ON A WEIGHTED TRUDINGER-MOSER TYPE INEQUALITY ON THE WHOLE SPACE AND RELATED MAXIMIZING PROBLEM

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Abstract. In this paper, we establish a weighted Trudinger-Moser type inequality with the full Sobolev norm constraint on the whole Euclidean space. Main tool is the singular Trudinger-Moser inequality on the whole space recently established by Adimurthi and Yang, and a transformation of functions. We also discuss the existence and non-existence of maximizers for the associated variational problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain with finite volume. Then the Sobolev embedding theorem assures that $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, however, as the function $\log(\log(e/|x|)) \in W_0^{1,N}(B)$, B the unit ball in $\mathbb{R}^N$, shows, the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold. Instead, functions in $W_0^{1,N}(\Omega)$ enjoy the exponential summability:

$$W_0^{1,N}(\Omega) \hookrightarrow \{u \in L^N(\Omega) : \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \quad \text{for any } \alpha > 0\},$$

see Yudovich [31], Pohozaev [26], and Trudinger [30]. Moser [22] improved the above embedding as follows, now known as the Trudinger-Moser inequality: Define

$$TM(N, \Omega, \alpha) = \sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_{L^N(\Omega)} \leq 1}} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) dx.$$

Then we have

$$TM(N, \Omega, \alpha) \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N, \end{cases}$$

here and henceforth $\alpha_N = N \omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} denotes the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. On the attainability of the supremum, Carleson-Chang [6], Flucher [13], and Lin [17] proved that $TM(N, \Omega, \alpha)$ is attained on any bounded domain for all $0 < \alpha \leq \alpha_N$.

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Later, Adimurthi-Sandeep [2] established a weighted (singular) Trudinger-Moser inequality as follows: Let $0 \le \beta < N$ and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$. Define

$$\widetilde{TM}(N,\Omega,\alpha,\beta) = \sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|\nabla u\|_{L^N(\Omega)}\le 1}} \frac{1}{|\Omega|}\int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}.$$

Then it is proved that

$$\widetilde{TM}(N,\Omega,\alpha,\beta)\begin{cases} <\infty, & \alpha \le \alpha_{N,\beta},\\ =\infty, & \alpha > \alpha_{N,\beta}.\end{cases}$$

On the attainability of the supremum, recently Csató-Roy [10], [11] proved that $\widetilde{TM}(2,\Omega,\alpha,\beta)$ is attained for $0 < \alpha \le \alpha_{2,\beta} = 2\pi(2-\beta)$ for any bounded domain $\Omega \subset \mathbb{R}^2$. For other types of weighted Trudinger-Moser inequalities, see for example, [7], [8], [9], [14], [18], [28], [29], [32], to name a few.

On domains with infinite volume, for example on the whole space $\mathbb{R}^N$, the Trudinger-Moser inequality does not hold as it is. However, several variants are known on the whole space. In the following, let

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2}\frac{t^j}{j!}$$

denote the truncated exponential function.

First, Ogawa [23], Ogawa-Ozawa [24], Cao [5], Ozawa [25], and Adachi-Tanaka [1] proved that the following inequality holds true, which we call Adachi-Tanaka type Trudinger-Moser inequality: Define

$$A(N,\alpha) = \sup_{\substack{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1}} \frac{1}{\|u\|^N_{L^N(\mathbb{R}^N)}}\int_{\mathbb{R}^N}\Phi_N(\alpha|u|^{\frac{N}{N-1}})dx. \tag{1.1}$$

Then

$$A(N,\alpha)\begin{cases} <\infty, & \alpha < \alpha_N,\\ =\infty, & \alpha \ge \alpha_N.\end{cases} \tag{1.2}$$

The functional in (1.1)

$$F(u) = \frac{1}{\|u\|^N_{L^N(\mathbb{R}^N)}}\int_{\mathbb{R}^N}\Phi_N(\alpha|u|^{\frac{N}{N-1}})dx$$

enjoys the scale invariance under the scaling $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$, i.e., $F(u_\lambda) = F(u)$ for any $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$. Note that the critical exponent $\alpha = \alpha_N$ is not allowed for the finiteness of the supremum. On the attainability of the supremum, Ishiwata-Nakamura-Wadade [16] proved that $A(N,\alpha)$ is attained for any $\alpha \in (0,\alpha_N)$. In this sense, Adachi-Tanaka type Trudinger-Moser inequality has a subcritical nature of the problem.

On the other hand, Ruf [27] and Li-Ruf [20] proved that the following inequality holds true: Define

$$B(N,\alpha) = \sup_{\substack{u\in W^{1,N}(\mathbb{R}^N) \\ \|u\|_{W^{1,N}(\mathbb{R}^N)}\le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})dx. \tag{1.3}$$

Then

$$B(N,\alpha)\begin{cases} <\infty, & \alpha \le \alpha_N, \\ =\infty, & \alpha > \alpha_N. \end{cases} \tag{1.4}$$

Here $\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|^N_{L^N(\mathbb{R}^N)} + \|u\|^N_{L^N(\mathbb{R}^N)}\right)^{1/N}$ is the full Sobolev norm. Note that the scale invariance ($u \mapsto u_\lambda$) does not hold for this inequality. Also the critical exponent $\alpha = \alpha_N$ is permitted to the finiteness of (1.3). Concerning the attainability of $B(N,\alpha)$, it is known that $B(N,\alpha)$ is attained for $0 < \alpha \le \alpha_N$ if $N \ge 3$ [27]. On the other hand when $N = 2$, there exists an explicit constant $\alpha_* > 0$ related to the Gagliardo-Nirenberg inequality in $\mathbb{R}^2$ such that $B(2,\alpha)$ is attained for $\alpha_* < \alpha \le \alpha_2 (= 4\pi)$ [27], [15]. However, if $\alpha > 0$ is sufficiently small, then $B(2,\alpha)$ is not attained [15]. The non-attainability of $B(2,\alpha)$ for α sufficiently small is attributed to the non-compactness of "vanishing" maximizing sequences, as described in [15].

In the following, we are interested in the weighted version of the Trudinger-Moser inequalities on the whole space. Let $N \ge 2$, $-\infty < \gamma < N$ and define the weighted Sobolev space $X^{1,N}_\gamma(\mathbb{R}^N)$ as

$$\begin{aligned} X^{1,N}_\gamma(\mathbb{R}^N) &= \dot{W}^{1,N}(\mathbb{R}^N) \cap L^N(\mathbb{R}^N, |x|^{-\gamma}dx) \\ &= \{u \in L^1_{loc}(\mathbb{R}^N) \,:\, \|u\|_{X^{1,N}_\gamma(\mathbb{R}^N)} = \left(\|\nabla u\|^N_N + \|u\|^N_{N,\gamma}\right)^{1/N} < \infty\}, \end{aligned}$$

where we use the notation $\|u\|_{N,\gamma}$ for $\left(\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\gamma} dx\right)^{1/N}$. We also denote by $X^{1,N}_{\gamma,rad}(\mathbb{R}^N)$ the subspace of $X^{1,N}_\gamma(\mathbb{R}^N)$ consisting of radial functions. We note that a special form of the Caffarelli-Kohn-Nirenberg inequality in [4]:

$$\|u\|_{N,\beta} \le C\|u\|^{\frac{N-\beta}{N-\gamma}}_{N,\gamma} \|\nabla u\|_N^{1-\frac{N-\beta}{N-\gamma}} \tag{1.5}$$

implies that $X^{1,N}_\gamma(\mathbb{R}^N) \subset X^{1,N}_\beta(\mathbb{R}^N)$ when $\gamma \le \beta$. From now on, we assume

$$N \ge 2, \quad -\infty < \gamma \le \beta < N \tag{1.6}$$

and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$.

Recently, Ishiwata-Nakamura-Wadade [16] proved that the following weighted Adachi-Tanaka type Trudinger-Moser inequality holds true: Define

$$\tilde{A}_{rad}(N,\alpha,\beta,\gamma) = \sup_{\substack{u\in X^{1,N}_{\gamma,rad}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1}} \frac{1}{\|u\|^{N(\frac{N-\beta}{N-\gamma})}_{N,\gamma}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}. \tag{1.7}$$

Then for N, β, γ satisfying (1.6), we have

$$\tilde{A}_{rad}(N,\alpha,\beta,\gamma) \begin{cases} < \infty, & \alpha < \alpha_{N,\beta}, \\ = \infty, & \alpha \geq \alpha_{N,\beta}. \end{cases} \tag{1.8}$$

Later, Dong-Lu [12] extends the result in the non-radial setting. Let

$$\tilde{A}(N,\alpha,\beta,\gamma) = \sup_{\substack{u \in X^{1,N}_\gamma(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1}} \frac{1}{\|u\|_{N,\gamma}^{N(\frac{N-\beta}{N-\gamma})}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}. \tag{1.9}$$

Then the corresponding result holds true also for $\tilde{A}(N,\alpha,\beta,\gamma)$. Attainability of the best constant (1.7), (1.9) is also considered in [16] and [12]: $\tilde{A}_{rad}(N,\alpha,\beta,\gamma)$ and $\tilde{A}(N,\alpha,\beta,\gamma)$ are attained for any $0 < \alpha < \alpha_{N,\beta}$.

First purpose of this note is to establish the weighted Li-Ruf type Trudinger-Moser inequality on the weighted Sobolev space $X^{1,N}_\gamma(\mathbb{R}^N)$ with N, β, γ satisfying (1.6). Define

$$\tilde{B}_{rad}(N,\alpha,\beta,\gamma) = \sup_{\substack{u \in X^{1,N}_{\gamma,rad}(\mathbb{R}^N) \\ \|u\|_{X^{1,N}_\gamma(\mathbb{R}^N)} \leq 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}, \tag{1.10}$$

$$\tilde{B}(N,\alpha,\beta,\gamma) = \sup_{\substack{u \in X^{1,N}_{\gamma}(\mathbb{R}^N) \\ \|u\|_{X^{1,N}_\gamma(\mathbb{R}^N)} \leq 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}. \tag{1.11}$$

Theorem 1. *(Weighted Li-Ruf type inequality) Assume (1.6) and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right)\alpha_N$. Then we have*

$$\tilde{B}_{rad}(N,\alpha,\beta,\gamma) \begin{cases} < \infty, & \alpha \leq \alpha_{N,\beta}, \\ = \infty, & \alpha > \alpha_{N,\beta}. \end{cases} \tag{1.12}$$

Furthermore if $0 \leq \gamma \leq \beta < N$, we have

$$\tilde{B}(N,\alpha,\beta,\gamma) \begin{cases} < \infty, & \alpha \leq \alpha_{N,\beta}, \\ = \infty, & \alpha > \alpha_{N,\beta}. \end{cases} \tag{1.13}$$

We also study the existence and non-existence of maximizers for the weighted Trudinger-Moser inequalities (1.12) and (1.13).

Theorem 2. *Assume (1.6). Then the following statements hold.*

(i) *If $N \geq 3$ then $\tilde{B}_{rad}(N,\alpha,\beta,\gamma)$ is attained for any $0 < \alpha \leq \alpha_{N,\beta}$.*
(ii) *If $N = 2$ then $\tilde{B}_{rad}(2,\alpha,\beta,\gamma)$ is attained for any $0 < \alpha \leq \alpha_{2,\beta}$ if $\beta > \gamma$, while there exists $\alpha_* > 0$ such that $\tilde{B}_{rad}(2,\alpha,\beta,\beta)$ is attained for any $\alpha_* < \alpha < \alpha_{2,\beta}$.*
(iii) *$\tilde{B}_{rad}(2,\alpha,\beta,\beta)$ is not attained for sufficiently small $\alpha > 0$.*

Theorem 3. *Let $N \geq 2$, $0 \leq \gamma \leq \beta < N$. Then the following statements hold.*

(i) *If $N \geq 3$ then $\tilde{B}(N,\alpha,\beta,\gamma)$ is attained for any $0 < \alpha \leq \alpha_{N,\beta}$.*

(ii) *If* $N = 2$ *then* $\tilde{B}(2, \alpha, \beta, \gamma)$ *is attained for any* $0 < \alpha \le \alpha_{2,\beta}$ *if* $\beta > \gamma$*, while there exists* $\alpha_* > 0$ *such that* $\tilde{B}(2, \alpha, \beta, \beta)$ *is attained for any* $\alpha_* < \alpha < \alpha_{2,\beta}$*.*

(iii) $\tilde{B}(2, \alpha, \beta, \beta)$ *is not attained for sufficiently small* $\alpha > 0$*.*

Next, we study the relation between the suprema of Adachi-Tanaka type and Li-Ruf type weighted Trudinger-Moser inequalities, along the line of Lam-Lu-Zhang [19]. Set $\tilde{B}_{rad}(N, \beta, \gamma) = \tilde{B}_{rad}(N, \alpha_{N,\beta}, \beta, \gamma)$ in (1.10), and $\tilde{B}(N, \beta, \gamma) = \tilde{B}(N, \alpha_{N,\beta}, \beta, \gamma)$ in (1.11), i.e.,

$$\tilde{B}_{rad}(N, \beta, \gamma) = \sup_{\substack{u \in X^{1,N}_{\gamma,rad}(\mathbb{R}^N) \\ \|u\|_{X^{1,N}_{\gamma}} \le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}, \tag{1.14}$$

$$\tilde{B}(N, \beta, \gamma) = \sup_{\substack{u \in X^{1,N}_{\gamma}(\mathbb{R}^N) \\ \|u\|_{X^{1,N}_{\gamma}} \le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}, \tag{1.15}$$

for N, β, γ satisfying (1.6). Then $\tilde{B}_{rad}(N, \beta, \gamma) < \infty$, and $\tilde{B}(N, \beta, \gamma) < \infty$ if $\gamma \ge 0$, by Theorem 1.

Theorem 4. *(Relation) Assume (1.6). Then we have*

$$\tilde{B}_{rad}(N, \beta, \gamma) = \sup_{\alpha \in (0, \alpha_{N,\beta})} \left(\frac{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{A}_{rad}(N, \alpha, \beta, \gamma).$$

Furthermore if $\gamma \ge 0$*, we have*

$$\tilde{B}(N, \beta, \gamma) = \sup_{\alpha \in (0, \alpha_{N,\beta})} \left(\frac{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{A}(N, \alpha, \beta, \gamma).$$

Note that this implies $\tilde{A}_{rad}(N, \alpha, \beta, \gamma) < \infty$ for N, β, γ satisfying (1.6), and $\tilde{A}(N, \alpha, \beta, \gamma) < \infty$ if $0 \le \gamma \le \beta < N$, by Theorem 1.

Furthermore, we prove how $\tilde{A}_{rad}(N, \alpha, \beta, \gamma)$ and $\tilde{A}(N, \alpha, \beta, \gamma)$ behaves as α approaches to $\alpha_{N,\beta}$ from the below:

Theorem 5. *(Asymptotic behavior of Adachi-Tanaka supremum) Assume (1.6). Then there exist positive constants* C_1, C_2 *(depending on* N*,* β*, and* γ*) such that for* α *close enough to* $\alpha_{N,\beta}$*, the estimate*

$$\left(\frac{C_1}{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \le \tilde{A}_{rad}(N, \alpha, \beta, \gamma) \le \left(\frac{C_2}{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}}$$

holds. Corresponding estimates hold true for $\tilde{A}(N, \alpha, \beta, \gamma)$ *if* $\gamma \ge 0$*.*

Note that the estimate from the above follows from Theorem 4. On the other hand, we will see that the estimate from the below follows from a computation using the Moser sequence.

The organization of the paper is as follows: In section 2, we prove Theorem 1. Main tools are a transformation which relates a function in $X^{1,N}_\gamma(\mathbb{R}^N)$ to a function in $W^{1,N}(\mathbb{R}^N)$, and the singular Trudinger-Moser type inequality recently proved by Adimurthi and Yang [3], see also de Souza and de O [29]. In section 3, we prove the existence part of Theorems 2, 3 (i) (ii). In section 4, we prove the nonexistence part of Theorem 2, 3 (iii). Finally in section 5, we prove Theorem 4 and Theorem 5. The letter C will denote various positive constant which varies from line to line, but is independent of functions under consideration.

2. Proof of Theorem 1.

In this section, we prove Theorem 1. We will use the following singular Trudinger-Moser inequality on the whole space $\mathbb{R}^N$: For any $\beta \in [0, N)$, define

$$\tilde{B}(N,\alpha,\beta,0) = \sup_{\substack{u\in W^{1,N}(\mathbb{R}^N),\\ \|u\|_{W^{1,N}}\le 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}. \tag{2.1}$$

Then it holds

$$\tilde{B}(N,\alpha,\beta,0) \begin{cases} <\infty, & \alpha \le \alpha_{N,\beta}, \\ =\infty, & \alpha > \alpha_{N,\beta}. \end{cases} \tag{2.2}$$

Here $\|u\|_{W^{1,N}} = \left(\|\nabla u\|_N^N + \|u\|_N^N\right)^{1/N}$ denotes the full norm of the Sobolev space $W^{1,N}(\mathbb{R}^N)$. Note that the inequality (2.2) was first established by Ruf [27] for the case $N = 2$ and $\beta = 0$. It then was extended to the case $N \ge 3$ and $\beta = 0$ by Li and Ruf [20]. The case $N \ge 2$ and $\beta \in (0, N)$ was proved by Adimurthi and Yang [3], see also de Souza and de O [29].

Proof of Theorem 1: We define the vector-valued function F by

$$F(x) = \left(\frac{N-\gamma}{N}\right)^{\frac{N}{N-\gamma}} |x|^{\frac{\gamma}{N-\gamma}} x.$$

Its Jacobian matrix is

$$\begin{aligned} DF(x) &= \left(\frac{N-\gamma}{N}\right)^{\frac{N}{N-\gamma}} |x|^{\frac{\gamma}{N-\gamma}} \left(Id_N + \frac{\gamma}{N-\gamma}\frac{x}{|x|}\otimes\frac{x}{|x|}\right) \\ &= \frac{N-\gamma}{N}|F(x)|^{\frac{\gamma}{N}} \left(Id_N + \frac{\gamma}{N-\gamma}\frac{x}{|x|}\otimes\frac{x}{|x|}\right). \end{aligned}$$

where Id_N denotes the $N\times N$ identity matrix and $v\otimes v = (v_iv_j)_{1\le i,j\le N}$ denotes the matrix corresponding to the orthogonal projection onto the line generated by the unit vector $v = (v_1, \cdots, v_N) \in \mathbb{R}^N$, i.e., the map $x \mapsto (x\cdot v)v$. Since a matrix of the form $I + \alpha v\otimes v$,

$\alpha \in \mathbb{R}$, has eigenvalues 1 (with multiplicity $N-1$) and $1+\alpha$ (with multiplicity 1), we see that

$$det(DF(x)) = \left(\frac{N-\gamma}{N}\right)^{N-1} |F(x)|^{\gamma}. \tag{2.3}$$

Let $u \in X^{1,N}_{\gamma}(\mathbb{R}^N)$ be such that $\|u\|_{X^{1,N}_{\gamma}} \leq 1$. We introduce a change of functions as follows.

$$v(x) = \left(\frac{N-\gamma}{N}\right)^{\frac{N-1}{N}} u(F(x)). \tag{2.4}$$

A simple calculation shows that

$$\begin{aligned} \nabla v(x) &= \left(\frac{N-\gamma}{N}\right)^{\frac{N-1}{N}} DF(x)(\nabla u(F(x))) \\ &= \left(\frac{N-\gamma}{N}\right)^{\frac{2N-1}{N}} |F(x)|^{\frac{\gamma}{N}} \left(\nabla u(F(x)) + \frac{\gamma}{N-\gamma}\left(\nabla u(F(x)) \cdot \frac{x}{|x|}\right)\frac{x}{|x|}\right), \end{aligned}$$

and hence

$$|\nabla v(x)|^2 = \left(\frac{N-\gamma}{N}\right)^{\frac{2(2N-1)}{N}} |F(x)|^{\frac{2\gamma}{N}} \left(|\nabla u(F(x))|^2 + \frac{\gamma(2N-\gamma)}{(N-\gamma)^2}\left(\nabla u(F(x)) \cdot \frac{x}{|x|}\right)^2\right).$$

Since $\left(\nabla u(F(x)) \cdot \frac{x}{|x|}\right)^2 \leq |\nabla u(F(x))|^2$, we then have

$$|\nabla v(x)| \leq \left(\frac{N-\gamma}{N}\right)^{\frac{N-1}{N}} |F(x)|^{\frac{\gamma}{N}} |\nabla u(F(x))| = (\det(DF(x)))^{\frac{1}{N}} |\nabla u(F(x))| \tag{2.5}$$

if $\gamma \geq 0$, with equality if and only if $\left(\nabla u(F(x)) \cdot \frac{x}{|x|}\right)^2 = |\nabla u(F(x))|^2$ when $\gamma > 0$. If $\gamma = 0$ the inequality (2.5) is an equality. Note that the inequality (2.5) does not hold if $\gamma < 0$ and u is not radial function. In fact, a reversed inequality occurs in this case. Moreover, (2.5) becomes an equality if u is a radial function for any $-\infty < \gamma < N$. Integrating both sides of (2.5) on $\mathbb{R}^N$, we obtain

$$\|\nabla v\|_N \leq \|\nabla u\|_N. \tag{2.6}$$

Moreover, for any function G on $[0, \infty)$, using the change of variables, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} G\left(|u(x)|^{\frac{N}{N-1}}\right) |x|^{-\delta} dx \\ &\qquad = \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\delta)}{N-\gamma}} \int_{\mathbb{R}^N} G\left(\frac{N}{N-\gamma}|v(y)|^{\frac{N}{N-1}}\right) |y|^{\frac{N(\gamma-\delta)}{N-\gamma}} dy. \end{aligned} \tag{2.7}$$

Consequently, by choosing $G(t) = t^{N-1}$ and $\delta = \gamma$, we get $\|u\|_{N,\gamma} = \|v\|_N$ and hence

$$\|u\|^N_{X^{1,N}_{\gamma}} = \|\nabla u\|^N_N + \int_{\mathbb{R}^N} |u(x)|^N |x|^{-\gamma} dx \geq \|\nabla v\|^N_N + \|v\|^N_N = \|v\|^N_{W^{1,N}}. \tag{2.8}$$

We remark again that (2.6) and (2.8) become equalities if u is radial function for any $\gamma < N$. Thus $\|v\|_{W^{1,N}} \leq 1$ if $\|u\|_{X^{1,N}_\gamma} \leq 1$. By choosing $G(t) = \Phi_N(\alpha t)$ and $\delta = \beta \geq \gamma$, we get

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha |u(x)|^{\frac{N}{N-1}}\right) |x|^{-\beta} dx = \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \int_{\mathbb{R}^N} \Phi_N\left(\frac{N}{N-\gamma}\alpha |v(y)|^{\frac{N}{N-1}}\right) |y|^{-\frac{N(\beta-\gamma)}{N-\gamma}} dy. \tag{2.9}$$

Denote

$$\tilde{\beta} = \frac{N(\beta-\gamma)}{N-\gamma} \in [0, N).$$

By using (2.8) and (2.9) and applying the singular Trudinger-Moser inequality (2.2), we get

$$\begin{aligned}
&\sup_{u \in X^{1,N}_\gamma(\mathbb{R}^N), \|u\|_{X^{1,N}_\gamma} \leq 1} \int_{\mathbb{R}^N} \Phi_N\left(\alpha |u(x)|^{\frac{N}{N-1}}\right) |x|^{-\beta} dx \\
&\leq \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \sup_{v \in W^{1,N}(\mathbb{R}^N), \|v\|_{W^{1,N}} \leq 1} \int_{\mathbb{R}^N} \Phi_N\left(\frac{N}{N-\gamma}\alpha |v(y)|^{\frac{N}{N-1}}\right) |y|^{-\tilde{\beta}} dy \\
&= \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right) \\
&< \infty,
\end{aligned}$$

since $\frac{N}{N-\gamma}\alpha \leq \frac{N}{N-\gamma}\alpha_{N,\beta} = \frac{N-\beta}{N-\gamma}\alpha_N = \left(\frac{N-\tilde{\beta}}{N}\right)\alpha_N = \alpha_{N,\tilde{\beta}}$.

If u is radial then so is v. In this case, (2.5), (2.6) become equalities, and hence so does (2.8). Then the conclusion follows again from the singular Trudinger-Moser inequality (2.2).

We finish the proof of Theorem 1 by showing that $\tilde{B}(N, \alpha, \beta, \gamma) = \infty$ and $\tilde{B}_{rad}(N, \alpha, \beta, \gamma) = \infty$ when $\alpha > \alpha_{N,\beta}$. Since $\tilde{B}_{rad}(N, \alpha, \beta, \gamma) \leq \tilde{B}(N, \alpha, \beta, \gamma)$, it is enough to prove that $\tilde{B}_{rad}(N, \alpha, \beta, \gamma) = \infty$. Suppose the contrary that $\tilde{B}_{rad}(N, \alpha, \beta, \gamma) < \infty$ for some $\alpha > \alpha_{N,\beta}$. Using again the transformation of functions (2.4) for radial functions $u \in X^{1,N}_\gamma$, we then have equalities in (2.5), (2.6), and hence in (2.8). Evidently, the transformation of functions (2.4) is a bijection between $X^{1,N}_{\gamma,rad}$ and $W^{1,N}_{rad}$ and preserves the equality in (2.8). Consequently, we have

$$\tilde{B}_{rad}(N, \alpha, \beta, \gamma) = \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \tilde{B}_{rad}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right),$$

with $\tilde{\beta} = \frac{N(\beta-\gamma)}{N-\gamma} \in [0, N)$. Hence $\tilde{B}_{rad}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right) < \infty$. By rearrangement argument, we have

$$\tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right) = \tilde{B}_{rad}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right) < \infty$$

which violates the result of Adimurthi and Yang since $\frac{N}{N-\gamma}\alpha > \alpha_{N,\tilde{\beta}}$.

For the later purpose, we also prove here directly $\tilde{B}_{rad}(N, \alpha, \beta, \gamma) = \infty$ when $\alpha > \alpha_{N,\beta}$ by using the weighted Moser sequence as in [16], [19]: Let $-\infty < \gamma \le \beta < N$ and for $n \in \mathbb{N}$ set

$$A_n = \left(\frac{1}{\omega_{N-1}}\right)^{1/N}\left(\frac{n}{N-\beta}\right)^{-1/N}, \quad b_n = \frac{n}{N-\beta},$$

so that $(A_n b_n)^{\frac{N}{N-1}} = n/\alpha_{N,\beta}$. Put

$$u_n = \begin{cases} A_n b_n, & \text{if } |x| < e^{-b_n}, \\ A_n \log(1/|x|), & \text{if } e^{-b_n} < |x| < 1, \\ 0, & \text{if } 1 \le |x|. \end{cases} \tag{2.10}$$

Then direct calculation shows that

$$\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1, \tag{2.11}$$

$$\|u_n\|_{N,\gamma}^N = \frac{N-\beta}{(N-\gamma)^{N+1}}\Gamma(N+1)(1/n) + o(1/n) \tag{2.12}$$

as $n \to \infty$. Thus $u_n \in X^{1,N}_{\gamma,rad}(\mathbb{R}^N)$. In fact for (2.12), we compute

$$\begin{aligned} \|u_n\|_{N,\gamma}^N &= \omega_{N-1}\int_0^{e^{-b_n}} (A_n b_n)^N r^{N-1-\gamma} dr + \omega_{N-1}\int_{e^{-b_n}}^1 A_n^N (\log(1/r))^N r^{N-1-\gamma} dr \\ &= I + II. \end{aligned}$$

We see

$$I = \omega_{N-1}(A_n b_n)^N \left[\frac{r^{N-\gamma}}{N-\gamma}\right]_{r=0}^{r=e^{-b_n}} = \omega_{N-1}\left(\frac{n}{\alpha_{N,\beta}}\right)^{N-1}\frac{e^{-(\frac{N-\gamma}{N-\beta})n}}{N-\gamma} = o(1/n)$$

as $n \to \infty$. Also

$$\begin{aligned} II &= \left(\frac{N-\beta}{n}\right)\int_{e^{-b_n}}^1 (\log(1/r))^N r^{N-1-\gamma} dr \\ &= \left(\frac{N-\beta}{n}\right)\int_0^{b_n} \rho^N e^{-(N-\gamma)\rho} d\rho = \frac{N-\beta}{(N-\gamma)^{N+1}}(1/n)\int_0^{(N-\gamma)b_n} \rho^N e^{-\rho} d\rho \\ &= \frac{N-\beta}{(N-\gamma)^{N+1}}(1/n)\Gamma(N+1) + o(1/n). \end{aligned}$$

Thus we obtain (2.12).

Now, put $v_n(x) = \lambda_n u_n(x)$ where u_n is the weighted Moser sequence in (2.10) and $\lambda_n > 0$ is chosen so that $\lambda_n^N + \lambda_n^N \|u_n\|_{N,\gamma}^N = 1$. Thus we have $\|\nabla v_n\|_{L^N}^N + \|v_n\|_{N,\gamma}^N = 1$ for any $n \in \mathbb{N}$. By (2.12) with $\beta = \gamma$, we see that $\lambda_n^N = 1 - O(1/n)$ as $n \to \infty$. For $\alpha > \alpha_{N,\beta}$, we calculate

$$\begin{aligned}
\int_{\mathbb{R}^N} \Phi_N(\alpha |v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} &\geq \int_{\{0 \leq |x| \leq e^{-b_n}\}} \Phi_N(\alpha |v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta} \\
&= \int_{\{0 \leq |x| \leq e^{-b_n}\}} \left(e^{\alpha |v_n|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!} |v_n|^{\frac{Nj}{N-1}} \right) \frac{dx}{|x|^\beta} \\
&\geq \left\{ \exp\left(\frac{n\alpha}{\alpha_{N,\beta}} \lambda_n^{\frac{N}{N-1}} \right) - O(n^{N-1}) \right\} \int_{\{0 \leq |x| \leq e^{-b_n}\}} \frac{dx}{|x|^\beta} \\
&\geq \left\{ \exp\left(\frac{n\alpha}{\alpha_{N,\beta}} \left(1 - O\left(\frac{1}{n^{\frac{1}{N-1}}} \right) \right) \right) - O(n^{N-1}) \right\} \left(\frac{\omega_{N-1}}{N - \beta} \right) e^{-n} \to +\infty
\end{aligned}$$

as $n \to \infty$. Here we have used that for $0 \leq |x| \leq e^{-b_n}$,

$$\alpha |v_n|^{\frac{N}{N-1}} = \alpha \lambda_n^{\frac{N}{N-1}} (A_n b_n)^{\frac{N}{N-1}} = \frac{n\alpha}{\alpha_{N,\beta}} \lambda_n^{\frac{N}{N-1}}$$

by definition of A_n and b_n. Also we used that for $0 \leq |x| \leq e^{-b_n}$,

$$|v_n|^{\frac{Nj}{N-1}} = \lambda_n^{\frac{Nj}{N-1}} (A_n b_n)^{\frac{Nj}{N-1}} \leq C n^j \leq C n^{N-1}$$

for $0 \leq j \leq N - 2$ and n is large. This proves Theorem 1 completely. □

3. Existence of maximizers for the weighted Trudinger-Moser inequality

As explained in the Introduction, the existence and non-existence of maximizers for (2.1) is well known. Now, let us recall it here.

Proposition 1. *The following statements hold,*

(i) *If* $N \geq 3$ *then* $\tilde{B}(N, \alpha, 0, 0)$ *is attained for any* $0 < \alpha \leq \alpha_N$ *(see* [15, 20]*).*
(ii) *If* $N = 2$*, there exists* $0 < \alpha_* < \alpha_2 = 4\pi$ *such that* $\tilde{B}(2, \alpha, 0, 0)$ *is attained for any* $\alpha_* < \alpha \leq \alpha_2$ *(see* [15, 27]*).*
(iii) *If* $\beta \in (0, N)$ *and* $N \geq 2$ *then* $\tilde{B}(N, \alpha, \beta, 0)$ *is attained for any* $0 < \alpha \leq \alpha_{N,\beta}$ *(see* [21]*).*
(iv) $\tilde{B}(2, \alpha, 0, 0)$ *is not attained for any sufficiently small* $\alpha > 0$ *(see* [15]*).*

The existence part (iii) of Proposition 1 is recently proved by X. Li, and Y. Yang [21] by a blow-up analysis.

Remark 1. By a rearrangement argument, the maximizers for (2.1), if exist, must be a decreasing spherical symmetric function if $\beta \in (0, N)$ and up to a translation if $\beta = 0$.

The proofs of the existence part (i) (ii) of Theorem 2 and 3 are completely similar by using the formula of change of functions (2.4) and the results on the existence of maximizers for (2.1). So we prove Theorem 3 only here. As we have seen from the proof of Theorem 1 that

$$\tilde{B}(N,\alpha,\beta,\gamma) \le \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right)$$

if $0 \le \gamma \le \beta < N$, where $\tilde{\beta} = N(\beta-\gamma)/(N-\gamma) \in [0,N)$. If N, α, β and γ satisfy the condition (i) and (ii) of Theorem 3, then N, $N\alpha/(N-\gamma)$ and $\tilde{\beta}$ satisfy the condition (i)–(iii) of Proposition 1, hence there exists a maximizer $v \in W^{1,N}(\mathbb{R}^N)$ for $\tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right)$ with $\|v\|_N^N + \|\nabla v\|_N^N = 1$ and

$$\int_{\mathbb{R}^N} \Phi_N\left(\frac{N}{N-\gamma}\alpha|v(y)|^{\frac{N}{N-1}}\right)|y|^{-\tilde{\beta}}dy = \tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right).$$

As mentioned in Remark 1, we can assume that v is a radial function. Let $u \in X_\gamma^{1,N}$ be a function defined by (2.4). Note that u is also a radial function, hence (2.5) becomes an equality. So do (2.6) and (2.8). Hence, we get

$$\|u\|_{X_\gamma^{1,N}}^N = \|\nabla v\|_N^N + \|v\|_N^N = 1,$$

and by (2.9)

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{\frac{N}{N-1}}\right)|x|^{-\beta}dx = \left(\frac{N-\gamma}{N}\right)^{N-1+\frac{N(\gamma-\beta)}{N-\gamma}} \tilde{B}\left(N, \frac{N}{N-\gamma}\alpha, \tilde{\beta}, 0\right).$$

This shows that u is a maximizer for $\tilde{B}(N,\alpha,\beta,\gamma)$. □

4. Non-existence of maximizers for the weighted Trudinger-Moser inequality

In this section, we prove the non-existence part (iii) of Theorem 3. The proof of (iii) of Theorem 2 is completely similar. We follow Ishiwata's argument in [15].

Assume $0 \le \beta < 2$, $0 < \alpha \le \alpha_{2,\beta} = 2\pi(2-\beta)$ and recall

$$\tilde{B}(2,\alpha,\beta,\beta) = \sup_{\substack{u \in X_\beta^{1,2}(\mathbb{R}^2) \\ \|u\|_{X_\beta^{1,2}(\mathbb{R}^2)} \le 1}} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)\frac{dx}{|x|^\beta}.$$

We will show that $\tilde{B}(2,\alpha,\beta,\beta)$ is not attained if $\alpha > 0$ sufficiently small. Set

$$M = \left\{u \in X_\beta^{1,2}(\mathbb{R}^2) \,:\, \|u\|_{X_\beta^{1,2}} = \left(\|\nabla u\|_2^2 + \|u\|_{2,\beta}^2\right)^{1/2} = 1\right\}$$

be the unit sphere in the Hilbert space $X_\beta^{1,2}(\mathbb{R}^2)$ and

$$J_\alpha : M \to \mathbb{R}, \quad J_\alpha(u) = \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)\frac{dx}{|x|^\beta}$$

be the corresponding functional defined on M. Actually, we will prove the stronger claim that J_α has no critical point on M when $\alpha > 0$ is sufficiently small.

Assume the contrary that there existed $v \in M$ such that v is a critical point of J_α on M. Define an orbit on M through v as

$$v_\tau(x) = \sqrt{\tau} v(\sqrt{\tau} x) \quad \tau \in (0, \infty), \quad w_\tau = \frac{v_\tau}{\|v_\tau\|_{X^{1,2}_\beta}} \in M.$$

Since $w_\tau|_{\tau=1} = v$, we must have

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) = 0. \tag{4.1}$$

Note that

$$\|\nabla v_\tau\|^2_{L^2(\mathbb{R}^2)} = \tau \|\nabla v\|^2_{L^2(\mathbb{R}^2)}, \quad \|v_\tau\|^p_{p,\beta} = \tau^{\frac{p+\beta-2}{2}} \|v\|^p_{p,\beta}$$

for $p > 1$. Thus,

$$\begin{aligned} J_\alpha(w_\tau) &= \int_{\mathbb{R}^2} \left(e^{\alpha w_\tau^2} - 1\right) \frac{dx}{|x|^\beta} = \int_{\mathbb{R}^2} \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{v_\tau^{2j}(x)}{\|v_\tau\|^{2j}_{X^{1,2}_\beta}} \frac{dx}{|x|^\beta} \\ &= \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{\|v_\tau\|^{2j}_{2j,\beta}}{\left(\|\nabla v_\tau\|_2^2 + \|v_\tau\|^2_{2,\beta}\right)^j} = \sum_{j=1}^\infty \frac{\alpha^j}{j!} \frac{\tau^{j-1+\frac{\beta}{2}} \|v\|^{2j}_{2j,\beta}}{\left(\tau \|\nabla v\|_2^2 + \tau^{\frac{\beta}{2}} \|v\|^2_{2,\beta}\right)^j}. \end{aligned}$$

By using an elementary computation

$$\begin{aligned} f(\tau) &= \frac{\tau^{j-1+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^j}, \quad a = \|\nabla v\|_2^2,\ b = \|v\|^2_{2,\beta},\ c = \|v\|^{2j}_{2j,\beta}, \\ f'(\tau) &= (1 - \frac{\beta}{2}) \frac{\tau^{j-2+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^{j+1}} \left\{-\tau a + (j-1) b\right\}, \end{aligned}$$

we estimate $\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau)$:

$$\begin{aligned} &\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) \\ &= \sum_{j=1}^\infty \left[\frac{\alpha^j}{j!} (1 - \frac{\beta}{2}) \frac{\tau^{j-2+\beta/2} \|v\|^{2j}_{2j,\beta}}{\left(\tau \|\nabla v\|_2^2 + \tau^{\beta/2} \|v\|^2_{2,\beta}\right)^{j+1}} \left\{-\tau \|\nabla v\|_2^2 + (j-1) \|v\|^2_{2,\beta}\right\}\right]_{\tau=1} \\ &= -\alpha (1 - \frac{\beta}{2}) \|\nabla v\|_2^2 \|v\|^2_{2,\beta} + \sum_{j=2}^\infty \frac{\alpha^j}{j!} (1 - \frac{\beta}{2}) \|v\|^{2j}_{2j,\beta} \left\{-\|\nabla v\|_2^2 + (j-1) \|v\|^2_{2,\beta}\right\} \\ &\le \alpha (1 - \frac{\beta}{2}) \|\nabla v\|_2^2 \|v\|^2_{2,\beta} \left\{-1 + \sum_{j=2}^\infty \frac{\alpha^{j-1}}{(j-1)!} \frac{\|v\|^{2j}_{2j,\beta}}{\|\nabla v\|_2^2 \|v\|^2_{2,\beta}}\right\}, \end{aligned} \tag{4.2}$$

since $-\|\nabla v\|_2^2 + (j-1) \|v\|^2_{2,\beta} \le j$.

Now, we state a lemma. Unweighted version of the next lemma is proved in [15]:Lemma 3.1, and the proof of the next is a simple modification of the one given there using the weighted Adachi-Tanaka type Trudinger-Moser inequality:

$$\tilde{A}(2,\alpha,\beta,\beta) = \sup_{\substack{u\in X^{1,2}_\beta(\mathbb{R}^2)\setminus\{0\} \\ \|\nabla u\|_{L^2(\mathbb{R}^2)}\le 1}} \frac{1}{\|u\|^2_{2,\beta}} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \frac{dx}{|x|^\beta} < \infty$$

for $\alpha \in (0, \alpha_{2,\beta})$ if $\beta \ge 0$, and the expansion of the exponential function.

Lemma 1. *For any $\alpha \in (0, \alpha_{2,\beta})$, there exists $C_\alpha > 0$ such that*

$$\|u\|^{2j}_{2j,\beta} \le C_\alpha \frac{j!}{\alpha^j} \|\nabla u\|_2^{2j-2} \|u\|^2_{2,\beta}$$

holds for any $u \in X^{1,2}_\beta(\mathbb{R}^2)$ and $j \in \mathbb{N}$, $j \ge 2$.

By this lemma, if we take $\alpha < \tilde{\alpha} < \alpha_{2,\beta}$ and put $C = C_{\tilde{\alpha}}$, we see

$$\frac{\|v\|^{2j}_{2j,\beta}}{\|\nabla v\|_2^2 \|v\|^2_{2,\beta}} \le C \frac{j!}{\tilde{\alpha}^j} \|\nabla v\|_{2j}^{2j-4} \le C \frac{j!}{\tilde{\alpha}^j}$$

for $j \ge 2$ since $v \in M$. Thus we have

$$\sum_{j=2}^\infty \frac{\alpha^{j-1}}{(j-1)!} \frac{\|v\|^{2j}_{2j,\beta}}{\|\nabla v\|_2^2 \|v\|^2_{2,\beta}} \le \sum_{j=2}^\infty \frac{C\alpha^{j-1}}{(j-1)!} \frac{j!}{\tilde{\alpha}^j} = (\frac{C\alpha}{\tilde{\alpha}^2}) \sum_{j=2}^\infty \left(\frac{\alpha}{\tilde{\alpha}}\right)^{j-2} j \le \alpha C'$$

for some $C' > 0$. Inserting this into the former estimate (4.2), we obtain

$$\frac{d}{d\tau}\Big|_{\tau=1} J_\alpha(w_\tau) \le (1 - \frac{\beta}{2}) \alpha \|\nabla v\|_2^2 \|v\|^2_{2,\beta} (-1 + C'\alpha) < 0$$

when $\alpha > 0$ is sufficiently small. This contradicts to (4.1). □

5. Proof of Theorem 4 and 5.

In this section, we prove Theorem 4 and Theorem 5. As stated in the Introduction, we follow the argument by Lam-Lu-Zhang [19]. First, we prepare several lemmata.

Lemma 2. *Assume (1.6) and set*

$$\widehat{A}(N,\alpha,\beta,\gamma) = \sup_{\substack{u\in X^{1,N}_\gamma(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)}\le 1 \\ \|u\|_{N,\gamma}=1}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) \frac{dx}{|x|^\beta}. \tag{5.1}$$

Let $\tilde{A}(N,\alpha,\beta,\gamma)$ be defined as in (1.9). Then $\tilde{A}(N,\alpha,\beta,\gamma) = \widehat{A}(N,\alpha,\beta,\gamma)$ for any $\alpha > 0$. Similarly, $\tilde{A}_{rad}(N,\alpha,\beta,\gamma) = \widehat{A}_{rad}(N,\alpha,\beta,\gamma)$ for any $\alpha > 0$, where $\widehat{A}_{rad}(N,\alpha,\beta,\gamma)$ is defined similar to (5.1) and $\widehat{A}_{rad}(N,\alpha,\beta,\gamma)$ is defined in (1.7).

Proof. For any $u \in X^{1,N}_\gamma(\mathbb{R}^N) \setminus \{0\}$ and $\lambda > 0$, we put $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}^N$. Then it is easy to see that

$$\begin{cases} \|\nabla u_\lambda\|^N_{L^N(\mathbb{R}^N)} = \|\nabla u\|^N_{L^N(\mathbb{R}^N)}, \\ \|u_\lambda\|^N_{N,\gamma} = \lambda^{-(N-\gamma)} \|u\|^N_{N,\gamma}. \end{cases} \tag{5.2}$$

Thus for any $u \in X^{1,N}_\gamma(\mathbb{R}^N) \setminus \{0\}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$, if we choose $\lambda = \|u\|_{N,\gamma}^{N/(N-\gamma)}$, then $u_\lambda \in X^{1,N}_\gamma(\mathbb{R}^N)$ satisfies

$$\|\nabla u_\lambda\|_{L^N(\mathbb{R}^N)} \le 1 \quad \text{and} \quad \|u_\lambda\|_{N,\gamma}^N = 1.$$

Thus

$$\widehat{A}(N,\alpha,\beta,\gamma) \ge \int_{\mathbb{R}^N} \Phi_N(\alpha|u_\lambda|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} = \frac{1}{\|u\|_{N,\gamma}^{\frac{N(N-\beta)}{N-\gamma}}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta}$$

which implies $\widehat{A}(N,\alpha,\beta,\gamma) \ge \tilde{A}(N,\alpha,\beta,\gamma)$. The opposite inequality is trivial. □

Lemma 3. *Assume (1.6) and set $\tilde{B}(N,\beta,\gamma)$ as in (1.15). Then we have*

$$\tilde{A}(N,\alpha,\beta,\gamma) \le \left(\frac{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{B}(N,\beta,\gamma)$$

for any $0 < \alpha < \alpha_{N,\beta}$. The same relation holds for $\tilde{A}_{rad}(N,\alpha,\beta,\gamma)$ in (1.7) and $\tilde{B}_{rad}(N,\beta,\gamma)$ in (1.14).

Proof. Choose any $u \in X^{1,N}_\gamma$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$ and $\|u\|_{N,\gamma} = 1$. Put $v(x) = Cu(\lambda x)$ where $C \in (0,1)$ and $\lambda > 0$ are defined as

$$C = \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{\frac{N-1}{N}} \quad \text{and} \quad \lambda = \left(\frac{C^N}{1-C^N}\right)^{1/(N-\gamma)}.$$

Then by scaling rules (5.2), we see

$$\begin{aligned}\|v\|_{X^{1,N}_\gamma}^N &= \|\nabla v\|_N^N + \|v\|_{N,\gamma}^N = C^N\|\nabla u\|_N^N + \lambda^{-(N-\gamma)}C^N\|u\|_{N,\gamma}^N \\ &\le C^N + \lambda^{-(N-\gamma)}C^N = 1.\end{aligned}$$

Also we have

$$\begin{aligned}\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|v|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} &= \lambda^{-(N-\beta)}\int_{\mathbb{R}^N} \Phi_N\left(\alpha_{N,\beta}C^{\frac{N}{N-1}}|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} \\ &= \lambda^{-(N-\beta)}\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.\end{aligned}$$

Thus testing $\tilde{B}(N,\beta,\gamma)$ by v, we see

$$\tilde{B}(N,\beta,\gamma) \ge \left(\frac{1-C^N}{C^N}\right)^{\frac{N-\beta}{N-\gamma}} \int_{\mathbb{R}^N} \Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.$$

By taking the supremum for $u \in X^{1,N}_\gamma$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \le 1$ and $\|u\|_{N,\gamma} = 1$, we have

$$\tilde{B}(N,\beta,\gamma) \ge \left(\frac{1 - \left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \widehat{A}(N,\alpha,\beta,\gamma).$$

Finally, Lemma 2 implies the result. The proof of

$$\tilde{B}_{rad}(N,\beta,\gamma) \geq \left(\frac{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \widehat{A}_{rad}(N,\alpha,\beta,\gamma)$$

is similar. □

Proof of Theorem 4: We prove the relation between $\tilde{B}(N,\beta,\gamma)$ and $\tilde{A}(N,\alpha,\beta,\gamma)$ only. The assertion that

$$\tilde{B}(N,\beta,\gamma) \geq \sup_{\alpha\in(0,\alpha_{N,\beta})} \left(\frac{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{A}(N,\alpha,\beta,\gamma)$$

follows from Lemma 3. Note that $\tilde{B}(N,\beta,\gamma) < \infty$ when $0 \leq \gamma \leq \beta < N$ by Theorem 1.

Let us prove the opposite inequality. Let $\{u_n\} \subset X^{1,N}_\gamma(\mathbb{R}^N)$, $u_n \neq 0$, $\|\nabla u_n\|^N_{L^N} + \|u_n\|^N_{N,\gamma} \leq 1$, be a maximizing sequence of $\tilde{B}(N,\beta,\gamma)$:

$$\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u_n|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} = \tilde{B}(N,\beta,\gamma) + o(1)$$

as $n \to \infty$. We may assume $\|\nabla u_n\|^N_{L^N(\mathbb{R}^N)} < 1$ for any $n \in \mathbb{N}$. Define

$$\begin{cases} v_n(x) = \frac{u_n(\lambda_n x)}{\|\nabla u_n\|_N}, \quad (x \in \mathbb{R}^N) \\ \lambda_n = \left(\frac{1-\|\nabla u_n\|^N_N}{\|\nabla u_n\|^N_N}\right)^{1/(N-\gamma)} > 0. \end{cases}$$

Thus by (5.2), we see

$$\|\nabla v_n\|^N_{L^N(\mathbb{R}^N)} = 1,$$

$$\|v_n\|^{\frac{N(N-\beta)}{N-\gamma}}_{N,\gamma} = \left(\frac{\lambda_n^{-(N-\gamma)}}{\|\nabla u_n\|^N_N}\|u_n\|^N_{N,\gamma}\right)^{\frac{N-\beta}{N-\gamma}} = \left(\frac{\|u_n\|^N_{N,\gamma}}{1-\|\nabla u_n\|^N_N}\right)^{\frac{N-\beta}{N-\gamma}} \leq 1,$$

since $\|\nabla u_n\|^N_N + \|u_n\|^N_{N,\gamma} \leq 1$. Thus, setting

$$\alpha_n = \alpha_{N,\beta}\|\nabla u_n\|^{\frac{N}{N-1}}_N < \alpha_{N,\beta}$$

for any $n \in \mathbb{N}$, we may test $\tilde{A}(N, \alpha_n, \beta, \gamma)$ by $\{v_n\}$, which results in

$$\begin{aligned}
\tilde{B}(N,\beta,\gamma) + o(1) &= \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}|u_n(y)|^{\frac{N}{N-1}})\frac{dy}{|y|^\beta} \\
&= \lambda_n^{N-\beta}\int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta}\|\nabla u_n\|_N^{\frac{N}{N-1}}|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&= \lambda_n^{N-\beta}\int_{\mathbb{R}^N} \Phi_N(\alpha_n|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&\le \lambda_n^{N-\beta}\left(\frac{1}{\|v_n\|_{N,\beta}^N}\right)^{\frac{N-\beta}{N-\gamma}}\int_{\mathbb{R}^N} \Phi_N(\alpha_n|v_n(x)|^{\frac{N}{N-1}})\frac{dx}{|x|^\beta} \\
&\le \lambda_n^{N-\beta}\tilde{A}(N,\alpha_n,\beta,\gamma) = \left(\frac{1-\|\nabla u_n\|_N^N}{\|\nabla u_n\|_N^N}\right)^{\frac{N-\beta}{N-\gamma}}\tilde{A}(N,\alpha_n,\beta,\gamma) \\
&= \left(\frac{1-\left(\frac{\alpha_n}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha_n}{\alpha_{N,\beta}}\right)^{N-1}}\right)^{\frac{N-\beta}{N-\gamma}}\tilde{A}(N,\alpha_n,\beta,\gamma) \\
&\le \sup_{\alpha\in(0,\alpha_{N,\beta})}\left(\frac{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}\right)^{\frac{N-\beta}{N-\gamma}}\tilde{A}(N,\alpha,\beta,\gamma).
\end{aligned}$$

Here we have used a change of variables $y = \lambda_n x$ for the second equality, and $\|v_n\|_{N,\gamma}^{\frac{N(N-\beta)}{N-\gamma}} \le 1$ for the first inequality. Letting $n \to \infty$, we have the desired result. $\square$

Proof of Theorem 5: Again, we prove theorem for $\tilde{A}(N,\alpha,\beta,\gamma)$ only. The assertion that

$$\tilde{A}(N,\alpha,\beta,\gamma) \le \left(\frac{C_2}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}\right)^{\frac{N-\beta}{N-\gamma}}$$

follows form Theorem 4 and the fact that $\tilde{B}(N,\beta,\gamma) < \infty$ when $0 \le \gamma \le \beta < N$.

For the rest, we need to prove that there exists $C > 0$ such that for any $\alpha < \alpha_{N,\beta}$ sufficiently close to $\alpha_{N,\beta}$, it holds that

$$\left(\frac{C}{1-\left(\frac{\alpha}{\alpha_{N,\beta}}\right)^{N-1}}\right)^{\frac{N-\beta}{N-\gamma}} \le \tilde{A}(N,\alpha,\beta,\gamma). \tag{5.3}$$

For that purpose, we use the weighted Moser sequence (2.10) again. By (2.12), we have $N_1 \in \mathbb{N}$ such that if $n \in \mathbb{N}$ satisfies $n \geq N_1$, then it holds

$$\|u_n\|_{N,\gamma}^N \leq \frac{2(N-\gamma)\Gamma(N+1)}{(N-\beta)^{N+1}}(1/n). \tag{5.4}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)})\frac{dx}{|x|^\beta} &\geq \omega_{N-1}\int_0^{e^{-b_n}} \Phi_N\left(\alpha(A_n b_n)^{N/(N-1)}\right) r^{N-1-\beta}dr \\ &= \frac{\omega_{N-1}}{N-\beta}\Phi_N\left((\alpha/\alpha_{N,\beta})n\right)\left[r^{N-\beta}\right]_{r=0}^{r=e^{-b_n}} \\ &= \frac{\omega_{N-1}}{N-\beta}\Phi_N\left((\alpha/\alpha_{N,\beta})n\right)e^{-n}.\end{aligned}$$

Note that there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $\Phi_N\left((\alpha/\alpha_{N,\beta})n\right) \geq \frac{1}{2}e^{(\alpha/\alpha_{N,\beta})n}$. Thus we have

$$\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)})\frac{dx}{|x|^\beta} \geq \frac{1}{2}\left(\frac{\omega_{N-1}}{N-\beta}\right)e^{-(1-\frac{\alpha}{\alpha_{N,\beta}})n}. \tag{5.5}$$

Combining (5.4) and (5.5), we have $C_1(N,\beta,\gamma) > 0$ such that

$$\frac{1}{\|u_n\|_{N,\gamma}^{\frac{N(N-\beta)}{N-\gamma}}}\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)})\frac{dx}{|x|^\beta} \geq C_1(N,\beta,\gamma)n^{\frac{N-\beta}{N-\gamma}}e^{-(1-\frac{\alpha}{\alpha_{N,\beta}})n} \tag{5.6}$$

holds when $n \geq \max\{N_1, N_2\}$.

Note that $\lim_{x\to 1}\left(\frac{1-x^{N-1}}{1-x}\right) = N-1$, thus

$$\frac{1-(\alpha/\alpha_{N,\beta})^{N-1}}{1-(\alpha/\alpha_{N,\beta})} \geq \frac{N-1}{2}$$

if $\alpha/\alpha_{N,\beta} < 1$ is very close to 1. Now, for any $\alpha > 0$ sufficiently close to $\alpha_{N,\beta}$ so that

$$\begin{cases} \max\{N_1, N_2\} < \left(\frac{2}{1-\alpha/\alpha_{N,\beta}}\right), \\ \frac{1-(\alpha/\alpha_{N,\beta})^{N-1}}{1-(\alpha/\alpha_{N,\beta})} \geq \frac{N-1}{2}, \end{cases} \tag{5.7}$$

we can find $n \in \mathbb{N}$ such that

$$\begin{cases} \max\{N_1, N_2\} \leq n \leq \left(\frac{2}{1-\alpha/\alpha_{N,\beta}}\right), \\ \left(\frac{1}{1-\alpha/\alpha_{N,\beta}}\right) \leq n. \end{cases} \tag{5.8}$$

We fix $n \in \mathbb{N}$ satisfying (5.8). Then by $1 \le n(1-\alpha/\alpha_{N,\beta}) \le 2$, (5.6) and (5.7), we have

$$\frac{1}{\|u_n\|_{N,\beta}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha |u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \ge C_1(N,\beta,\gamma) n^{\frac{N-\beta}{N-\gamma}} e^{-2}$$

$$\ge C_2(N,\beta,\gamma) \left(\frac{1}{1-(\alpha/\alpha_{N,\beta})} \right)^{\frac{N-\beta}{N-\gamma}} \ge \frac{N-1}{2} C_2(N,\beta,\gamma) \left(\frac{1}{1-(\alpha/\alpha_{N,\beta})^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}}$$

$$= C_3(N,\beta,\gamma) \left(\frac{1}{1-(\alpha/\alpha_{N,\beta})^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}},$$

where $C_2(N,\beta,\gamma) = e^{-2} C_1(N,\beta,\gamma)$ and $C_3(N,\beta,\gamma) = \frac{N-1}{2} C_2(N,\beta,\gamma)$. Thus we have (5.3) for some $C > 0$ independent of α which is sufficiently close to $\alpha_{N,\beta}$. □

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