only for radial functions, (3) has the quasi-scale invariance. The key tools of the proof of Theorem 1 (iii) are the rearrangement technique and the quasi-scale invariance only for radial functions. For Theorem 1 (ii), in the case $1 < a < e$, we can not apply the rearrangement technique, because the potential function $(|x| \log \frac{aR}{|x|})^{-N}$ is not radially decreasing on $B_R(0)$. Instead of the rearrangement technique, we take some spherical averaging (26) in order to use the quasi-scale invariance. In §3, we give the proof of Theorem 2. Theorem 2 (i) says that (3) can not hold except for the Hardy case ($q = \beta = N$) if $\partial\Omega$ touches $\partial B_R(0)$ enough. However, Theorem 2 (ii) implies that (3) can hold if $\partial\Omega$ touches $\partial B_R(0)$ at only one point very sharply. In order to show these, we make several test functions, which concentrate on the boundary or the origin, and the weighted inequality in a domain with a sharp corner. In §4, we firstly explain the transformation which introduced by [53] in order to prove Theorem 3. Roughly speaking, this transformation says that the Hardy inequality with the optimal constant is equivalent to the critical Hardy inequality with it in the radial case. By showing that this transformation also connects the Hardy-Sobolev type inequality and the generalized critical Hardy inequality (3), we shall reveal the explicit optimal constant and its minimizers of $G_{\rm rad}$ . In §5, we state some Propositions about our inequality (3).

Before stating the proof, we fix several notations: $B_R(a)$ and $B_R^N(a)$ are balls centered $a$ with radius $R$ in $\mathbb{R}^N$. Especially, when $a$ is the origin, we write $B(R)$ for the sake of simplicity. $\omega_N$ is an area of the unit sphere in $\mathbb{R}^N$. $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$. The Schwarz symmetrization $u^\# : \mathbb{R}^N \to [0, \infty]$ is given by $u^\#(|x|) = \inf \left\{ \tau > 0 \,:\, |\{x \in \mathbb{R}^N \,:\, |u(x)| > \tau\}| \le |B_{|x|}(0)| \right\}$. $X_{\rm rad}$ is a set of radial functions in a functional space $X$. Throughout the paper, if $u$ is a radial function in $\mathbb{R}^N$, then we can write as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$ in $\mathbb{R}_+$. Then we write $u(x) = u(|x|)$ with admitting some ambiguity. We hope no confusion occurs by this abbreviation. And also, we use $C$ as a general constant.

## 2 Non-sharp case : $a > 1$

At first, we confirm that the necessary and sufficient condition for the validity of the inequality (3) is the assumption (6) in Theorem 1. To do so, we shall show Corollary 1 from Theorem A proved by Machihara-Ozawa-Wadade [43]. They proved more general critical Hardy inequality in Sobolev-Lorentz space $H^s_{p,q}(\mathbb{R}^N)$ as follows (see also [15], [27], [28], [47], [48], [49]).

**Theorem A** *([43] Theorem 1.1.) Let $N \in \mathbb{N}, 1 < p < \infty, 1 < r \le \infty$ and $1 < \alpha, \beta < \infty$. Then there exists a constant $C > 0$ such that for all $u \in H^{\frac{N}{p}}_{p,r}(\mathbb{R}^N)$, the inequality*

$$\left( \int_{B(\frac{1}{2})} \frac{|u(x)|^\alpha}{|x|^N (\log \frac{1}{|x|})^\beta} dx \right)^{\frac{1}{\alpha}} \le C \|u\|_{H^{\frac{N}{p}}_{p,r}(\mathbb{R}^N)} \tag{11}$$

*holds true if and only if one of the following conditions (i)∼(iii) is fulfilled*

$$\begin{cases} (i) & 1 + \alpha - \beta < 0, \\ (ii) & 1 + \alpha - \beta \ge 0 \text{ and } r < \frac{\alpha}{1+\alpha-\beta}, \\ (iii) & 1 + \alpha - \beta > 0, r = \frac{\alpha}{1+\alpha-\beta}, \text{ and } \alpha \ge \beta. \end{cases} \tag{12}$$

Since $H^1_{N,N}(\mathbb{R}^N)$ is equivalent to the Sobolev space $W^{1,N}(\mathbb{R}^N)$, we can obtain the following Corollary.

**Corollary 1** *Let $a > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$, $R = \sup_{x\in\Omega} |x|$, $N \ge 2$ and $q, \beta > 1$. Then there exists a constant $C > 0$ such that the inequality (3) holds for all $u \in W^{1,N}_0(\Omega)$ if and only if $\beta$ is fulfilled (6).*

*Remark 3* In fact, we can obtain the following inequality with the only radial derivative term

$$C \left( \int_\Omega \frac{|u(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}} \le \int_\Omega \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \tag{13}$$

if $q < N$ and $\beta > \frac{N-1}{N} q + 1$. In contrast, if $q > N$, the inequality (13) does not hold for all $u \in W^{1,N}_0(\Omega)$, see Proposition 2 in §5.

*Proof of Corollary 1.* Let $\beta$ be fulfilled (6). For $u \in W^{1,N}_0(\Omega)$, set $\tilde{u}(x) = u(y)\,(y = 2Rx)$. Then $\tilde{u} \in W^{1,N}_0(\frac{1}{2R}\Omega)$ and $\frac{1}{2R}\Omega \subset B(\frac{1}{2})$. If we take $(p, r, N, \alpha, \beta) = (N, N, N, q, \beta)$, then we can check that a set $(p, r, N, \alpha, \beta)$ in Theorem A satisfies (12). Thus we can apply Theorem A for $\tilde{u}$. Then there exists a constant $C > 0$ such that the inequality

$$C \left( \int_{\frac{\Omega}{2R}} \frac{|\tilde{u}(x)|^q}{|x|^N (\log \frac{1}{|x|})^\beta} dx \right)^{\frac{N}{q}} \le \int_{\frac{\Omega}{2R}} |\nabla \tilde{u}|^N dx$$

holds. Therefore we obtain the following inequality

$$C\left(\int_{\Omega}\frac{|u|^q}{|y|^N(\log\frac{2R}{|y|})^\beta}dy\right)^{\frac{N}{q}}\leq\int_{\Omega}|\nabla u|^N dy.$$

Since $\log\frac{2R}{|y|}\geq C\log\frac{aR}{|y|}$ for any $y\in B_R(0)$ and some $C=C(a)>0$, the inequality

$$C\left(\int_{\Omega}\frac{|u|^q}{|y|^N(\log\frac{aR}{|y|})^\beta}dy\right)^{\frac{N}{q}}\leq\int_{\Omega}|\nabla u|^N dy$$

holds for all $u\in W_0^{1,N}(\Omega)$.

On the other hand, if $\beta$ does not satisfy (6), we consider the following test function $u_s\in W_0^{1,N}(\Omega)$: We choose $b>0$ which satisfies $B_{baR}(0)\subset\Omega$. Let $s<\frac{N-1}{N}$ be a positive parameter. We define

$$u_s(x):=\begin{cases}\left(\log\frac{aR}{|x|}\right)^s & \text{if }\ 0\leq|x|\leq\frac{baR}{2}\\ \left(\log\frac{2}{b}\right)^s\frac{2(baR-|x|)}{baR} & \text{if }\ \frac{baR}{2}\leq|x|\leq baR\\ 0 & \text{if }\ baR\leq|x|.\end{cases} \tag{14}$$

Then direct calculation shows that

$$\begin{aligned}\left(\int_{\Omega}\frac{|u_s|^q}{|x|^N\left(\log\frac{aR}{|x|}\right)^\beta}dx\right)^{\frac{N}{q}}&\geq\left(\omega_N\int_0^{\frac{baR}{2}}\left(\log\frac{aR}{r}\right)^{sq-\beta}\frac{dr}{r}\right)^{\frac{N}{q}}\\&=\left(\omega_N\int_{\log\frac{2}{b}}^{\infty}t^{sq-\beta}\,ds\right)^{\frac{N}{q}}\\&=\begin{cases}C\left(\frac{N-1}{N}-s\right)^{-\frac{N}{q}}, & \text{if }\ \beta=\frac{N-1}{N}q+1\ \text{and}\ q<N,\\ \infty, & \text{if }\ \beta<\frac{N-1}{N}q+1,\end{cases}\\ \int_{\Omega}|\nabla u_s|^N dx\leq\frac{\omega_N s^N}{N\left(\frac{N-1}{N}-s\right)}\left(\log\frac{2}{b}\right)^{-N\left(\frac{N-1}{N}-s\right)}&+C\end{aligned}$$

for $s$ close to $\frac{N-1}{N}$. Thus we obtain

$$\frac{\int_{\Omega}|\nabla u_s|^N\,dx}{\left(\int_{\Omega}\frac{|u_s|^q}{|x|^N(\log\frac{aR}{|x|})^\beta}\right)^{\frac{N}{q}}}\to 0\quad\text{as}\quad s\to\frac{N-1}{N}.$$

Hence the inequality (3) holds for all $u\in W_0^{1,N}(\Omega)$ if and only if $\beta$ is fulfilled (6). □

In order to prove Theorem 1, we prepare the following Lemmas and Proposition.

**Lemma 1** *(Compactness of the embedding) Let $a > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$, $R = \sup_{x\in\Omega} |x|, N \geq 2$, and $q,\beta > 1$ satisfy (6). Set $f_{a,\beta}(x) = |x|^{-N}\left(\log \frac{aR}{|x|}\right)^{-\beta}$. Then the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$ is*

*(i) compact if $\beta > \frac{N-1}{N}q + 1$,*

*(ii) non-compact if $\beta = \frac{N-1}{N}q + 1$ and $q \geq N$.*

*Proof of Lemma 1.* (i) Let $(u_m)_{m=1}^\infty \subset W_0^{1,N}(\Omega)$ be a bounded sequence. Then there exists a subsequence $(u_{m_k})_{k=1}^\infty$ such that

$$\begin{aligned} &u_{m_k} \rightharpoonup u \text{ in } W_0^{1,N}(\Omega),\\ &u_{m_k} \to u \text{ in } L^r(\Omega) \quad \text{for any } 1 \leq r < \infty. \end{aligned} \tag{15}$$

Let $\alpha$ satisfy $\frac{N-1}{N}q + 1 < \alpha < \beta$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left(\log \frac{aR}{|x|}\right)^{\alpha-\beta} < \varepsilon \quad \text{for all } x \in B(\delta). \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned} \int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N(\log \frac{aR}{|x|})^\beta}dx &\leq \varepsilon \int_{B(\delta)} \frac{|u_{m_k} - u|^q}{|x|^N(\log \frac{aR}{|x|})^\alpha}dx + \delta^{-N}\left(\log \frac{aR}{\delta}\right)^{-\beta} \|u_{m_k} - u\|_{L^q(\Omega)}\\ &\leq \varepsilon C\|\nabla(u_{m_k} - u)\|_{L^N(\Omega)}^q + C\|u_{m_k} - u\|_{L^q(\Omega)}\\ &\leq C\varepsilon + C\|u_{m_k} - u\|_{L^q(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0, k \to \infty. \end{aligned}$$

Therefore, the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$ is compact if $\beta > \frac{N-1}{N}q + 1$.
(ii) Let $\varepsilon > 0$ satisfy $B(\varepsilon) \subset \Omega$, $u \in W_0^{1,N}(B(\varepsilon))$ be a positive radial function and $u_\lambda \in W_0^{1,N}(B(a^{1-\frac{1}{\lambda}}\varepsilon))$ be defined by (9). For $0 < \lambda \leq 1$, we set

$$\widetilde{u_\lambda}(x) := \begin{cases} u_\lambda(x) & \text{if } x \in B(a^{1-\frac{1}{\lambda}}\varepsilon),\\ 0 & \text{if } x \in \Omega \setminus B(a^{1-\frac{1}{\lambda}}\varepsilon). \end{cases}$$

By applying Proposition 3 in §5 and $\beta = \frac{N-1}{N}q + 1$, we obtain the sequence $\{\widetilde{u_{\frac{1}{m}}}\}_{m=1}^\infty \subset W_0^{1,N}(\Omega)$ such that

$$\begin{aligned} \int_\Omega |\nabla\widetilde{u_{\frac{1}{m}}}(x)|^N\, dx &= \int_{B(\varepsilon)} |\nabla u(y)|^N\, dy < \infty,\\ \int_\Omega \frac{|\widetilde{u_{\frac{1}{m}}}(x)|^q}{|x|^N(\log \frac{eR}{|x|})^\beta}dx &= \int_{B(\varepsilon)} \frac{|u(x)|^q}{|x|^N(\log \frac{eR}{|x|})^\beta}dx > 0. \end{aligned} \tag{17}$$

If we assume that the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$ is compact, then there exist a subsequence $\{\widetilde{u_{\frac{1}{m_k}}}\}_{k=1}^\infty$ and $u_0 \in L^q(\Omega; f_{a,\beta}(x)dx)$ such that $\widetilde{u_{\frac{1}{m_k}}} \to u_0$ in $L^q(\Omega; f_{a,\beta}(x)dx)$. Therefore $\widetilde{u_{\frac{1}{m_k}}} \to u_0$ a.e. in $\Omega$. By the definition of $\widetilde{u_{\frac{1}{m}}}$, we have $u_0 = 0$. However this contradicts (17). Hence the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$ is non-compact.

□

**Lemma 2** *(Positivity of minimizers) Let $q \geq N, a > 1$, and $\beta = \frac{N-1}{N}q + 1$. If $u \in W_0^{1,N}(B(R))$ is a nonnegative minimizer of $G(B(R); a, q, \beta)$, then $u \in C^1(B(R) \setminus \{0\})$ and $u > 0$ in $B(R) \setminus \{0\}$.*

*Proof of Lemma 2.* Let $u \in W_0^{1,N}(B(R))$ be a nonnegative minimizer of $G(B(R); a, q, \beta)$. Then, by the method of Lagrange multiplier, there exists $A \in \mathbb{R}$ such that $u$ is a weak solution of the Euler-Lagrange equation

$$\begin{cases} -\Delta_N u = A \frac{u^{q-1}}{|x|^N \left(\log \frac{aR}{|x|}\right)^\beta} & \text{in } B(R) \\ \quad u = 0 & \text{on } \partial B(R), \end{cases}$$

where $\Delta_N u := \mathrm{div}(|\nabla u|^{N-2}\nabla u)$. We observe that $A > 0$. For $\varepsilon > 0$,

$$\text{the function } |x|^{-N}\left(\log \frac{aR}{|x|}\right)^{-\beta} \text{ is bounded in } B(R) \setminus B(\varepsilon). \tag{18}$$

Furthermore the Sobolev inequality yields that

$$u \in L^r(B(R)) \text{ for all } 1 \leq r < \infty. \tag{19}$$

Thus we see that $\Delta_N u \in L^r(B(R) \setminus B(\varepsilon))$ for all $1 \leq r < \infty$ by (18) and (19). If we take large $r$ , then $u \in C^1(B(R) \setminus B(\varepsilon))$. (see [14]) Hence, by applying the strong maximum principle for the distributional solution $u \in C^1(B(R)\setminus B(\varepsilon))$ to the inequality $-\Delta_N u \geq 0$ in $B(R) \setminus B(\varepsilon)$, we obtain $u > 0$ in $B(R) \setminus B(\varepsilon)$. (see [50] Theorem 2.5.1.) Since $\varepsilon > 0$ is arbitrary, we have proved that $u \in C^1(B(R) \setminus \{0\})$ and $u > 0$ in $B(R) \setminus \{0\}$. Therefore nonnegative minimizers of $G(B(R); a, q, \beta)$ are positive in $B(R) \setminus \{0\}$.

□

**Proposition 1** *(Non-existence of radial minimizers) Let $q \geq N, a > 1, R > 0$, and $\beta = \frac{N-1}{N}q + 1$. For $\tilde{R} \in (0, aR)$, set*

$$G_{rad}(B(\tilde{R}); a, q, \beta) = \inf_{u \in W_{0,rad}^{1,N}(B(\tilde{R}))\setminus\{0\}} \frac{\int_{B(\tilde{R})} |\nabla u|^N \, dx}{\left(\int_{B(\tilde{R})} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}.$$

*Then $G_{rad}(B(\tilde{R}); a, q, \beta)$ is independent of $\tilde{R}$, and is not attained for any $\tilde{R} \in (0, aR)$ and $a > 1$.*

*Proof of Proposition 1.* For $u \in W_{0,\mathrm{rad}}^{1,N}(B(R))$, we consider the scaled function $u_\lambda \in W_{0,\mathrm{rad}}^{1,N}(B(a^{1-\frac{1}{\lambda}}R))$ which is given by (9). Thanks to the quasi-scale invariance for radial functions (see Proposition 3 in §5), we have

$$\frac{\int_{B(R)} |\nabla u|^N \, dx}{\left(\int_{B(R)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}} = \frac{\int_{B(a^{1-\frac{1}{\lambda}}R)} |\nabla u_\lambda|^N \, dx}{\left(\int_{B(a^{1-\frac{1}{\lambda}}R)} \frac{|u_\lambda|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}$$

which yields that

$$G_{\rm rad}(B(R);a,q,\beta) = G_{\rm rad}(B(\tilde{R});a,q,\beta) \tag{20}$$

for any $\tilde{R} \in (0, aR)$. Therefore $G_{\rm rad}(B(\tilde{R});a,q,\beta)$ is independent of $\tilde{R}$.

Next we shall show that $G_{\rm rad}(B(\tilde{R});a,q,\beta)$ is not attained by a contradiction. Assume that there exists a minimizer $u \in W_0^{1,N}(B(\tilde{R}))$ of $G_{\rm rad}(B(\tilde{R});a,q,\beta)$. Then the scaled function $u_\lambda \in W_{0,{\rm rad}}^{1,N}(B(a^{1-\frac{1}{\lambda}}\tilde{R}))$ is also a minimizer of $G_{\rm rad}(B(\tilde{R});a,q,\beta)$ for $\lambda \in (0,1)$, because it holds that $W_{0,{\rm rad}}^{1,N}(B(a^{1-\frac{1}{\lambda}}\tilde{R})) \subset W_{0,{\rm rad}}^{1,N}(B(\tilde{R}))$ for $\lambda \in (0,1)$. Note that $u_\lambda \equiv 0$ on $B(\tilde{R}) \setminus B(a^{1-\frac{1}{\lambda}}\tilde{R})$. This contradicts Lemma 2.

□

*Proof of Theorem 1*. (i) If $\beta > \frac{N-1}{N}q + 1$, we can easily show that $G$ is attained from Lemma 1 (i). We omit the proof.
(iii) [Step 1] First, we shall show that $G(\Omega;a,q,\beta)$ is independent of $\Omega$ if $\beta = \frac{N-1}{N}q + 1, q > N$, and $a \geq e^{\frac{\beta}{N}}$. For $u \in W_0^{1,N}(B(R))$, it is known the Pólya-Szegö inequality (see e.g. [40]):

$$\int_{B(R)} |\nabla u|^N \, dx \geq \int_{B(R)} |\nabla u^\#|^N \, dx \tag{21}$$

and the Hardy-Littlewood inequality (see e.g. [40]):

$$\int_{B(R)} \frac{|u(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \leq \int_{B(R)} \left( \frac{1}{|x|^N (\log \frac{aR}{|x|})^\beta} \right)^\# |u^\#(x)|^q dx \tag{22}$$

hold true. By the assumption $a \geq e^{\frac{\beta}{N}}$, the potential function $|x|^{-N}(\log \frac{aR}{|x|})^{-\beta}$ is radially decreasing on $B(R)$. Therefore it holds that

$$\left( \frac{1}{|x|^N (\log \frac{aR}{|x|})^\beta} \right)^\# = \frac{1}{|x|^N (\log \frac{aR}{|x|})^\beta} \quad (x \in B(R)). \tag{23}$$

From (21), (22), and (23), we obtain

$$\frac{\int_{B(R)} |\nabla u|^N \, dx}{\left( \int_{B(R)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}}} \geq \frac{\int_{B(R)} |\nabla u^\#|^N \, dx}{\left( \int_{B(R)} \frac{|u^\#|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \right)^{\frac{N}{q}}}$$

which yields that

$$G(B(R);a,q,\beta) = G_{\rm rad}(B(R);a,q,\beta). \tag{24}$$

Especially, if we take small $\tilde{R} > 0$ such that $B(\tilde{R}) \subset \Omega$, then we obtain

$$G_{\rm rad}(B(\tilde{R});a,q,\beta) \geq G(\Omega;a,q,\beta) \geq G(B(R);a,q,\beta) \tag{25}$$

by zero extension. From (24), (25), and Proposition 1, we observe that $G(\Omega;a,q,\beta)$ is independent of $\Omega$.

[Step 2] In order to show that $G(\Omega; a, q, \beta)$ is not attained, assume that there exists a minimizer $u \in W_0^{1,N}(\Omega)$ of $G(\Omega; a, q, \beta)$. We shall deduce a contradiction. If $\Omega \subsetneq B(R)$, then the zero-extended function $u \in W_0^{1,N}(B(R))$ is a minimizer of $G(B(R); a, q, \beta)$ since $G(\Omega; a, q, \beta)$ is independent of $\Omega$ from Step 1. However, $u \equiv 0$ in $B(R) \setminus \Omega$. This contradicts Lemma 2. Therefore we suppose that $\Omega = B(R)$. In this case, $u^\# \in W_{0,\mathrm{rad}}^{1,N}(B(R))$ becomes a radial minimizer of $G(B(R); a, q, \beta)$ from (24) in Step 1. However this contradicts Proposition 1. Hence $G(\Omega; a, q, \beta)$ is not attained.
(ii) Now we consider the case $\beta = q = N$. Let $x = r\omega$ ($r = |x|, \omega \in S^{N-1}$) for $x \in B(R)$. For $u \in W_0^{1,N}(B(R))$, we consider the following radial function $U$:

$$U(r) = \left( \omega_N^{-1} \int_{S^{N-1}} |u(r\omega)|^N dS_\omega \right)^{\frac{1}{N}}. \tag{26}$$

Then we can check that

$$U'(r) \leq \left( \omega_N^{-1} \int_{S^{N-1}} \left| \frac{\partial}{\partial r} u(r\omega) \right|^N dS_\omega \right)^{\frac{1}{N}}$$

which yields that

$$\int_{B(R)} |\nabla U|^N dx \leq \int_{B(R)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx. \tag{27}$$

And also we have

$$\int_{B(R)} \frac{|U|^N}{|x|^N (\log \frac{aR}{|x|})^N} dx = \int_{B(R)} \frac{|u|^N}{|x|^N (\log \frac{aR}{|x|})^N} dx \tag{28}$$

for all $a \geq 1$. Therefore, from (27), (28), we obtain

$$G(B(R); a, N, N) = G_{\mathrm{rad}}(B(R); a, N, N),$$

for all $a \geq 1$. If we recall that $G(B(R); a, N, N) = (\frac{N-1}{N})^N$ in [3] Lemma 2.5, the rest of the proof is the same as (iii).

The proof of Theorem 1 is now complete. □

From Theorem 1 (ii), we obtain the following Corollary.

**Corollary 2** $G(\Omega; 1, N, N) = (\frac{N-1}{N})^N$ *is not attained for any bounded domains $\Omega$ with $0 \in \Omega$.*

*Remark 4* In the two dimensional case $N = 2$, the above result is already known by [21] Theorem 9.1.4.

*Proof of Corollary 2.* Since $\log \frac{R}{|x|} \leq \log \frac{aR}{|x|}$ for any $x \in B_R(0)$ and $a > 1$, we have $\left(\frac{N-1}{N}\right)^N \leq G(\Omega; 1, N, N) \leq G(\Omega; a, N, N)$. From Theorem 1 (ii), it holds that $G(\Omega; a, N, N) = \left(\frac{N-1}{N}\right)^N$. Therefore we obtain $G(\Omega; 1, N, N) = \left(\frac{N-1}{N}\right)^N$ independent of $\Omega$. Therefore we observe that if there exists a minimizer $u$ of $G(\Omega; 1, N, N)$, then $u$ is also a minimizer of $G(B_R(0); 1, N, N)$. However it is known that $G(B_R(0); 1, N, N)$ is not attained (see [32]). This is a contradiction. □

## 3 Sharp case : $a = 1$

In this section, we consider the sharp case. Key tools of the proof of Theorem 2 are the test function method and the Hardy inequality on the half space.

*Proof of Theorem 2.*
(i) Let $\Gamma$ be a neighborhood in $\partial B(R)$, which satisfies $\Gamma \subset \partial\Omega \cap \partial B(R)$. First we show that $G(\Omega; 1, q, \beta) = 0$ if $\beta > \frac{N-1}{N}q + 1$. Set $x = r\omega\,(r = |x|, \omega \in S^{N-1})$ for $x \in \Omega$. Let $\delta > 0$ satisfy $\left\{(r, \omega) \in [0, R) \times S^{N-1} \,|\, R - 2\delta \le r \le R, \omega \in \frac{1}{R}\Gamma\right\} \subset \Omega$. Define

$$u_s(x) = \begin{cases} \left(\log \frac{R}{r}\right)^s \psi(\omega) & \text{if } R - \delta \le r \le R \\ \text{smooth} & \text{if } R - 2\delta \le r \le R - \delta \\ 0 & \text{if } 0 \le r \le R - 2\delta, \end{cases}$$

where $\psi \in C_0^\infty(\frac{1}{R}\Gamma)$. Then we obtain

$$\begin{aligned}
\int_{B(R)} |\nabla u_s|^N dx &= \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r}\omega + \frac{1}{r}\nabla_{S^{N-1}} u_s \right|^N r^{N-1}\, dr dS_\omega \\
&\le 2^{N-1} \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r} \right|^N r^{N-1} + |\nabla_{S^{N-1}} u_s|^N r^{-1}\, dr dS_\omega \\
&\le s^N C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{(s-1)N} \frac{dr}{r} + C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{sN} \frac{dr}{r} + C \\
&\le s^N C \int_0^{\log \frac{R}{R-\delta}} t^{(s-1)N}\, dt + C \\
&= C \frac{s^N}{N} \left(s - \frac{N-1}{N}\right)^{-1} \left(\log \frac{R}{R-\delta}\right)^{s-N+1} + C < \infty.
\end{aligned}$$

Thus $u_s \in W_0^{1,N}(\Omega)$ for all $s > \frac{N-1}{N}$. However, direct calculation shows that

$$\int_\Omega \frac{|u_s|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \ge C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{sq-\beta} \frac{dr}{r} = C \int_0^{\log \frac{R}{R-\delta}} t^{sq-\beta}\, dt$$

which implies that

$$\int_\Omega \frac{|u_s|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx = \infty$$

for $s$ close to $\frac{N-1}{N}$ since $\beta > \frac{N-1}{N}q + 1$. Therefore we proved that

$$G(\Omega; 1, q, \beta) = 0 \quad \text{if} \quad \beta > \frac{N-1}{N}q + 1. \tag{29}$$

Next we show that $G(\Omega; 1, q, \beta) = 0$ if $\beta > N$. Set $x_\varepsilon = (R - 2\varepsilon)\frac{y}{R}$. Note that $B_\varepsilon(x_\varepsilon) \subset \Omega$ for small $\varepsilon > 0$. Then we define the test function $u_\varepsilon$ as follows:

$$u_\varepsilon(x) = \begin{cases} v\left(\frac{|x - x_\varepsilon|}{\varepsilon}\right) & \text{if } x \in B_\varepsilon(x_\varepsilon) \\ 0 & \text{if } x \in \Omega \setminus B_\varepsilon(x_\varepsilon), \end{cases}$$

where

$$v = v(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2} \\ 2(1-t) & \text{if } \frac{1}{2} < t \le 1. \end{cases} \tag{30}$$

Since $\log t \le t - 1$ for $t \ge 1$, we obtain

$$\begin{aligned} \int_\Omega |\nabla_x u_\varepsilon(x)|^N \, dx &= \int_{B(1)} |\nabla_y v(|y|)|^N \, dy < \infty, \\ \int_\Omega \frac{|u_\varepsilon(x)|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx &\ge C \int_{B_\varepsilon(x_\varepsilon)} \frac{|u_\varepsilon(x)|^q}{(R-|x|)^\beta} dx \\ &\ge \frac{C}{(3\varepsilon)^\beta} \int_{B_{\frac{\varepsilon}{2}}(x_\varepsilon)} dx = C\, \varepsilon^{N-\beta} \to \infty \end{aligned}$$

as $\varepsilon \to 0$ when $\beta > N$. Hence we proved that

$$G(\Omega; 1, q, \beta) = 0 \quad \text{when} \quad \beta > N. \tag{31}$$

From (29), (31), and the assumption (6), we showed that $G(\Omega; 1, q, \beta) > 0$ if and only if $q = \beta = N$.
(ii) Let $0 < \alpha \le 1$. Set $x_\varepsilon = (R - 2\varepsilon)\frac{y}{R}$ and $Q_\delta := B_\delta^{N-1}(0) \times (-R, -R + \delta)$ for small $\varepsilon > 0$ and $\delta > 0$. Since $\partial\Omega$ is represented by the graph $x_N = \phi(x')$ for $x' \in B_\delta^{N-1}(0)$ and $\phi$ satisfies $C_1|x'|^\alpha - R \le \phi(x') \le C_2|x'|^\alpha - R$ for $x' \in B_\delta^{N-1}(0)$, we obtain

$$\{(x', x_N) \in Q_\delta \mid x_N \ge C_2|x'|^\alpha - R\} \subset \Omega \cap Q_\delta \subset \{(x', x_N) \in Q_\delta \mid x_N \ge C_1|x'|^\alpha - R\}. \tag{32}$$

First we shall show that $G(\Omega; 1, q, \beta) = 0$ if $\beta > \frac{N}{\alpha}$. From (32), we can observe that $B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon) \subset \Omega$ for small $\varepsilon, A > 0$. Then we define the test function $w_\varepsilon$ as follows:

$$w_\varepsilon(x) = \begin{cases} v\left(\frac{|x - x_\varepsilon|}{A\varepsilon^{\frac{1}{\alpha}}}\right) & \text{if } x \in B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon) \\ 0 & \text{if } x \in \Omega \setminus B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon), \end{cases}$$

where $v$ is given by (30). In the same way as (i), we have

$$\begin{aligned} &\int_\Omega |\nabla_x w_\varepsilon(x)|^N \, dx < \infty, \\ &\int_\Omega \frac{|w_\varepsilon(x)|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \ge C\, \varepsilon^{\frac{N}{\alpha} - \beta} \to \infty \end{aligned}$$

as $\varepsilon \to 0$ if $\beta > \frac{N}{\alpha}$. Therefore we obtain

$$G(\Omega; 1, q, \beta) = 0 \quad \text{if at least} \quad \beta > \frac{N}{\alpha}. \tag{33}$$

Next we shall show that $G(\Omega; 1, q, \beta) > 0$ if $\beta < \frac{N-1}{\alpha} + 1$. For $u \in W_0^{1,N}(\Omega)$, we divide the domain into three parts as follows:

$$\int_\Omega \frac{|u(x)|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx = \int_{\Omega \cap B(\frac{R}{2})} + \int_{\Omega \setminus \left(B(\frac{R}{2}) \cup Q_\delta\right)} + \int_{\Omega \cap Q_\delta} =: I_1 + I_2 + I_3. \tag{34}$$

From Theorem A, we obtain

$$I_1 \le C \left( \int_\Omega |\nabla u|^N dx \right)^{\frac{q}{N}}. \tag{35}$$

Since the potential function $|x|^{-N}(\log \frac{R}{|x|})^{-\beta}$ does not have any singularity in $\Omega \setminus \left(B(\frac{R}{2}) \cup Q_\delta\right)$, Sobolev inequality yields that

$$I_2 \le C \int_\Omega |u|^q dx \le C \left( \int_\Omega |\nabla u|^N dx \right)^{\frac{q}{N}}. \tag{36}$$

Finally, we shall estimate $I_3$ as above. Since $\log t \ge \frac{1}{2}(t-1)\,(1 \le t \le 2)$, we obtain

$$I_3 \le C \int_{\Omega \cap Q_\delta} \frac{|u(x)|^q}{(R-|x|)^\beta} dx \le C \int_{z_N=0}^{z_N=\delta} \int_{z_N \ge C_1 |z'|^\alpha} \frac{|\tilde{u}(z', z_N)|^q}{|z|^\beta} dz, \tag{37}$$

where $u(x) = \tilde{u}(z)\,(z = x + (0, \cdots, 0, R))$. If $\beta < \frac{N-1}{\alpha} + 1$, then there exists $\varepsilon > 0$ and $p > \frac{N}{N-\varepsilon}$ such that $(\beta - \varepsilon)p < \frac{N-1}{\alpha} + 1$. By using Hölder inequality and Sobolev inequality, we obtain

$$\begin{aligned}
&\int_{z_N=0}^{z_N=\delta} \int_{z_N \ge C_1 |z'|^\alpha} \frac{|\tilde{u}(z', z_N)|^q}{|z|^\beta} dz \\
&= \int \int \frac{|\tilde{u}|^\varepsilon}{|z|^\varepsilon} |\tilde{u}|^{q-\varepsilon} |z|^{\beta-\varepsilon} dz \\
&\le \left( \int \int \frac{|\tilde{u}|^N}{|z|^N} dz \right)^{\frac{\varepsilon}{N}} \left( \int \int |\tilde{u}|^{(q-\varepsilon)\frac{Np}{Np-N-p\varepsilon}} dz \right)^{\frac{Np-N-p\varepsilon}{Np}} \left( \int \int |z|^{-(\beta-\varepsilon)p} dz \right)^{\frac{1}{p}} \\
&\le C \left( \int \int \frac{|\tilde{u}|^N}{|z_N|^N} dz \right)^{\frac{\varepsilon}{N}} \left( \int_\Omega |\nabla \tilde{u}|^N dz \right)^{\frac{q-\varepsilon}{N}} \left( \int \int_{|z'| \le \left(\frac{z_N}{C_1}\right)^{\frac{1}{\alpha}}} z_N^{-(\beta-\varepsilon)p} dz \right)^{\frac{1}{p}} \\
&\le C \left( \int \int \frac{|\tilde{u}|^N}{|z_N|^N} dz \right)^{\frac{\varepsilon}{N}} \left( \int_\Omega |\nabla \tilde{u}|^N dz \right)^{\frac{q-\varepsilon}{N}} \left( \int_{z_N=0}^{z_N=\delta} z_N^{\frac{N-1}{\alpha}-(\beta-\varepsilon)p} dz_N \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\frac{N-1}{\alpha} - (\beta-\varepsilon)p > -1$, $\int_0^\delta z_N^{\frac{N-1}{\alpha}-(\beta-\varepsilon)p} dz_N < \infty$. Furthermore, applying the Hardy inequality on the half space $\mathbb{R}^N_+$:

$$\left( \frac{r-1}{r} \right)^r \int_{\mathbb{R}^N_+} \frac{|u|^r}{|x_N|^r} dx \le \int_{\mathbb{R}^N_+} |\nabla u|^r dx \quad (1 \le r < \infty)$$

yields that

$$\int_{z_N=0}^{z_N=\delta}\int_{z_N\geq C_1|z'|^\alpha}\frac{|\tilde{u}(z',z_N)|^q}{|z|^\beta}dz\leq C\left(\int_{\Omega+(0,\cdots,0,R)}|\nabla\tilde{u}|^N dz\right)^{\frac{q}{N}}. \tag{38}$$

By (37) and (38), we obtain

$$I_3\leq C\left(\int_\Omega|\nabla u|^N dx\right)^{\frac{q}{N}}. \tag{39}$$

Therefore, from (34), (35), (36), and (39), the inequality

$$C\left(\int_\Omega\frac{|u(x)|^q}{|x|^N\left(\log\frac{R}{|x|}\right)^\beta}dx\right)^{\frac{N}{q}}\leq\int_\Omega|\nabla u|^N dx$$

holds for all $u\in W_0^{1,N}(\Omega)$. Hence

$$G(\Omega;1,q,\beta)>0\quad\text{if at least}\quad\beta<\frac{N-1}{\alpha}+1. \tag{40}$$

From (33) and (40), there exists $\beta^*\in[\frac{N-1}{\alpha}+1,\frac{N}{\alpha}]$ such that $G(\Omega;1,q,\beta)>0$ if $\beta$ satisfies (6) and $\beta<\beta^*$, on the other hand $G(\Omega;1,q,\beta)=0$ if $\beta>\beta^*$.

Lastly we shall show that $G(\Omega;1,q,\beta)$ is attained if $\frac{N-1}{N}q+1<\beta<\beta^*$. In order to show it, we should prove that the continuous embedding $W_0^{1,N}(\Omega)\hookrightarrow L^q(\Omega;f_{1,\beta}(x)dx)$ is compact if $\frac{N-1}{N}q+1<\beta<\beta^*$, where $f_{a,\beta}(x)=|x|^{-N}(\log\frac{aR}{|x|})^{-\beta}$. Let $(u_m)_{m=1}^\infty\subset W_0^{1,N}(\Omega)$ be a bounded sequence. Then there exist a subsequence $(u_{m_k})_{k=1}^\infty$ such that

$$\begin{aligned}&u_{m_k}\rightharpoonup u\ \text{in}\ W_0^{1,N}(\Omega),\\&u_{m_k}\to u\ \text{in}\ L^r(\Omega)\quad\text{for all}\ 1\leq r<\infty.\end{aligned} \tag{41}$$

We divide the domain into two parts as follows:

$$\int_\Omega\frac{|u_{m_k}-u|^q}{|x|^N\left(\log\frac{R}{|x|}\right)^\beta}dx=\int_{\Omega\setminus Q_\delta}+\int_{\Omega\cap Q_\delta}=:J_1(u_{m_k}-u)+J_2(u_{m_k}-u). \tag{42}$$

Since $\log\frac{R}{|x|}\geq C\log\frac{aR}{|x|}$ in $\Omega\setminus Q_\delta$ for some $a>1$ and $C=C(a,\delta,R)>0$, it holds that

$$J_1(u_{m_k}-u)\leq C\int_{\Omega\setminus Q_\delta}\frac{|u_{m_k}-u|^q}{|x|^N\left(\log\frac{aR}{|x|}\right)^\beta}dx\leq C\int_\Omega\frac{|u_{m_k}-u|^q}{|x|^N\left(\log\frac{aR}{|x|}\right)^\beta}dx.$$

Note that the continuous embedding $W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega;f_{a,\beta}(x)dx)$ is compact from Lemma 1 and the assumption $\beta>\frac{N-1}{N}q+1$, we obtain

$$J_1(u_{m_k}-u)\to 0\quad\text{as}\ k\to\infty. \tag{43}$$

On the other hand, for any $\varepsilon > 0$, we take $\gamma > 0$ which satisfies $\beta < \gamma < \beta^*$ and $(\log \frac{R}{|x|})^{\gamma-\beta} < \varepsilon$ for $x \in Q_\delta$ (If necessary, we take small $\delta > 0$ again.). Then we have

$$J_2(u_{m_k} - u) \leq \varepsilon \int_{\Omega \cap Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\gamma} dx \leq C\varepsilon \left( \int_\Omega |\nabla(u_{m_k} - u)|^N \, dx \right)^{\frac{q}{N}} \leq C\varepsilon. \quad (44)$$

From (42), (43), and (44), we obtain

$$\int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \to 0 \quad \text{as } k \to \infty.$$

Therefore the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{1,\beta}(x)dx)$ is compact if $\frac{N-1}{N}q + 1 < \beta < \beta^*$. In conclusion, we have showed that $G(\Omega; 1, q, \beta)$ is attained if $\frac{N-1}{N}q + 1 < \beta < \beta^*$.

□

## 4 Optimal constant and its minimizers

In this section, we discuss the explicit value of the optimal constant and its minimizers of $G_{\mathrm{rad}}(B(R); a, q, \beta)$. In order to show Theorem 3, we need some results about the following type inequality:

$$C \left( \int_\Omega |x|^\alpha |u|^q dx \right)^{\frac{p}{q}} \leq \int_\Omega |\nabla u|^p dx \quad (45)$$

for radial functions, where $\Omega$ is a domain in $\mathbb{R}^n$, $\alpha \geq -p$, and $q > 1$. The necessary and sufficient condition for the validity of one-dimensional weighted inequality is known as follows:

**Theorem B** *( [6] Bradley, [45] Muckenhoupt ) Let $1 < \rho \leq \sigma < \infty$ and let $U$ and $V$ be measurable weights. Then there exists a constant $C > 0$ such that the inequality*

$$\left( \int_0^\infty \left| U(t) \int_t^\infty |\psi(s)| ds \right|^\sigma dt \right)^{\frac{1}{\sigma}} \leq C \left( \int_0^\infty |V(t)\psi(t)|^\rho dt \right)^{\frac{1}{\rho}} \quad (46)$$

*holds for all measurable functions $\psi$ such that the integral on the right hand side of (46) is finite if and only if*

$$\sup_{r>0} \left( \int_0^r |U(t)|^\sigma dt \right)^{\frac{1}{\sigma}} \left( \int_r^\infty |V(t)|^{-\rho'} dt \right)^{\frac{1}{\rho'}} < +\infty.$$

Especially, if we take $u(t) = \int_t^\infty |\psi(s)| ds, U(t) = t^{\frac{n+\alpha-1}{\sigma}}, \sigma = q, \rho = p$, and $V(t) = t^{\frac{n-1}{\rho}}$ in Theorem B, then we show that

$$\sup_{r>0} \left( \int_0^r |U(t)|^\sigma dt \right)^{\frac{1}{\sigma}} \left( \int_r^\infty |V(t)|^{-\rho'} dt \right)^{\frac{1}{\rho'}} = \sup_{r>0} \left( \int_0^r t^{n+\alpha-1} dt \right)^{\frac{1}{q}} \left( \int_r^\infty t^{-\frac{n-1}{p-1}} dt \right)^{\frac{p-1}{p}} < +\infty$$

if $n + \alpha = \frac{(n-p)q}{p}$. Thus we obtain the following Corollary.

**Corollary 3** *Let $\Omega = \mathbb{R}^n$. Then the inequality (45) hold for all $u \in W^{1,p}_{0,rad}(\mathbb{R}^n)$ if and only if $q = p^*(\alpha) := \frac{p(n+\alpha)}{n-p}$.*

*Remark 5* Note that even if $q = p^*(\alpha)$, the inequality (45) can not hold for all $u \in W^{1,p}_0(\mathbb{R}^n)$ when $\alpha > 0$ when $\Omega = \mathbb{R}^n$ (see [9]).

Set

$$H_{\rm rad}(p,\alpha,n) = \inf_{u \in W^{1,p}_{0,\rm rad}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left( \int_\Omega |x|^\alpha |u|^{p^*(\alpha)} dx \right)^{\frac{p}{p^*(\alpha)}}}, \tag{47}$$

where $\Omega$ is a ball or $\mathbb{R}^n$. For the minimization problem (47), the following is known. We refer to [57], [39], [13], [26], [30].

**Theorem C** *The following statements (i)$\sim$(iii) hold.*
*(i) $H_{rad}(p,\alpha,n)$ is independent of $\Omega$.*
*(ii) When $\Omega \neq \mathbb{R}^n$, $H_{rad}(p,\alpha,n)$ is not attained. On the other hand, when $\Omega = \mathbb{R}^n$, $H_{rad}(p,\alpha,n)$ is attained if and only if $\alpha > -p$.*
*(iii) When $\alpha > -p$ and $\Omega = \mathbb{R}^n$, the explicit optimal constant $H_{rad}(p,\alpha,n)$ is given by*

$$H_{rad}(p,\alpha,n) = \frac{\omega_n^{\frac{p+\alpha}{n+\alpha}} (n+\alpha)(n-p)^{p-\frac{n-p}{n+\alpha}} (p-1)^{\frac{n+2\alpha+p}{n+\alpha}-p}}{(p+\alpha)^{\frac{2(p+\alpha)}{n+\alpha}}} \left( \frac{\Gamma\left(\frac{(p-1)(n+\alpha)}{p+\alpha}\right)\Gamma\left(\frac{n-p}{p+\alpha}\right)}{\Gamma\left(\frac{p(n+\alpha)}{p+\alpha}\right)} \right)^{\frac{p+\alpha}{n+\alpha}},$$

*and the minimizers are given by the family of functions*

$$W(x) = C\lambda^{\frac{n-p}{p}} \left(1 + |\lambda x|^{\frac{p+\alpha}{p-1}}\right)^{-\frac{n-p}{p+\alpha}}, \ C \in \mathbb{R}\setminus\{0\}, \lambda > 0.$$

When $\Omega = \mathbb{R}^n$ and $\alpha = -p$, $H_{\rm rad}(p,-p,n) = (\frac{n-p}{p})^p$ is not attained, because the function $|x|^{-\frac{n-p}{p}}$ which is the solution of the Euler-Lagrange equation

$$-{\rm div}\left(|\nabla u|^{p-2}\nabla u\right) = \left(\frac{n-p}{p}\right)^p \frac{|u|^{p-2}u}{|x|^p} \quad \text{in } \mathbb{R}^n$$

is not in suitable functional space $W^{1,p}_0(\mathbb{R}^n)$, see also Remark 2.

In the radial case, the authors in [53] proved that the critical Hardy inequality (2) on the ball $B^N_R(0) \subset \mathbb{R}^N$ is equivalent to the subcritical Hardy inequality (45) with exponent $\alpha = -p$, $p = N(< n)$ on $\mathbb{R}^n$. Concretely, they showed the following.

**Theorem D** *([53] Sano-Takahashi) Let $n, N \in \mathbb{N}$ satisfy $n > N \geq 2$. Then for any $w \in C^1_{rad}(B^N_R(0)\setminus\{0\})$ (resp. $u \in C^1_{rad}(\mathbb{R}^n\setminus\{0\})$), there exists $u \in C^1_{rad}(\mathbb{R}^n\setminus\{0\})$ (resp. $w \in C^1_{rad}(B^N_R(0)\setminus\{0\})$) such that the equality*

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla u|^N \, dx - \left(\frac{n-N}{N}\right)^N \int_{\mathbb{R}^n} \frac{|u|^N}{|x|^N} \, dx \\ &= \frac{\omega_n}{\omega_N} \left(\frac{m-N}{N-1}\right)^{N-1} \left( \int_{B^N_R(0)} |\nabla w|^N \, dy - \left(\frac{N-1}{N}\right)^N \int_{B^N_R(0)} \frac{|w|^N}{|y|^N \left(\log \frac{R}{|y|}\right)^N} \, dy \right) \end{aligned} \tag{48}$$

*holds true.*

A key ingredient of their proof is the transformation (49) which connects two Hardy inequalities. The transformation (49) also plays an important role on the generalized critical Hardy inequality (3). Indeed, the transformation (49) connects our inequality (3) with $a = 1$ on the ball $B_R^N(0)$ and the inequality (45) on $\mathbb{R}^n$. Moreover (52) also connects (3) with $a > 1$ on the ball $B_R^N(0)$ and (45) on $B_R^n(0)$.

*Proof of Theorem 3.* First we consider the sharp case $a = 1$. Let $x \in \mathbb{R}^n, r = |x|, y \in \mathbb{R}^N, t = |y|$ and $N < n$. For a nonnegative radial function $w = w(y) \in C^1(B_R^N(0) \setminus \{0\})$, we define a radial function $u = u(x) \in C_0^1(\mathbb{R}^n)$ as follows:

$$u(r) = w(t), \text{ where } r = r(t) = \left(\log \frac{R}{t}\right)^{-\frac{N-1}{n-N}}. \tag{49}$$

Note that $r'(t) > 0$ for any $t \in [0, R)$ and $r(0) = 0, r(R) = +\infty$. Also $u(r) \equiv 0$ near $r = +\infty$ since $w(t) \equiv 0$ near $t = R$. Furthermore we obtain

$$\frac{dr}{r} = \frac{N-1}{n-N}\frac{dt}{t\log\frac{R}{t}}.$$

Let $\alpha > -N$ satisfy $q = \frac{N(n+\alpha)}{n-N}$. Direct calculation shows that

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^\alpha |u|^q dx &= \omega_n \int_0^\infty |u(r)|^q r^{n+\alpha}\frac{dr}{r}\\ &= \omega_n \frac{N-1}{n-N}\int_0^R |w(t)|^q \left(\log\frac{R}{t}\right)^{-\frac{(N-1)(n+\alpha)}{n-N}-1}\frac{dt}{t}\\ &= \frac{\omega_n}{\omega_N}\frac{N-1}{n-N}\int_{B_R^N(0)} \frac{|w|^q}{|y|^N(\log\frac{R}{|y|})^\beta}dy,\end{aligned}$$

since $\beta = \frac{N-1}{N}q + 1 = \frac{(N-1)(n+\alpha)}{n-N} + 1$. In the same way as above, we have

$$\int_{\mathbb{R}^n} |\nabla u|^N dx = \frac{\omega_n}{\omega_N}\left(\frac{n-N}{N-1}\right)^{N-1}\int_{B_R^N(0)} |\nabla w|^N dy.$$

Therefore the following equality holds.

$$\frac{\int_{B_R^N(0)} |\nabla w|^N\, dy}{\left(\int_{B_R^N(0)} \frac{|w|^q}{|y|^N(\log\frac{R}{|y|})^\beta}dy\right)^{\frac{N}{q}}} = \left(\frac{\omega_N}{\omega_n}\right)^{1-\frac{N}{q}}\left(\frac{N-1}{n-N}\right)^{N-1+\frac{N}{q}}\frac{\int_{\mathbb{R}^n} |\nabla u|^N\, dx}{\left(\int_{\mathbb{R}^n} |x|^\alpha |u|^q dx\right)^{\frac{N}{q}}}.$$

Thus we obtain

$$G_{\mathrm{rad}}(B(R); 1, q, \beta) = \left(\frac{\omega_N}{\omega_n}\right)^{1-\frac{N}{q}}\left(\frac{N-1}{n-N}\right)^{N-1+\frac{N}{q}} H_{\mathrm{rad}}(N, \alpha, n). \tag{50}$$

Concretely, by using equalities $n-N=\frac{N}{q}(n+\alpha), N+\alpha=(1-\frac{N}{q})(n+\alpha)$, and (iii) in Theorem C, we have

$$
\begin{aligned}
&H_{\rm rad}(N,\alpha,n)\\
&=\frac{\omega_n^{\frac{N+\alpha}{n+\alpha}}(n+\alpha)(n-N)^{N-\frac{n-N}{n+\alpha}}(N-1)^{\frac{n+2\alpha+N}{n+\alpha}-N}}{(N+\alpha)^{\frac{2(N+\alpha)}{n+\alpha}}}\left(\frac{\Gamma\left(\frac{(N-1)(n+\alpha)}{N+\alpha}\right)\Gamma\left(\frac{n-N}{N+\alpha}\right)}{\Gamma\left(\frac{N(n+\alpha)}{N+\alpha}\right)}\right)^{\frac{N+\alpha}{n+\alpha}},\\
&=\frac{\omega_n^{1-\frac{N}{q}}(n+\alpha)\left(\frac{N}{q}(n+\alpha)\right)^{N-\frac{N}{q}}(N-1)^{2-\frac{N}{q}-N}}{\left(\left(1-\frac{N}{q}\right)(n+\alpha)\right)^{2\left(1-\frac{N}{q}\right)}}\left(\frac{\Gamma\left(\frac{q(N-1)}{q-N}\right)\Gamma\left(\frac{N}{q-N}\right)}{\Gamma\left(\frac{qN}{q-N}\right)}\right)^{1-\frac{N}{q}}.
\end{aligned}\tag{51}
$$

By (50) and (51), we observe that

$$
\begin{aligned}
G_{\rm rad}(B(R);1,q,\beta)&=\left(\frac{\omega_N}{\omega_n}\right)^{1-\frac{N}{q}}\left(\frac{q(N-1)}{N(n+\alpha)}\right)^{N-1+\frac{N}{q}}H_{\rm rad}(N,\alpha,n)\\
&=\omega_N^{1-\frac{N}{q}}(N-1)\left(\frac{N}{q}\right)^{1-\frac{2N}{q}}\left(1-\frac{N}{q}\right)^{-2+\frac{2N}{q}}\left(\frac{\Gamma\left(\frac{q(N-1)}{q-N}\right)\Gamma\left(\frac{N}{q-N}\right)}{\Gamma\left(\frac{qN}{q-N}\right)}\right)^{1-\frac{N}{q}}
\end{aligned}
$$

which is independent of exponents $n,\alpha$. Also we obtain the minimizer $U$ of $G_{\rm rad}(B(R);1,q,\beta)$ from the minimizer $W$ of $H_{\rm rad}(N,\alpha,n)$. Indeed, by the transformation (49) and the equality $\frac{n-N}{N+\alpha}=\frac{N}{q-N}$, it holds

$$
W(x)=\lambda^{\frac{n-N}{N}}\left(1+|\lambda x|^{\frac{N+\alpha}{N-1}}\right)^{-\frac{n-N}{N+\alpha}}=\mu^{-\frac{N-1}{N}}\left(1+\left(\mu\log\frac{R}{|y|}\right)^{-\frac{q-N}{N}}\right)^{-\frac{N}{q-N}}=U(y),
$$

where $\lambda^{n-N}=\mu^{1-N}$. Thus we have (iii) in Theorem 3.

On the other hand, in the case $a>1$, if we modify the transformation (49) a little as follows:

$$
u(r)=w(t),\ \text{ where }\ \frac{r}{R}=\left(\frac{\log\frac{aR}{t}}{\log a}\right)^{-\frac{N-1}{n-N}},\tag{52}
$$

then we also obtain a similar result as (50):

$$
G_{\rm rad}(B_R^N(0);a,q,\beta)=\left(\frac{\omega_N}{\omega_n}\right)^{1-\frac{N}{q}}\left(\frac{N-1}{n-N}\right)^{N-1+\frac{N}{q}}H_{\rm rad}(B_R^n(0);N,\alpha,n).\tag{53}
$$

Namely, we can observe that $G_{\rm rad}$ with $a>1$ is equivalent to $H_{\rm rad}$ on a ball. Hence we can see that (i) in Theorem 3 follows from (50), (53), and (i) in Theorem C. And also (ii) in Theorem 3 follows from (50), Corollary 3, (53), and (ii) in Theorem C.

□

## 5 Appendix

First we shall show that on the generalized critical Hardy inequalities (3), we can not replace the derivative term $\int_\Omega |\nabla u|^N dx$ by the radial derivative term $\int_\Omega |\nabla u \cdot (x/|x|)|^N dx$ in general.

**Proposition 2** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$ and $R = \sup_{x\in\Omega} |x|$. If $q > N$, then*

$$\inf_{0\neq u\in W_0^{1,N}(\Omega)} \frac{\int_\Omega \left|\nabla u \cdot \frac{x}{|x|}\right|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{R}{|x|})^\beta}\right)^{\frac{N}{q}}} = 0.$$

The proof is inspired by the idea of Musina [46].

*Proof* We use polar coordinate $(r, \theta_1, \cdots, \theta_{N-1}) \in [0,\infty) \times [0,\pi)^{N-2} \times [0, 2\pi)$ of $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$ as follows:

$$\begin{cases} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1 \cos\theta_2, \\ &\vdots \\ x_{N-1} &= r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2}\cos\theta_{N-1}, \\ x_N &= r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-2}\sin\theta_{N-1}. \end{cases}$$

Moreover its Jacobian is given by

$$J\left(\frac{\partial(x_1, x_2, \cdots, x_N)}{\partial(r, \theta_1, \cdots, \theta_{N-1})}\right) = r^{N-1} \prod_{i=1}^{N-2} (\sin\theta_i)^{N-1-i}. \tag{54}$$

Let $\delta > 0$ satisfy $B(\delta) \subset \Omega$. Then we consider the following test function $u_\mu \in W_0^{1,N}(B(\delta))$ for all $\mu > 1$:

$$u_\mu(x) = u_\mu(r, \theta_{N-1}) := f(r) g_\mu(\theta_{N-1}) \quad (0 < r < \delta, 0 \le \theta_{N-1} \le 2\pi)$$

where

$$g_\mu(\theta_{N-1}) := \begin{cases} g(\mu\theta_{N-1}) & \text{if } 0 \le \theta_{N-1} \le \frac{2\pi}{\mu}, \\ 0 & \text{if } \frac{2\pi}{\mu} \le \theta_{N-1} < 2\pi, \end{cases}$$

$f \in C_0^\infty((0,\delta))$ and $g \in C_0^\infty((0, 2\pi))$. Since $q > N$ and the Jacobian $J\left(\frac{\partial(x_1,x_2,\cdots,x_N)}{\partial(r,\theta_1,\cdots,\theta_{N-1})}\right)$ is independent of $\theta_{N-1}$, we obtain

$$\frac{\int_\Omega \left|\nabla u_\mu \cdot \frac{x}{|x|}\right|^N dx}{\left(\int_\Omega \frac{|u_\mu|^q}{|x|^N (\log \frac{R}{|x|})^\beta}\right)^{\frac{N}{q}}} = C \frac{\int_0^{\frac{2\pi}{\mu}} |g_\mu(\theta_{N-1})|^N \, d\theta_{N-1}}{\left(\int_0^{\frac{2\pi}{\mu}} |g_\mu(\theta_{N-1})|^q \, d\theta_{N-1}\right)^{\frac{N}{q}}} = C\, \mu^{\frac{N}{q}-1} \to 0 \quad \text{as } \mu \to \infty.$$

Hence we have

$$\inf_{0\neq u\in W_0^{1,N}(\Omega)} \frac{\int_\Omega \left|\nabla u\cdot\frac{x}{|x|}\right|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{R}{|x|})^\beta}\right)^{\frac{N}{q}}} = 0.$$

□

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, $a\geq 1$, and $R := \sup_{x\in\Omega}|x|$. Under the suitable setting concerning $q,\beta,\Omega$, and $u$ (see Theorem 2, Corollary 1, Remark 3, Theorem 3, Proposition 2), we can see that inequalities

$$\int_\Omega \left|\nabla u\cdot\frac{x}{|x|}\right|^N dx \geq C\left(\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{aR}{|x|})^\beta}dx\right)^{\frac{N}{q}}, \tag{55}$$

$$\int_\Omega |\nabla u|^N dx \geq C\left(\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{aR}{|x|})^\beta}dx\right)^{\frac{N}{q}} \tag{56}$$

hold for all $u\in W_0^{1,N}(\Omega)$. In the next Proposition, we discuss the scale and the quasi-scale invariance of the generalized critical Hardy inequalities (55), (56) under the scaling (9).

**Proposition 3** *(i) If $\beta=\frac{N-1}{N}q+1$, then the inequality (55) has the quasi-scale invariance under the scaling (9). Furthermore, in the case $a=1$, if $\Omega=B(R)$ or $\Omega$ is a open cone, then (55) has the scale invariance under the scaling (9).*
*(ii) The inequality (56) does not have the quasi-scale invariance under the scaling (9) for any $q,\beta,a$, and $\Omega$. However, only for radial functions, (56) also satisfies (i).*

*Proof* (i) Let $r=|x|, s=|y|, y=\left(\frac{|x|}{aR}\right)^{\lambda-1}x\in\Omega$ and $\Omega_\lambda=\{x\in\mathbb{R}^N \mid y\in\Omega\}$. Then we can easily check that $s=r^\lambda(aR)^{1-\lambda}$ and $\frac{ds}{dr}r=\lambda s$. Also we observe that $\Omega_\lambda\subset B(a^{1-\frac{1}{\lambda}}R)$ since $\Omega\subset B(R)$. For $u=u(y)\in W_0^{1,N}(\Omega)\subset W_0^{1,N}(B(R))$, we obtain

$$\begin{aligned}
\int_{\Omega_\lambda}\left|\nabla_x u_\lambda(x)\cdot\frac{x}{|x|}\right|^N dx &= \int_{B(a^{1-\frac{1}{\lambda}}R)}\left|\frac{\partial}{\partial r}u_\lambda(x)\right|^N dx\\
&= \lambda^{1-N}\int_{S^{N-1}}\int_0^{a^{1-\frac{1}{\lambda}}R}\left|\frac{\partial}{\partial r}u(s\omega)\right|^N r^{N-1}\,drdS_\omega\\
&= \lambda^{1-N}\int_{S^{N-1}}\int_0^R\left|\frac{\partial}{\partial s}u(s\omega)\right|^N\left(\frac{ds}{dr}r\right)^{N-1} dsdS_\omega\\
&= \int_{S^{N-1}}\int_0^R\left|\frac{\partial}{\partial s}u(s\omega)\right|^N s^{N-1}\,dsdS_\omega\\
&= \int_\Omega\left|\nabla_y u(y)\cdot\frac{y}{|y|}\right|^N dy.
\end{aligned}\tag{57}$$

In the same manner as above, we have

$$\int_{\Omega_\lambda}\frac{|u_\lambda(x)|^q}{|x|^N(\log\frac{aR}{|x|})^\beta}dx = \lambda^{\beta-\frac{N-1}{N}q-1}\int_\Omega\frac{|u(y)|^q}{|y|^N(\log\frac{aR}{|y|})^\beta}dy. \tag{58}$$

The assumption $\beta = \frac{N-1}{N}q + 1$, (57), and (58) yield that (55) has the quasi-scale invariance under the scaling (9). Furthermore, in the case $a = 1$, if $\Omega = B(R)$ or $\Omega$ is a open cone, then we can easily check that $\Omega_\lambda = \Omega$. Therefore (56) has the scale invariance under the scaling (9).

(ii) In the same way as (i), we obtain

$$\int_{\Omega_\lambda} |\nabla_x u_\lambda(x)|^N dx = \lambda^{1-N} \int_{S^{N-1}} \int_0^{a^{1-\frac{1}{\lambda}} R} \left| \frac{\partial}{\partial r} u(s\omega)\omega + \frac{1}{r} \nabla_{S^{N-1}} u(s\omega) \right|^N r^{N-1}\, dr dS_\omega.$$

$$= \lambda^{1-N} \int_{S^{N-1}} \int_0^{R} \left| \frac{\partial}{\partial s} u(s\omega)\omega + \left(\frac{ds}{dr} r\right)^{-1} \nabla_{S^{N-1}} u(s\omega) \right|^N \left(\frac{ds}{dr} r\right)^{N-1} ds dS_\omega.$$

$$= \int_{S^{N-1}} \int_0^{R} \left| \frac{\partial}{\partial s} u(s\omega)\omega + \frac{1}{\lambda s} \nabla_{S^{N-1}} u(s\omega) \right|^N s^{N-1}\, ds dS_\omega.$$

Therefore, if $u$ is a non-radial function, then we can see that

$$\int_{\Omega_\lambda} |\nabla_x u_\lambda(x)|^N dx \neq \int_{\Omega} |\nabla_y u(y)|^N dy.$$

for $\lambda \neq 1$. Therefore (56) does not have the quasi-scale invariance under the scaling (9) in general. However, only for radial functions $u$, it holds

$$\left| \nabla u(x) \cdot \frac{x}{|x|} \right| = |\nabla u(x)|.$$

Since inequalities (55), (56) are same in the radial case, (56) also satisfies (i) only for radial functions.

□

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