Maximization problem on Trudinger-Moser inequality involving Lebesgue norm

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Abstract

We study the maximization problem of the Trudinger-Moser inequality. In this study, we consider the effect of the Lebesgue norm on attainability of the best constant. By the Lebesgue norm, there exists a borderline related to existence and non-existence.

Keywords: maximization problem, Trudinger-Moser inequality, two dimension, variational problem

1. Introduction

Assume that $N \geq 2$, $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the surface area of $(N-1)$-dimensional unit sphere. The classical Trudinger-Moser inequality asserts that for any bounded domain $\Omega \subset \mathbb{R}^N$,

$$\sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_N \leq 1}} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}} dx < +\infty.$$

Due to [11], α_N is the largest possible constant. In [3], they showed that maximizer exists when Ω is a unit ball. In general bounded domain case, the existence result was shown in 2-dimensional case by [6]. In the N-dimensional general bounded domain case, existence of the maximizer was shown by [9]. Recently, [12] was proved the existence of the maximizer by using the blow up analysis different from the technique in [3]. In addition to these, there

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are many researches on the Trudinger-Moser inequality. For example, whole space case is due to [1, 14, 8, 7, 4], the Trudinger-Moser inequality on $W^{1,N}(\Omega)$ is due to [15], and so on.

In this study, we consider the case when $N = 2$, $\Omega = B$, where B is a unit ball. We investigate existence and non-existence of maximizer for

$$C(\lambda, p) := \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2 \leq 1}} \int_B \left(e^{\alpha_2 u^2} - \lambda |u|^p \right) dx,$$

where λ is a positive parameter and $p \geq 1$. We can regard this problem as the perturbed Trudinger-Moser inequality

$$I_\alpha^g := \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2^2 \leq \alpha}} \int_B (1 + g(u)) e^{u^2} dx,$$

and our problem is the case when $\alpha = \alpha_2$ and $g(u) = -\lambda \frac{|u|^p}{\alpha_2^{p/2} e^{u^2}}$. In [12], they consider I_α^g under the following conditions:

$$\alpha \leq \alpha_2, \quad g \in C^1(\mathbb{R}), \quad \inf_{\mathbb{R}} g > -1, \quad g(t) = g(-t), \quad \lim_{|t| \to \infty} g(t) = 0.$$

For example, $g \equiv 0$ satisfies the above conditions, and thus the maximization problem on the classical Trudinger-Moser inequality is included this. They studied the blow up analysis for $H_0^1(B)$-norm of the non-compact sequence on the Trudinger-Moser functional. Their results are useful to study the maximizing problem I_α^g. In our problem, the perturbation g does not satisfy the condition $\inf_{\mathbb{R}} g > -1$. This fact causes the problem that there is the possibility of $I_\alpha^g = I_{\alpha_2}^g$ for some $\alpha < \alpha_2$, because the function $e^{\alpha_2 |\cdot|^2} - \lambda |\cdot|^p$ is not a increasing function.

In [5], they studied the case when $p = 2$. They showed existence of maximizer for $\lambda < \alpha_2$ and they supposed that the maximizer does not exists for $\lambda \geq \alpha_2$. Here, we can see that this term $\alpha_2 |\cdot|^2$ in their conjecture is the second term of $e^{\alpha_2 |\cdot|^2}$, that is, we observe

$$e^{\alpha_2 |\cdot|^2} = 1 + \alpha_2 |\cdot|^2 + \sum_{k=2}^{\infty} \frac{\alpha_2^k |\cdot|^{2k}}{k!},$$

and the term is appeared. We forecast that there is a relation between this term and the attainability of the maximization problem. Therefore, we consider the maximization problem $C(\lambda, p)$, and we study the effect of the L^p-term. The main theorem is as follows:

Theorem 1.1. *(I) If $p > 2$, then for any λ, there exists a maximizer of $C(\lambda, p)$.*

(II) If $p \in [1, 2]$, then there exists a positive constant $\lambda_* = \lambda_*(p)$ such that

(i) For $\lambda < \lambda_*$, there exists a maximizer of $C(\lambda, p)$.

(ii) For $\lambda > \lambda_*$, maximizer of $C(\lambda, p)$ does not exist.

(III) If $p \in [1, 2)$, then there exists a maximizer of $C(\lambda_*, p)$, where λ_* is a positive constant obtained in the part (II).

From these results, we can see that 2 is the borderline on the exponent p essentially. In addition, we can show the conjecture of [5] partially. In order to prove this theorem, we use the blow up analysis. We apply the techniques in [8], [15], and [12]. The techniques of [8], [15] are similar. However, the strategy of [12] are different from those of the others. In [8] and [15], they study the behavior of the Trudinger-Moser functional on the concentrating sequence. On the other hand, in [12], as we referred to before, they investigated the behavior of $H_0^1(B)$-norm of the concentrating sequence. We will combine these two techniques.

This paper is organized as follows. In Section 2, we prepare some lemmas and propositions. Especially, we investigate the properties of $C(\lambda, p)$ on the parameter λ. In Section 3, we prove the theorem. To prove this, we prepare two important propositions. In Section 4, we show these two propositions by applying the technique in [8], [15], and [12].

2. Preliminaries

First, we fix some notation. The $L^q(B)$-norm is written as $\|\cdot\|_q$. The constant C_0 is defined by $C(0, p)$. For simplicity, sometimes we write $v(r)$ as the radially symmetric function $v(x)$ by supposing that $r = |x|$. For a function v, we define v_+ and v_- as $v_+ := \max\{v, 0\}$ and $v_- := \min\{v, 0\}$.

We prepare some lemmas and proposition to prove Theorem 1.1. We set

$$C_{rad}(\lambda, p) = \sup_{\substack{u \in H^1_{0,rad}(B) \\ \|\nabla u\|_2 \le 1}} \int_B \left(e^{\alpha_2 u^2} - \lambda |u|^p \right) dx,$$

where $H^1_{0,rad}(B)$ is the set of radially symmetric functions in $H_0^1(B)$. By the symmetrization of function in $H_0^1(B)$, we can see that $C(\lambda, p) = C_{rad}(\lambda, p)$

and existence of maximizer of $C(\lambda, p)$ is equivalent to existence of maximizer of $C_{rad}(\lambda, p)$.

We take a sequence $\{u_n\}$ satisfying

$$\{u_n\} \subset H^1_{0,rad}(B), \quad \|\nabla u_n\|_2 \leq 1, \quad u_n \rightharpoonup 0 \text{ weakly in } H^1_0(B) \tag{1}$$

$$\lim_{n\to\infty} \|\nabla u_n\|_2 \to 1, \quad \lim_{n\to\infty} \|\nabla u_n\|_{L^2(B\setminus B_\varepsilon)} = 0 \quad \text{for any } \varepsilon > 0. \tag{2}$$

We call $\{u_n\}$ satisfying (1), (2) a normalized concentrating sequence. Then we have the following upper bound:

Proposition 2.1 ([3])**.** *For any normalized concentrating sequence $\{u_n\}$, we have*

$$\limsup_{n\to\infty} \int_B e^{\alpha_2 u_n^2} dx \leq |B| + e\pi.$$

Proposition 2.2 ([5])**.** *There exists a normalized concentrating sequence $\{y_n\}$ such that*

$$\lim_{n\to\infty} \int_B e^{\alpha_2 y_n^2} dx = |B| + e\pi.$$

More precisely, for sufficiently large n, y_n satisfies

$$\int_B e^{\alpha_2 y_n^2} dx = |B| + e\pi + \varepsilon_n,$$

where ε_n is a positive constant such that $\varepsilon_n \to 0$ as $n \to \infty$.

The following lemma follows from the definition of $C(\lambda, p)$ and Proposition 2.2.

Lemma 2.3. *(i) $C(\lambda, p)$ is continuous and non-increasing with respect to λ.*

(ii) We have $C(\lambda, p) \geq |B| + e\pi$ for any λ and p.

Proposition 2.4. *For any $t \in [0, 1)$, we have*

$$\sup_{\substack{u\in H^1_0(B) \\ \|\nabla u\|_2 \leq t}} \int_B \left(e^{\alpha_2 u^2} - \lambda |u|^p \right) dx < C(\lambda, p).$$

Proof. By the part (ii) of Lemma 2.3, we can see that 0 is not maximizer. Set

$$C_t(\lambda,p)=\sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2\le t}}\int_B\left(e^{\alpha_2u^2}-\lambda|u|^p\right)dx$$

and assume that $C_t(\lambda,p)=C(\lambda,p)$. We take a maximizing sequence $\{u_n\}\subset H_0^1(B)$, that is,

$$\|\nabla u_n\|_2\le t,\quad \lim_{n\to\infty}\int_B\left(e^{\alpha_2u_n^2}-\lambda|u_n|^p\right)dx=C(\lambda,p).$$

Then we have $u_n\rightharpoonup u_\infty$ weakly in $H_0^1(B)$ and $\|\nabla u_\infty\|_2=\tilde t\le t$. Moreover, by the compactness of the Trudinger-Moser functional and the Sobolev embedding, it follows that

$$\int_B\left(e^{\alpha_2u_\infty^2}-\lambda|u_\infty|^p\right)dx=\lim_{n\to\infty}\int_B\left(e^{\alpha_2u_n^2}-\lambda|u_n|^p\right)dx=C(\lambda,p).$$

In addition, we may assume that $u_\infty\ge0$. Since u_∞ is also the maximizer of

$$\sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2=\tilde t}}\int_B\left(e^{\alpha_2u^2}-\lambda|u|^p\right)dx,$$

there exists the Lagrange multiplier L such that

$$L\int_B\nabla u_\infty\nabla\phi dx-\int_B\left(\alpha_2u_\infty e^{\alpha_2u_\infty^2}-\frac{p}{2}\lambda u_\infty^{p-1}\right)\phi dx=0$$

for any $\phi\in H_0^1(B)$. On the other hand, for $s\in[0,1/\tilde t]$ we set

$$f(s):=\int_B\left[e^{\alpha_2(su_\infty)^2}-\lambda(s|u_\infty|)^p\right]dx.$$

Then since $f'(s)|_{s=1}=0$ we have

$$\int_B\left(2\alpha_2u_\infty^2e^{\alpha_2u_\infty^2}-p\lambda u_\infty^p\right)dx=0,$$

and hence $L=0$. From this, it follows that

$$e^{\alpha_2u_\infty^2}-\frac{p}{2\alpha_2}\lambda u_\infty^{p-2}=0.$$

for any $x\in B$. However, for any λ and p, this equality does not hold for x near ∂B since $u_\infty|_{\partial B}=0$. $\square$

Lemma 2.5. *(i) If $C(\lambda,p) > |B| + e\pi$, then maximizer of $C(\lambda,p)$ exists.*

(ii) If there exists λ_* such that $C(\lambda_*,p) = |B| + e\pi$, then for $\lambda > \lambda_*$ maximizer does not exist.

Proof. We prove (i). Assume that $\{u_n\}$ is a maximizing sequence of $C(\lambda,p)$, namely, $\{u_n\}$ satisfies

$$\{u_n\} \subset H^1_{0,rad}(B), \quad \|\nabla u_n\|_2 \le 1, \quad \lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda |u_n|^p\right) dx = C(\lambda,p).$$

Since $\{u_n\}$ is bounded sequence, there exists u_∞ such that up to a subsequence $u_n \rightharpoonup u_\infty$ weakly in $H^1_0(B)$, and $\|\nabla u_\infty\|_2 \le 1$. By the assumption and Proposition 2.1, we can see that $\{u_n\}$ is not normalized concentrating sequence. Therefore by the theorem in [10] and the Sobolev embedding, we have

$$\lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda |u_n|^p\right) dx = \int_B \left(e^{\alpha_2 u_\infty^2} - \lambda |u_\infty|^p\right) dx.$$

Consequently u_∞ is the maximizer.

We prove (ii). Assume that $\lambda > \lambda_*$ and $u_\lambda \in H^1_{0,rad}(B)$ is a maximizer of $C(\lambda,p)$. Then we have

$$\begin{aligned} |B| + e\pi &\le C(\lambda,p) = \int_B \left(e^{\alpha_2 u_\lambda^2} - \lambda |u_\lambda|^p\right) dx \\ &< \int_B \left(e^{\alpha_2 u_\lambda^2} - \lambda_* |u_\lambda|^p\right) dx \le C(\lambda_*,p) = |B| + e\pi. \end{aligned}$$

This is a contradiction. □

Proposition 2.6 (Proposition B.1 in [2]). *We define the function with positive parameter t, which is introduced in [11] originally.*

$$m_t(r) = t^{\frac{1}{2}} m_1(r^{\frac{1}{t}}) = \omega_1^{-\frac{1}{2}} (\log 2)^{\frac{1}{2}} t^{\frac{1}{2}} \min\left\{\frac{\log \frac{1}{r}}{t(\log 2)}, 1\right\},$$

where

$$m_1(r) := \omega_1^{-\frac{1}{2}} (\log 2)^{\frac{1}{2}} \min\left\{\frac{\log \frac{1}{r}}{\log 2}, 1\right\}.$$

Assume that $u_n \in H^1_{0,rad}(B)$ satisfying $\|\nabla u_n\| \le 1$, $u_n \rightharpoonup 0$ weakly in $H^1_0(B)$. In addition, u_n satisfies

$$\liminf_{n\to\infty} \int_B e^{\alpha_2 u_n^2} dx > |B|.$$

Then, there exists a sequence $\{t_n\} \subset (0,1)$ *such that*

$$u_n - m_{t_n} \to 0 \quad \text{strongly in } H_0^1(B).$$

Remark 2.1. In [2], they wrote that this proposition holds for all dimension. However, the author can confirm the validity of this proposition only when $N = 2$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We prepare the following two propositions. We will prove these in the next section.

Proposition 3.1. *We assume that* $p \in [1,2]$*, and that there exists a maximizer of* $C(\lambda, p)$ *for sufficiently large* λ*. We write* u_λ *as the maximizer for* λ*. Then, there exist positive constants* $C_1 = C_1(p), C_2 = C_2(p)$ *such that for sufficient large* λ *we have*

$$\|\nabla u_\lambda\|_2^2 \leq 1 - \lambda \frac{C_1}{\|u_\lambda\|_\infty^{p+2}} + \frac{C_2}{\|u_\lambda\|_\infty^4} + o(\|u_\lambda\|_\infty^{-4}).$$

Proposition 3.2. *We fix* λ *and* p*. Assume that* u_k *is a maximizer of*

$$C_k(\lambda, p) := \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2 \leq 1}} \int_B \left(e^{\alpha_k u^2} - \lambda |u|^p \right) dx,$$

where α_k *is a sequence of real numbers such that* $\alpha_k \nearrow \alpha_2$ *as* $k \to \infty$*. If* $\sup_{x \in B} u_k(x) = u(0) \to \infty$ *as* $k \to \infty$*, then there exists a positive constant* $C = C(p)$ *such that*

$$\|\nabla u_k\|_2^2 \geq \frac{\alpha_2}{\alpha_k} \left(1 + \frac{1}{\alpha_2^2} \frac{1}{\|u_k\|_\infty^4} - \lambda \frac{C}{\|u_k\|_\infty^{p+2}} \right) + o(\|u_k\|_\infty^{-4}).$$

First, we prove the part (I). Since it follows that

$$\lim_{\varepsilon \to 0} \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2 \leq 1}} \int_B \left(e^{(\alpha_2 - \varepsilon) u^2} - \lambda |u|^p \right) dx = C(\lambda, p),$$

u_k defined in Proposition 3.2 is a maximizing sequence of $C(\lambda, p)$. In addition, since $\|\nabla u_k\|_2 \leq 1$, there exists $u_\infty \in H_0^1(B)$ such that up to a subsequence

$u_n \rightharpoonup u_\infty$ weakly in $H_0^1(B)$, and $\|\nabla u_\infty\|_2 \leq 1$. For $p > 2$, if $\sup_{x\in B} u_k(x) \to \infty$ as $k \to \infty$, then $\|\nabla u_k\|_2^2 > 1$ for sufficient large k by Proposition 3.2, which contradicts that $\|\nabla u_k\|_2 \leq 1$. Thus $\sup_{x\in B} u_k(x)$ is bounded for k uniformly. By the dominated convergence theorem, we have

$$\int_B \left(e^{\alpha_2 u_\infty^2} - \lambda|u_\infty|^p\right) dx = \lim_{k\to\infty} \int_B \left(e^{\alpha_k u_k^2} - \lambda|u_k|^p\right) dx = \lim_{k\to\infty} C_k(\lambda, p) = C(\lambda, p).$$

Consequently u_∞ is the maximizer of $C(\lambda, p)$.

Next, we prove the part (II). In order to show existence of λ_* we show nonexistence of maximizer for large λ. For $p \in [1, 2]$, we assume that u_λ is a maximizer for sufficiently large λ. By Proposition 2.4, we have $\|\nabla u_\lambda\|_2 = 1$. However, by Proposition 3.1 we have $\|\nabla u_\lambda\|_2 < 1$ for sufficiently large λ, which is a contradiction. Therefore maximizer does not exists for sufficiently large λ. By part (i) of Lemma 2.5 we can find that $C(\lambda, p) = |B| + e\pi$ for sufficiently large λ. We define $\lambda_* \in [0, \infty)$ as

$$\lambda_* := \inf\left\{\lambda > 0 | C(\lambda, p) = |B| + e\pi\right\}.$$

By Lemma 2.3, we can see that $C(\lambda, p) = |B| + e\pi$ for $\lambda \geq \lambda_*$ and $C(\lambda, p) > |B| + e\pi$ for $\lambda \in (0, \lambda_*)$. Consequently, this λ_* is the borderline of the result of part (II) by Lemma 2.5. To finish the proof of the part (II), we have to confirm that $\lambda_* > 0$. By Proposition 2.2, we have $C_0 > |B| + e\pi$. Thus we have $C(\lambda, p) > |B| + e\pi$ for sufficiently small λ and hence $\lambda_* > 0$.

Finally, we prove the part (III). We set a sequence λ_n such that $\lambda_n \nearrow \lambda_*$ as $n \to \infty$. By the part (II), maximizer of $C(\lambda_n, p)$ exists and we write this as u_n. By the part of (i) of Lemma 2.3, we can see that u_n is a maximizing sequence of $C(\lambda_*, p)$. If $\sup_{x\in B} u_n(x) \to \infty$ as $\lambda \to \infty$, we can show that $\|\nabla u_n\|_2 < 1$ for sufficiently large n by preparing similar proposition to Proposition 3.1. This contradicts that $\|\nabla u_n\|_2 = 1$. Consequently $\sup_{x\in B} u_n(x)$ is bounded uniformly for sufficiently large n, and by the dominated convergence theorem we can show existence of maximizer.

4. Proof of Proposition 3.1 and 3.2

Since the proof of Proposition 3.2 is close to the proof of Proposition 3.1 we only prove Proposition 3.1. Fix $p \in [1, 2]$ and set a sequence λ_k such that $\lambda_k \to \infty$ as $k \to \infty$ (a suitable subsequence is also written by λ_k). Assume that $u_k := u_{\lambda_k} \in H_0^1(B)$ is the maximizer of $C(\lambda_k, p)$. From Proposition 2.4, we have

$$\|\nabla u_k\|_2 = 1, \quad u_k \in H^1_{0,rad}(B), \quad u_k \geq 0, \quad \frac{\partial u_k}{\partial r} \text{ is a decreasing function.}$$

By the part (ii) of Lemma 2.3 we have

$$|B| + e\pi \leq \lim_{k\to\infty} \int_B \left(e^{\alpha_2 u_k^2} - \lambda_k |u_k|^p \right) dx \leq C(0,p) - \lim_{k\to\infty} \lambda_k \int_B |u_k|^p dx.$$

Thus up to a subsequence

$$\int_B |u_k|^p dx = O(\lambda_k^{-1}), \quad u_k(x) \to 0 \quad \text{for } x \in B \setminus \{0\}.$$

In addition, since

$$|B| + e\pi \leq \lim_{k\to\infty} \int_B \left(e^{\alpha_2 u_k^2} - \lambda_k |u_k|^p \right) dx \leq \lim_{k\to\infty} \int_B e^{\alpha_2 u_k^2} dx \tag{3}$$

we have

$$\lim_{k\to\infty} \sup_{x\in B} u_k = \lim_{k\to\infty} u_k(0) = \infty.$$

By the Lagrange multiplier theorem, u_k is a solution of

$$\begin{cases} -\Delta u = \frac{\alpha_2}{M_k} \left(u e^{\alpha_2 u^2} - \frac{p}{2\alpha_2} \lambda_k u^{p-1} \right), \quad u > 0, & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where

$$M_k := \alpha_2 \int_B \left(u_k^2 e^{\alpha_2 u_k^2} - \frac{p}{2\alpha_2} \lambda_k u_k^p \right) dx.$$

By setting $v_k := \alpha_2^{1/2} u_k$, v_k satisfies

$$\begin{cases} -\Delta v_k = \frac{\alpha_2}{M_k} \left(v_k e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k v_k^{p-1} \right), \quad v_k > 0, & \text{in } B, \\ v_k = 0 & \text{on } \partial B, \end{cases} \tag{4}$$

and

$$\|\nabla v_k\|_2^2 = \alpha_2, \quad M_k = \int_B \left(v_k^2 e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k v_k^p \right) dx. \tag{5}$$

We note that $\lim_{k\to\infty} v_k = 0$ in $B \setminus \{0\}$ and $\lim_{k\to\infty} v_k(0) = \infty$. For simplicity, we set

$$c_k := v_k(0) = \sup_{x\in B} v_k(x).$$

Lemma 4.1. *We have*

$$\lim_{k\to\infty} \lambda_k \int_B v_k^p dx = 0.$$

Proof. We show that $\lim_{k\to\infty} \lambda_k \int_B u_k^p dx = 0$. By the part (ii) of Lemma 2.3 and (3), we can find that u_k is a normalized concentrating sequence. Thus by Proposition 2.1, it follows that

$$\lim_{k\to\infty} \int_B e^{\alpha_2 u_k^2} dx = |B| + e\pi.$$

This observation and (3) yield the equality of the lemma. □

Lemma 4.2. *For sufficient large k, we have*

$$M_k > \alpha_2 e\pi + o(1).$$

Proof. By the part (ii) of Lemma 2.3 and Lemma 4.1, we have

$$\begin{aligned}
|B| + e\pi &\leq \int_B \left(e^{\alpha_2 u_k^2} - \lambda_k u_k^p\right) dx \\
&\leq \int_{[u_k \leq 1]} e^{\alpha_2 u_k^2} dx + \int_{[u_k > 1]} e^{\alpha_2 u_k^2} dx - \lambda_k \int_B u_k^p dx \\
&\leq |B| + o(1) + \int_{[u_k > 1]} u_k^2 e^{\alpha_2 u_k^2} dx - \lambda_k \int_B u_k^p dx \\
&\leq |B| + \frac{M_k}{\alpha_2} + o(1).
\end{aligned}$$

□

Lemma 4.3. *For sufficiently large k, we have*

$$M_k \leq c_k^2 \left(|B| + e\pi + o(1)\right).$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
M_k &= \int_B \left(v_k^2 e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k v_k^p\right) dx \\
&\leq c_k^2 \int_B e^{v_k^2} dx + o(1) \\
&= c_k^2 \left(|B| + e\pi + o(1)\right).
\end{aligned}$$

□

We set

$$r_k := \frac{\sqrt{M_k}}{\sqrt{\pi} c_k} e^{-\frac{c_k^2}{2}},$$

and

$$\begin{cases} \phi_k(y) := c_k(v_k(r_k y) - c_k), \\ \psi_k(y) := c_k^{-1} v_k(r_k y). \end{cases}$$

Then, we have

$$-\Delta \phi_k = 4 \left[\psi_k e^{\phi_k(1+\psi_k)} - \frac{p}{2\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2} \lambda \psi_k^{p-1} \right] \quad \text{in } B_{1/r_k} \tag{6}$$

$$-\Delta \psi_k = \frac{4}{c_k^2} \left[\psi_k e^{c_k^2(\psi_k - 1)} - \frac{p}{2\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2} \lambda_k \psi_k \right] \quad \text{in } B_{1/r_k} \tag{7}$$

Multiplying (7) by $c_k^2 \psi_k$, integrating on B_{1/r_k}, and we have

$$\alpha_2 = 4 \left[\int_{B_{1/r_k}} \psi_k^2 e^{c_k^2(\psi_k^2 - 1)} dx - \frac{p}{2\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2} \lambda_k \int_{B_{1/r_k}} \psi_k^2 dx \right],$$

and thus

$$\frac{2p}{\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2} \lambda_k = \frac{4 \int_{B_{1/r_k}} \psi_k^2 e^{c_k^2(\psi_k^2-1)} dx - \alpha_2}{\int_{B_{1/r_k}} \psi_k^2 dx} \le 4 - \frac{\alpha_2}{\int_{B_{1/r_k}} \psi_k^2 dx}. \tag{8}$$

Thus we can use the elliptic regularity theory. We have

$$\psi_k \to 1 \quad \text{in } C^2_{loc}(\mathbb{R}^2). \tag{9}$$

On the other hand, in (8), by Lemma 4.1 we have

$$4 \int_{B_{1/r_k}} \psi_k^2 e^{c_k^2(\psi_k^2-1)} dx = \frac{\alpha_2}{M_k} \int_B v_k^2 e^{v_k^2} dx = \alpha_2 + \frac{\alpha_2}{M_k} \frac{p}{2\alpha_2^{p/2}} \lambda_k \int_B v_k^p = \alpha_2 + o(1),$$

and by (9)

$$\lim_{k\to\infty} \int_{B_{1/r_k}} \psi_k^2 dx \ge \lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R} \psi_k^2 dx = \lim_{R\to\infty} |B_R| = \infty.$$

Hence

$$\frac{p}{2\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2} \lambda_k = o(1).$$

Concerning the equation (6), by the elliptic regularity theory,

$$\phi_k \to \phi_\infty = -\log(1+|x|^2) \quad \text{in } C^2_{loc}(\mathbb{R}^2),$$

$$-\Delta\phi_\infty = 4e^{2\phi_\infty} \quad \text{in } \mathbb{R}^2.$$

For a constant $\rho > 1$ we set

$$v_{k,\rho} := \min\left\{\frac{c_k}{\rho}, v_k\right\}. \tag{10}$$

Lemma 4.4.

$$\lim_{k\to\infty} \int_B |\nabla v_{k,\rho}|^2 dx = \frac{\alpha_2}{\rho}$$

The strategy is same as the proof of Lemma 3.3 in [8] and the proof of Lemma 3.6 in [15].

Lemma 4.5. *We have*

$$\liminf_{k\to\infty} \frac{M_k}{c_k^2} \geq e\pi.$$

Proof. By Lemma 4.4 we have

$$\begin{aligned}
\int_B e^{v_k^2} dx &= \int_{[v_k < c_k/\rho]} e^{v_k^2} dx + \int_{[v_k \geq c_k/\rho]} e^{v_k^2} dx \\
&\leq \int_B e^{v_{k,\rho}^2} + \frac{\rho^2}{c_k^2} \int_B v_k^2 e^{v_k^2} dx \\
&= |B| + \frac{\rho^2}{c_\lambda^2} \frac{M_k}{\alpha_2} \left[\frac{\alpha_2}{M_k} \int_B \left(v_k^2 e^{v_k^2} dx - \frac{p}{2\alpha_2^{p/2}} \lambda_k v_k^p \right) dx + \frac{\alpha_2}{M_k} \frac{p}{2\alpha_2^{p/2}} \lambda_k \int_B v_k^p dx \right] + o(1) \\
&= |B| + \frac{\rho^2}{c_\lambda^2} M_k + o(1).
\end{aligned}$$

On the left hand side, we have

$$\int_B e^{v_k^2} dx = \int_B e^{\alpha_2 u_k^2} dx = |B| + e\pi + o(1).$$

Hence we obtain the inequality of the lemma. □

Combining Lemma 4.3 and 4.5, for sufficiently large k

$$e\pi + o(1) \le \frac{M_k}{c_k^2} \le |B| + e\pi + o(1). \tag{11}$$

Lemma 4.6. *For any $\phi \in C^\infty(B)$ we have*

$$\lim_{k\to\infty} \frac{1}{M_k} \int_B c_k v_k e^{v_k^2} \phi dx = \phi(0).$$

We can prove in the same way as the proof of Lemma 3.6 in [8] and the proof of Lemma 3.9 in [15].

Proposition 4.7. *There exists positive constants C_1, C_2 such that for sufficiently large k we have*

$$\frac{C_1}{c_k^p} \le \int_B v_k^p dx \le \frac{C_2}{c_k^p}$$

Proof. We consider the equation:

$$\begin{cases} -\Delta(c_k v_k) = \frac{\alpha_2}{M_k}\left[(c_k v_k)e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}}\lambda_k c_k v_k^{p-1}\right] & \text{in } B, \\ v_k = 0 & \text{on } \partial B. \end{cases}$$

It follows that

$$\begin{aligned} \int_B (\Delta c_k v_k)_+ dx &= -\frac{\alpha_2}{M_k}\int_{[\Delta c_k v_k > 0]} \left(c_k v_k e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k c_k v_k^p \right) dx \\ &\le \frac{\alpha_2}{M_k} \int_B \frac{p}{2\alpha_2^{p/2}} \lambda_k c_k v_k^{p-1} dx \\ &= \alpha_2 + o(1) - \frac{\alpha_2}{M_\lambda}\left(\int_B c_k v_k e^{v_k^2} dx - \int_B \frac{p}{2\alpha_2^{p/2}} \lambda_k c_k v_k^{p-1} dx \right) \\ &= \alpha_2 + o(1) + \int_B \Delta(c_k v_k) dx \\ &= \alpha_2 + o(1) + c_k \int_{\partial B} \frac{\partial v_k}{\partial \nu} d\sigma \\ &\le \alpha_2 + o(1), \end{aligned}$$

and

$$
\begin{aligned}
-\int_B (\Delta c_k v_k)_- dx &= \frac{\alpha_2}{M_\lambda} \int_{[\Delta c_k v_k \le 0]} \left(c_k v_k e^{v_k^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k c_k v_k^{p-1} \right) dx \\
&\le \frac{\alpha_2}{M_k} \int_B c_k v_k e^{v_k^2} dx \\
&= \alpha_2 + o(1).
\end{aligned}
$$

Thus we have $\int_B |\Delta c_\lambda v_k| dx < 2\alpha_2 + o(1)$ and hence there exists $w \in W^{2,1}(B)$ such that

$$c_k v_k \rightharpoonup w \quad \text{weakly in} \quad W^{2,1}(B).$$

From this,

$$c_k v_k \to w \quad \text{weakly in} \quad W^{1,q}(B)$$

for any $q \in [1, 2)$, and we have

$$\int_B (c_k v_k)^p dx \to \int_B w^p dx. \tag{12}$$

Moreover, the compact embedding yields that

$$c_k v_k \to w \quad \text{strongly in} \quad C^{0,\alpha}(B_1 \setminus B_\varepsilon) \tag{13}$$

for any $\varepsilon > 0$. To end the proof, we show that $w \neq 0$.

By Proposition 2.6 there exists a sequence $\{t_k\}$ such that

$$\int_B \left| \nabla (v_k - \alpha_2^{1/2} m_{t_k}) \right|^2 dx \to 0,$$

where m_{t_k} is defined as in Proposition 2.6. In addition, since H_0^1-norm has the scaling invariance, it follows that

$$\int_B \left| \nabla \left(t_k^{-\frac{1}{2}} v_k(|x|^{t_k}) - \alpha_2^{1/2} m_1(|x|) \right) \right|^2 dx \to 0.$$

Thus we have

$$t_k^{-\frac{1}{2}} v_k(r^{t_k}) - \alpha_2^{1/2} \omega_1^{-\frac{1}{2}} (\log 2)^{\frac{1}{2}} \to 0$$

for each $r \in (0, 1/2)$. Since v_k is a radially decreasing function, we can find that $t_k = O(c_k^2)$. We go back to Proposition 2.6 and it follows that

$$\int_B \left| \nabla \left(\alpha_2^{1/2} \tilde{m}_{t_k}(|x|) - t_k^{\frac{1}{2}} v_k(r^{\frac{1}{t_k}}) \right) \right|^2 dx \to 0,$$

where

$$\tilde{m}_{t_k}(r) = t_k^{\frac{1}{2}} m_{t_k}(r^{\frac{1}{t_k}}) = t_k m_1(r^{\frac{1}{t_k^2}}).$$

From this, we have

$$t_k^{\frac{1}{2}} v_k(r^{\frac{1}{t_k}}) - \alpha_2^{1/2} \omega_1^{-\frac{1}{2}} (\log 2)^{-\frac{1}{2}} t_k^{-1} \left(\log \frac{1}{r} \right) \to 0,$$

for each $r \in (e^{-(\log 2) t_k^2}, 1)$. Especially, this convergence holds for $r \in (e^{-K t_k}, 1)$ with a positive constant K. Setting $r^{1/t_k} = s$, and we have

$$t_k^{\frac{1}{2}} v_k(s) - \alpha_2^{1/2} \omega_1^{-\frac{1}{2}} (\log 2)^{-\frac{1}{2}} \left(\log \frac{1}{s} \right) \to 0 \tag{14}$$

for each $s \in (e^{-K}, 1)$, in the sense of the pointwise convergence on $r = s^{t_k}$. If $t_k = o(c_k^2)$ holds, then

$$o(1) c_k v_k(s) - \alpha_2^{1/2} \omega_1^{-\frac{1}{2}} (\log 2)^{-\frac{1}{2}} \left(\log \frac{1}{s} \right) \to 0.$$

This contradicts (13). Hence it follows that

$$t_k = c_k^2 \left(C + o(1) \right). \tag{15}$$

Combining (12), (14) and (15), we have

$$t_k = c_k^2 \left(\alpha_2^{-1} \omega_1 (\log 2)^{-1} + o(1) \right), \quad \text{and} \quad w = \alpha_2^{1/2} \omega_1^{-1} \left(\log \frac{1}{|x|} \right).$$

Consequently, we show that $w \neq 0$. $\square$

We note that

$$\frac{\lambda_k}{c_k^p} \to 0. \tag{16}$$

Indeed, from Lemma 4.1 we have

$$o(1) = \lambda_k \int v_k^p dx = \frac{\lambda_k}{c_k^p} \int_B (c_k v_k)^p dx = \frac{\lambda_k}{c_k^p} \left[\int_B w^p dx + o(1) \right].$$

We set $\delta_k \in (0,1)$ as the point such that

$$e^{v_k(\delta_k)^2} - \frac{p}{2\alpha_2^{p/2}} \lambda_k v_k(\delta_k)^{p-2} = 0.$$

Since v_k is decreasing function with respect to r, this point δ_k is unique and

$$e^{v_k(r)^2} - \frac{p}{2\alpha_2^{p/2}}\lambda_k v_k(r)^{p-2} \geq 0 \quad \text{for } r \in [0, \delta_k]$$

$$e^{v_k(r)^2} - \frac{p}{2\alpha_2^{p/2}}\lambda_k v_k(r)^{p-2} < 0 \quad \text{for } r \in (\delta_k, 1).$$

We observe that

$$\frac{\alpha_2}{M_k}\int_B v_k^2 e^{v_k^2} dx = \frac{\alpha_2}{M_k}\int_{B_{\delta_k}} v_k^2 e^{v_k^2} dx + \frac{\alpha_2}{M_k}\int_{B_1\setminus B_{\delta_k}} v_k^2 e^{v_k^2} dx = I_1 + I_2. \quad (17)$$

First, we show that

$$I_2 \leq \frac{C}{c_k^4} \quad (18)$$

for some positive constant C. For $\theta > 1$, we have

$$e^{\left(\frac{c_k}{\theta}\right)^2} - \frac{p}{2\alpha_2^{p/2}}\lambda_k \left(\frac{c_k}{\theta}\right)^{p-2} > e^{\left(\frac{c_k}{\theta}\right)^2} - \frac{p}{2\alpha_2^{p/2}\theta^{p-2}} c_k^{2(p-1)} \to \infty$$

as $k \to \infty$, where we used (16). Thus we have $v_k(\delta_k) \leq c_k/\theta$. We define $v_{k,\theta}$ in the same way as (10). Then by using Lemma 4.4, we have

$$I_2 \leq \frac{\alpha_2}{M_k}\int_B v_{k,\theta}^2 e^{v_{k,\theta}^2} dx \leq \frac{\alpha_2}{M_k}\left(\int_B v_{k,\theta}^{\frac{2\theta}{1-\theta}}\right)^{\frac{1-\theta}{\theta}} \left(\int_B e^{\theta v_{k,\theta}^2} dx\right)^{\frac{1}{\theta}} \leq \frac{C}{c_k^4}.$$

Next, we show that

$$I_1 \leq 4\pi + \frac{6\pi}{c_k^4} + o(c_k^{-4}). \quad (19)$$

In order to prove this we apply the strategy of blow up analysis in [12].

We go back to the equation (4). Recall that as follows:

The function ϕ_k is defined by $\phi_k(y) := c_k(v_k(r_k y) - c_k)$ and ϕ_k satisfies

$$-\Delta_y \phi_k = 4\left(1 + \frac{\phi_k}{c_k^2}\right) e^{\phi_k\left(2+\frac{\phi_k}{c_k^2}\right)} - \frac{2p}{\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2}\lambda_k \left(1 + \frac{\phi_k}{c_k^2}\right)^{p-1} \quad \text{in } B_{1/r_k}. \quad (20)$$

Then $\phi_k \to \phi_\infty := -\log(1 + |x|^2)$ in $C^2_{loc}(\mathbb{R}^2)$ and ϕ_∞ satisfies

$$-\Delta\phi_\infty = 4e^{2\phi_\infty} \quad \text{in } \mathbb{R}^2.$$

From the following lemma to Proposition 4.11, since the proofs are same as those in [13] and [12], we introduce only the statements.

Lemma 4.8. *Set $\eta_k := c_k^2(\phi_k - \phi_\infty)$. Then there exists η_∞ such that $\eta_k \to \eta_\infty$ in $C^2_{loc}(\mathbb{R}^N)$, η_∞ is written by*

$$\eta_\infty(r) = \phi_\infty(r) + \frac{2r^2}{1+r^2} - \frac{1}{2}\phi_\infty^2(r) + \frac{1-r^2}{1+r^2}\int_1^{1+r^2} \frac{\log t}{1-t}dt,$$

and η_∞ is the unique solution of

$$\begin{cases} -\Delta w = 4e^{2\phi_\infty}\left(\phi_\infty + \phi_\infty^2 + 2w\right) & \text{in } \mathbb{R}^2, \\ w(0) = w'(0) = 0. \end{cases}$$

Moreover, $\eta_\infty = \phi_\infty + O(1)$ as $r \to \infty$ and $\int_{\mathbb{R}^2} -\Delta\eta_\infty dx = 4\pi$.

We write

$$\phi_k = \phi_\infty + \frac{\eta_\infty}{c_k^2} + \frac{z_k}{c_k^4}$$

On the second term in the right hand side, we have

$$\frac{2p}{\alpha_2^{p/2}} c_k^{p-2} e^{-c_k^2}\lambda_k = o(c_k^{2p-2}e^{-c_k^2}).$$

Thus we have

$$\begin{aligned} -\Delta\phi_k &= 4\left(1 + \frac{\phi_k}{c_k^2}\right) e^{\phi_k\left(2+\frac{\phi_k}{c_k^2}\right)} + o(c_k^{2p-2}e^{-c_k^2}) \\ &= 4e^{2\phi_\infty}\left[1 + \frac{\phi_\infty + \phi_\infty^2 + 2\eta_\infty}{c_k^2} + \frac{\eta_\infty + 2\eta_\infty^2 + 4\phi_\infty\eta_\infty + 2\eta_\infty\phi_\infty^2 + \phi_\infty^3 + \frac{1}{2}\phi_\infty^4 + 2z_k}{c_k^4}\right] \\ &\quad + O(c_k^{-6}). \end{aligned}$$

From this, we have $z_k \to z_\infty$ in $C^2_{loc}(\mathbb{R}^2)$ and

$$\begin{cases} -\Delta z_\infty = 4e^{2\phi_\infty}\left(\eta_\infty + 2\eta_\infty^2 + 4\phi_\infty\eta_\infty + 2\eta_\infty\phi_\infty^2 + \phi_\infty^3 + \frac{1}{2}\phi_\infty^4 + 2z_k\right) & \text{in } \mathbb{R}^2, \\ z_\infty(0) = z_\infty'(0) = 0. \end{cases}$$

We can observe that

$$z_\infty(r) = \beta\log(r) + O(1) \quad \text{as } r \to \infty,$$

where β is represented by

$$\beta = \frac{1}{2\pi}\int_{\mathbb{R}^2} -\Delta z_\infty dx = 6 + \frac{\pi^2}{3}.$$

We rewrite

$$\phi_k = \phi_\infty + \frac{\eta_\infty}{c_k^2} + \frac{z_\infty}{c_k^4} + \frac{\tau_k}{c_k^6}$$

Lemma 4.9. *Let $0 \leq S \leq s_k \leq e^{c_k}$ and $\tau : [S, s_k \to \mathbb{R}]$ be given so that $\tau = o(c_k^6)$ uniformly on $[S, s_k]$. Set*

$$\phi = \phi_\infty + \frac{\eta_\infty}{c_k^2} + \frac{z_\infty}{c_k^4} + \frac{\tau}{c_k^6},$$

and

$$\mathcal{T}_k(r, \tau) := c_k^6 \left[4\left(1 + \frac{\phi}{c_k^2} e^{2\phi + \frac{\phi}{c_k^2}}\right) + \Delta\phi_\infty + \frac{\Delta\eta_\infty}{c_k^2} + \frac{\Delta z_\infty}{c_k^4} \right].$$

Then

$$\mathcal{T}_k(r, \tau) = 4e^{2\phi_\infty} \left(2\tau + o(1)\tau + O(c_k^{-2}\xi^2)\tau + O(\xi^6)\right)$$

uniformly for $r \in [S, s_k]$, where

$$\xi(r) := 1 + \log(1 + r) \tag{21}$$

Proposition 4.10. *There exist $M > 0$, $T > 0$, and large constant $K(M, T)$ such that*

$$|\tau_k| \leq M\xi(r) \quad \text{for } r \in [0, e^{c_k}], \quad |\tau_k'(r)| \leq \frac{M}{r} \quad \text{for } r \in [T, e^{c_k}],$$

for any $k \geq K(M, T)$, where ξ is defined in (21).

Proposition 4.11. *Given a sequence $\{s_k\}$ with $s_k \in [c_k^q, e^{c_k}]$ for some $q > 2$, we have*

$$\frac{\alpha_2}{M_k} \int_{B_{s_k r_k}} v_k^2 e^{v_k^2} dx = 4\pi + \frac{4\pi}{c_k^4} + o(c_k^{-4}). \tag{22}$$

Lemma 4.12. *We have*

$$\phi_k \leq \phi_\infty \quad \text{in} \quad [\mu_k^2, \delta_k r_k^{-1}]$$

for sufficient large k.

Proof. By Lemma 4.8 we have

$$\eta_\infty(c_k^2) \leq -1, \quad \text{and} \quad \int_{\mathbb{R}^2} \Delta\eta_\infty dx < 0.$$

Moreover, by the definition of $\mathcal{T}_k$ in Lemma 4.9 we can see that

$$-\Delta\tau_k(r) = \mathcal{T}_k(r, \tau_k),$$

and thus we have

$$\sup_{(0,c_k^2)}\left|\frac{z_\infty}{c_k^4}+\frac{\tau_k}{c_k^6}\right|=o(\mu_k^{-2}),\quad \text{and}\quad \int_{B_{c_k^2}}\left|\Delta\left(\frac{z_\infty}{c_k^4}+\frac{\tau_k}{c_k^6}\right)\right|dx=o(c_k^{-2})$$

by Proposition 4.10. From these facts, it follows that

$$\begin{aligned}\int_{B_{c_k^2}}\Delta\phi_k dx &= \int_{B_{c_k^2}}\Delta\phi_\infty dx+\frac{1}{c_k^2}\int_{B_{c_k^2}}\Delta\eta_\infty dx+o(c_k^{-2})\\ &= -4\pi-\frac{c}{c_k^2}+o(c_k^{-2})\\ &< -4\pi.\end{aligned}$$

Since $\Delta\phi_k\leq 0$ in $[0,\delta_k r_k^{-1}]$, for $r\in[c^2,\delta_k r_k^{-1}]$ it follows that

$$\int_{B_r}\Delta\phi_k dx\leq\int_{B_{c_k^2}}\Delta\phi_k dx<-4\pi<\int_{B_r}\Delta\phi_\infty dx.$$

By this inequality and $\phi_k(c_k^2)<\phi_\infty(c_k^2)$, we have $\phi_k\leq\phi_\infty$. $\square$

We can calculate in the same way as the proof of Proposition 14 in [12]. Therefore for s_k given in Proposition 4.11 we have

$$\frac{\alpha_2}{M_k}\int_{B_{\delta_k}\setminus B_{s_k r_k}}\leq\frac{2\pi}{c_k^4}+o(c_k^{-4}). \tag{23}$$

Hence combining Proposition 4.11 and (23), we finish the estimate of (19).

Finally, by (11), Proposition 4.7, (17), (18), and (19) we prove the Proposition 3.1.

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