MINIMAL MASLOV NUMBER OF R-SPACES CANONICALLY EMBEDDED IN EINSTEIN-KÄHLER C-SPACES

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ABSTRACT. The R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that each R-space has the canonical embedding into a Kähler C-space as a real form, and thus a compact embedded totally geodesic Lagrangian submanifold. The minimal Maslov number of Lagrangian submanifolds in symplectic manifolds is one of invariants under Hamiltonian isotopies and very fundamental to study the Floer homology for intersections of Lagrangian submanifolds. In this paper we show a Lie theoretic formula for the minimal Maslov number of R-spaces canonically embedded in Einstein-Kähler C-spaces, and provide some examples of the calculation by the formula.

INTRODUCTION

The minimal Maslov number of a Lagrangian submanifold in a symplectic manifold is one of invariants under Hamiltonian isotopies and very fundamental to study the Floer homology for intersections of Lagrangian submanifolds, especially *monotone* Lagrangian submanifolds ([9]). It is known that any compact minimal Lagrangian submanifold of an Einstein-Kähler manifold with positive Einstein constant is monotone ([4], [13]) and a nice formula of minimal Maslov number for a monotone Lagrangian submanifold of a simply connected positive Einstein-Kähler manifold was shown by H. Ono [13].

The R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that each R-space can be canonical embedded into a Kähler C-space as a real form which is by definition the fixed point subset by an anti-holomorphic involutive isometry. R-spaces constitute a nice class of compact embedded totally geodesic Lagrangian submanifolds of Kähler manifolds. Any R-space can be canonically embedded in an Einstein-Kähler C-space and particularly it is a compact embedded monotone Lagrangian submanifold.

Y.-G. Oh has worked on the Floer homology of $(\mathbb{C}P^n; \mathbb{R}P^n)$ ([10]) and real forms of Hermitian symmetric spaces of compact type ([11]), which are nothing but canonically embedded symmetric R-spaces. Recently the intersection

Date: March 1, 2019.

2000 *Mathematics Subject Classification.* Primary: 53C40; Secondary: 53C42, 53D12.

Key words and phrases. R-spaces, Einstein-Kähler C-spaces, monotone Lagrangian submanifolds, minimal Maslov number.

This work was partly supported by JSPS KAKENHI Grant Numbers JP17H06127, JP18K03307, JP18H03668.

theory and Floer homology for *two* real forms of Hermitian symmetric spaces of compact type are intensively studied by [22], [6], [19], [20], [21], and more recently its generalization to general R-spaces is discussed in [7], [5].

The purpose of this paper is to provide a Lie theoretic formula (see Theorem 3.1) for the minimal Maslov number of R-spaces canonically embedded in Einstein-Kähler C-spaces and to discuss some examples of the calculation by our formula.

This paper is organized as follows: In Section 1 we recall basic definitions and related properties for the minimal Maslov number and the monotonicity of Lagrangian submanifolds in symplectic geometry and the formula of H. Ono for monotone Lagrangian submanifolds of Einstein-Kähler manifolds. In Section 2 we explain the construction of the canonical embedding of an R-space into a Kähler C-space from a given compact Riemannian symmetric pair. We describe the induced invariant symplectic structure, complex structure and Kähler structure and related properties. The canonical embedding of an R-space into an Einstein-Kähler C-space is characterized in terms of the root system. In Section 3 as a main theorem we show the Lie theoretic formula for minimal Maslov number of R-spaces canonically embedded canonically embedded in Einstein-Kähler C-spaces. In Section 4 we provide some examples calculated by that formula, including a list of the minimal Maslov number for all irreducible symmetric R-spaces canonically embedded in irreducible Hermitian symmetric spaces of compact type. More related examples will be discussed in the forthcoming paper.

1. Minimal Maslov number of Lagrangian submanifolds in symplectic manifolds

Let (M,ω) be a symplectic manifold of dimension $2n$ with a symplectic form ω. A smooth immersion (resp. embedding) $\iota : L \to M$ is called a *Lagrangian immersion* (resp. *Lagrangian embedding*) if $\dim L = n$ and $\iota^*\omega = 0$. Then L is a *Lagrangian submanifold* immersed (resp. embedded) in M.

Let L be a Lagrangian submanifold immersed in a symplectic manifold (M,ω). Define two kinds of group homomorphisms

$$I_{\mu,L} : \pi_2(M,L) \to \mathbb{Z} \quad \text{and} \quad I_{\omega,L} : \pi_2(M,L) \to \mathbb{R}.$$

For a smooth map $u : (D^2, \partial D^2) \to (M,L)$ with $A = [u] \in \pi_2(M,L)$, choose a trivialization of the pull-back bundle as a symplectic vector bundle (which is unique up to the homotopy) $u^{-1}TM \cong D^2 \times \mathbb{C}^n$. This gives a smooth map $\tilde{u} : S^1 = \partial D^2 \to \Lambda(\mathbb{C}^n)$. Here $\Lambda(\mathbb{C}^n)$ denotes the Grassmann manifold of Lagrangian vector subspaces of $\mathbb{C}^n$. Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n),\mathbb{Z}) \cong \mathbb{Z}$, we define a group homomorphism $I_{\mu,L} : \pi_2(M,L) \to \mathbb{Z}$ by $I_{\mu,L}(A) := \mu(\tilde{u})$.

Definition 1.1. If $I_{\mu,L} = 0$, we define $\Sigma_L = 0$. Assume that $I_{\mu,L} \neq 0$. The we denote by $\Sigma_L \in \mathbb{Z}_+$ the positive generator of an additive subgroup $\mathrm{Im}(I_{\mu,L}) \subset \mathbb{Z}$. Then such an integer Σ_L is called the *minimal Maslov number* of L.

Another group homomorphism $I_{\omega,L} : \pi_2(M,L) \to \mathbb{R}$ is defined by $I_{\omega,L}(A) := \int_{D^2} u^*\omega$. It is known that $I_{\mu,L}$ is invariant under symplectic isotopies and $I_{\omega,L}$ is invariant under Hamiltonian isotopies but not invariant under symplectic isotopies.

Definition 1.2. A Lagrangian submanifold L of (M^{2n},ω) is called *monotone* if $I_{\mu,L} = \lambda\, I_{\omega,L}$ for some $\lambda > 0$.

Based on Floer's works, Y.-G. Oh ([9], [10], [11]) introduced the concept of the monotonicity for Lagrangian submanifolds and developed the Floer theory for the intersection of monotone Lagrangian submanifolds. For monotone Lagrangian submanifolds of $\Sigma_L \geq 3$ or $\Sigma_L = 2$, the Floer homology and its Hamiltonian invariance were established by Y.-G. Oh. The minimal Maslov number Σ_L play a crucial role in the theory. If a given monotone Lagrangian submanifold L is Hamiltonian deformed in a Weinstein neighborhood by a suitable Morse-Smale function on L, then the Floer boundary operator ∂_J can be decomposed into by ∂_0 the Morse boundary operator as

$$\partial_J = \partial_0 + \partial_1 + \cdots + \partial_\nu, \qquad \text{where} \;\; \nu = \left[\frac{n+1}{\Sigma_L}\right]$$

and it constructs the spectral sequence of Floer homology for monotone Lagrangian submanifolds. ([12], [1]).

Cieliebak-Goldstein [4] and Hajime Ono [13] showed useful results on the monotonicity and minimal Maslov number of Lagrangian submanifolds in Kähler manifolds as follows:

Proposition 1.1 ([4], [13])**.** *Assume that (M,ω,J,g) is an Einstein-Kähler manifold with positive Einstein constant. Then any compact minimal Lagrangian submanifold L of M is monotone*

Proposition 1.2 ([13])**.** *Assume that (M,ω,J,g) is simply connected Einstein-Kähler manifold with positive Einstein constant. Then the minimal Maslov number of a compact monotone Lagrangian submanifold L of M is given by the formula*

$$n_L \Sigma_L = 2\gamma_{c_1}. \tag{1.1}$$

Here

$\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\}$,

$n_L := \min\{k \in \mathbb{Z}^+ \mid \otimes^k E|_L \text{ is trivial as a flat complex line bundle}\}$.

$$\begin{tikzcd} E|_L \arrow[r] \arrow[d, "\pi_L"', "\text{flat}"] & E \;\text{ cplx. line bdle.} \arrow[d, "\pi_E"', "U(1)\text{-connection } \nabla"] \\ L \arrow[r, "\text{Lag.}"'] & M \;\text{ Einstein-Kähler mfd.} \end{tikzcd}$$

and E is equipped with a $U(1)$-connection such that $\frac{1}{\gamma}\omega = c_1(E,\nabla)$ for some $\gamma > 0$.

2. R-spaces canonically embedded in Einstein-Kähler C-spaces

In this section we review fundamental geometric properties on R-spaces and their canonical embeddings into Kähler C-spaces. We use some related arguments and notations from [2], [14], [15], [16], [17], [18] and so on.

Let (G, K, θ) be a Riemannian symmetric pair with an involutive automorphism θ. Suppose that G is a connected compact Lie group with Lie algebra $\mathfrak{g}$ and K is a connected compact Lie subgroup of G with Lie algebra $\mathfrak{k}$. We choose an AdG- and θ-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{g}$.

We begin with the preparation of the Lie algebraic setting related to R-spaces. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the canonical decomposition of $\mathfrak{g}$ with respect to (G, K, θ). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we know that

$$\mathfrak{t} = \mathfrak{b} + \mathfrak{a}, \quad \mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}, \quad \mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$$

and $\mathfrak{t}$ is invariant by θ. Let $(\ ,\ )$ denote an inner product of $\mathfrak{t}$ which is a restriction of $\langle\ ,\ \rangle$ to $\mathfrak{t}$. The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$ is given as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}\xi(X) = \sqrt{-1}(\alpha, \xi)X \ (\forall \xi \in \mathfrak{t})\}$$

and $\Sigma(\mathfrak{g}) \subset \mathfrak{t}$ denotes the set of all roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$. Set

$$\Sigma_0(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \mathfrak{b}.$$

We define an involutive orthogonal transformation $\sigma \in O(\mathfrak{t})$ by

$$\sigma(H_{\mathfrak{b}} + H_{\mathfrak{a}}) := -H_{\mathfrak{b}} + H_{\mathfrak{a}}, \quad (H_{\mathfrak{b}} \in \mathfrak{b}, H_{\mathfrak{a}} \in \mathfrak{a}).$$

Note that $-\sigma = \theta|_{\mathfrak{t}}$. We choose a σ-order $>$ on $\mathfrak{t}$, that is, a linear order of $\mathfrak{t}$ lexicographical along $\mathfrak{a}$ and $\mathfrak{b}$, so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma\alpha > 0$ and thus $\theta\alpha = -\sigma\alpha < 0$ ([14]). Set $\Sigma^+(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0\}$ and $\Sigma_0^+(\mathfrak{g}) := \Sigma_0(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$. We choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Sigma(\mathfrak{g})$ such that

$$[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \quad \langle E_\alpha, E_{-\alpha}\rangle = 1, \quad \overline{E_\alpha} = E_{-\alpha} \quad \text{for } \alpha \in \Sigma(\mathfrak{g})$$

and let $\{\omega^\alpha \mid \alpha \in \Sigma(\mathfrak{g})\}$ be the linear forms on $\mathfrak{g}^{\mathbb{C}}$ dual to $\{E_\alpha \mid \alpha \in \Sigma(\mathfrak{g})\}$ so that

$$\omega^\alpha(\mathfrak{t}^{\mathbb{C}}) = \{0\}, \quad \omega^\alpha(E_\beta) = \delta_{\alpha\beta} \ \text{ for } \alpha, \beta \in \Sigma(\mathfrak{g}).$$

We fix an arbitrary element $H \in \mathfrak{a}$ of $\mathfrak{a}$. H is called *reqular* if $(H, \alpha) \neq 0$ for all $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$. Define closed subgroups G_H and K_H of G by

$$G_H := C_G(H) = \{a \in G \mid \mathrm{Ad}(a)(H) = H\}$$

and

$$K_H := C_K(H) = \{a \in K \mid \mathrm{Ad}(a)(H) = H\} = K \cap G_H.$$

Denote by $\mathfrak{g}_H$ and $\mathfrak{k}_H$ Lie algebras of G_H and K_H, respectively. It is well-know that G_H is always connected.

Definition 2.1. The compact homogeneous space $L := K/K_H$ is called an *R-space*, and it has the *standard imbedding* into $\mathfrak{p}$

$$\varphi_H : L = K/K_H \ni aK_H \longmapsto \mathrm{Ad}(a)(H) \in \mathrm{Ad}(K)(H) \subset \mathfrak{p}. \tag{2.1}$$

If H is a regular element of $\mathfrak{a}$, then $L := K/K_H$ is called a *regular R-space.* Set another compact homogeneous space $M := G/G_H$, which is called a *generalized flag manifold* or *Kähler C-space*, and it also has the *standard imbedding* into $\mathfrak{g}$

$$\psi_H : M = G/G_H \ni aG_H \longmapsto \mathrm{Ad}(a)(H) \in \mathrm{Ad}(G)(H) \subset \mathfrak{g}. \tag{2.2}$$

As mentioned in the next section it is known that $M = G/G_H$ admits a G-invariant Kähler metric. We can regard each Kähler C-space $M = G/G_H$ as an R-space $\Delta G/\Delta G_{(H,-H)}$ associated to a compact symmetric pair $(G \times G, \Delta G)$.

Definition 2.2. The *canonical embedding* of K/K_H into G/G_H is a smooth map defined by

$$\iota_H : L = K/K_H \ni aK_H \longmapsto aG_H \in G/G_H = M. \tag{2.3}$$

We take the orthogonal direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{k}$ as

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_H + \mathfrak{m}, \quad \mathfrak{m} \cong T_{eG_H}M, \\ \mathfrak{k} &= \mathfrak{k}_H + \mathfrak{l}, \quad \mathfrak{l} \cong T_{eK_H}L. \end{aligned}$$

Note that $\mathfrak{k}_H = \mathfrak{k} \cap \mathfrak{g}_H$. We observe that

$$\theta(G_H) = G_H \ \text{ and } \ \theta(\mathfrak{g}_H) = \mathfrak{g}_H. \tag{2.4}$$

Thus we have an orthogonal direct sum decomposition of $\mathfrak{g}$ as

$$\begin{aligned} \mathfrak{g} &= (\mathfrak{g}_H \cap \mathfrak{k}) + (\mathfrak{g}_H \cap \mathfrak{m}) + (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p}) \\ &= \mathfrak{k}_H + \mathfrak{l} + (\mathfrak{g}_H \cap \mathfrak{p}) + (\mathfrak{m} \cap \mathfrak{p}) \end{aligned}$$

We have $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{k} + \mathfrak{m} \cap \mathfrak{p}$, $\mathfrak{l} = \mathfrak{m} \cap \mathfrak{k}$. Since

$$(\mathrm{ad}H) : \mathfrak{m} \cap \mathfrak{k} \longrightarrow \mathfrak{m} \cap \mathfrak{p}, \quad (\mathrm{ad}H) : \mathfrak{m} \cap \mathfrak{p} \longrightarrow \mathfrak{m} \cap \mathfrak{k}$$

are injective and thus $\dim \mathfrak{m} \cap \mathfrak{k} = \dim \mathfrak{m} \cap \mathfrak{p}$. Hence we obtain

$$2 \dim L = \dim M. \tag{2.5}$$

For such H, we define a skew-symmetric bilinear form ω_H on $\mathfrak{g}$ by

$$\omega_H(X, Y) := \langle [H, X], Y \rangle \ \text{ for each } X, Y \in \mathfrak{g}.$$

Then it induces a G-invariant symplectic form on $M = G/G_H$, which is denoted also by ω_H, and ω_H has expression

$$\omega_H = -\sqrt{-1} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} (H, \alpha)\, \omega^{-\alpha} \wedge \overline{\omega^{-\alpha}}.$$

For each $X, Y \in \mathfrak{l}$, since $\omega_H(X, Y) = \langle [H, X], Y \rangle = 0$, we have $\iota_H^* \omega_H = 0$. Hence we know that

Proposition 2.1. *The canonical embedding*

$$\iota_H : L = K/K_H \ni aK_H \longmapsto aG_H \in G/G_H = M. \tag{2.6}$$

is a Lagrangian embedding with respect to ω_H.

Since $\theta(G_H) = G_H$, the involution automorphism θ of G induces an involution diffeomorphism

$$\hat{\theta}_H : M = G/G_H \ni aG_H \longmapsto \theta(a)G_H \in G/G_H = M \tag{2.7}$$

which is equivariant with respect to the Lie group automorphism $\theta : G \to G$. Since

$$\omega_H(\theta(X), \theta(Y)) = -\omega_H(X, Y)$$

for each $X, Y \in \mathfrak{m}$, we have

Proposition 2.2. *$\hat{\theta}_H : G/G_H \to G/G_H$ is anti-symplectic with respect to ω_H, that is,*

$$\hat{\theta}_H^* \omega_H = -\omega_H.$$

Define the fixed point subset of M by $\hat{\theta}_H$ as

$$\mathrm{Fix}(M, \hat{\theta}_H) := \{p \in M \mid \hat{\theta}_H(p) = p\}. \tag{2.8}$$

Then we have

$$\iota_H(K/K_H) \subset \mathrm{Fix}(M, \hat{\theta}_H) \tag{2.9}$$

which is a connected component of $\mathrm{Fix}(M, \hat{\theta}_H)$.

We give attention to the moment maps of the actions of G and K on G/G_H relative to ω_H. The natural left action of G on a symplectic manifold $(M = G/G_H, \omega_H)$ is Hamiltonian with the moment map

$$\mu_G := \psi_H : G/G_H \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*. \tag{2.10}$$

Moreover the natural left action of $K \subset G$ on a symplectic manifold $(M = G/G_H, \omega_H)$ is also Hamiltonian with the moment map

$$\mu_K := \pi_{\mathfrak{k}} \circ \mu_G = \pi_{\mathfrak{k}} \circ \psi_H : G/G_H \longrightarrow \mathfrak{k} \cong \mathfrak{k}^*.$$

Here $\pi_{\mathfrak{k}} : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \longrightarrow \mathfrak{k}$ denotes the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{k}$.

The relations of the anti-symplectic involution $\hat{\theta}_H$ are the moment maps μ_G and μ_K are as follows:

Proposition 2.3.

$$\mu_G \circ \hat{\theta}_H = -\theta \circ \mu_G, \quad \mu_K \circ \hat{\theta}_H = -\mu_K.$$

Proof. For each point $aG_H \in G/G_H$ we compute

$$\begin{aligned}
\mu_G(\hat{\theta}_H(aG_H)) =& \psi_H(\theta(a)G_H) \\
=& \mathrm{Ad}(\theta(a))(H) \\
=& \theta(\mathrm{Ad}(a)\theta(H)) \\
=& -\theta(\mathrm{Ad}(a)H) \\
=& -\theta(\psi_H(aG_H)) \\
=& -\theta(\mu_G(aG_H))
\end{aligned}$$

and

$$\begin{aligned}\mu_K(\hat{\theta}_H(aG_H)) =&(\pi_{\mathfrak{k}} \circ \psi_H)(\theta(a)G_H) \\ =&-(\pi_{\mathfrak{k}} \circ \theta)(\mu_G(aG_H)) \\ =&-(\pi_{\mathfrak{k}} \circ \mu_G)(aG_H)) \\ =&-\mu_K(aG_H).\end{aligned}$$

□

It follows from Proposition 2.3 that

Lemma 2.1.

$$\mathrm{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

Proof. For any point $aG_H \in G/G_H$ we have

$$\begin{aligned} aG_H \in \mathrm{Fix}(M, \hat{\theta}_H) &\iff \theta(\psi_H(aG_H)) = -\psi_H(aG_H) \\ &\iff \psi_H(aG_H) \in \mathfrak{p} \\ &\iff \mu_G(aG_H) \in \mathfrak{p} \\ &\iff aG_H \in \mu_K^{-1}(0). \end{aligned}$$

□

Since K and M are compact, by a result of Kirwan ([8, p.549, (3.1)]) we see that $\mu_K^{-1}(0)$ is connected. Thus $\mathrm{Fix}(M, \hat{\theta}_H)$ is also connected. Therefore we obtain

Proposition 2.4.

$$\iota_H(K/K_H) = \mathrm{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

By the action of the Weyl group $W(G,K) = N_K(\mathfrak{a})/C_K(\mathfrak{a})$, we may assume that $H \in \mathfrak{a} \subset \mathfrak{t}$ satisfies

$$(\alpha, H) \geq 0 \quad \text{for } \forall \alpha \in \Sigma^+(\mathfrak{g}).$$

Set

$$\begin{aligned}\Sigma_H(\mathfrak{g}) &:= \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, H) = 0\}, \\ \Sigma_H^+(\mathfrak{g}) &:= \Sigma_H(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g}), \\ \Pi_H(\mathfrak{g}) &:= \Pi(\mathfrak{g}) \cap \Sigma_H^+(\mathfrak{g}).\end{aligned}$$

We describe an invariant complex structure on G/G_H corresponding to H. The Lie algebra $\mathfrak{g}_H$ of G_H is nothing but the centralizer $\mathfrak{c}_{\mathfrak{g}}(H)$ of $\mathfrak{g}$ to H. By the maximality of $\mathfrak{t}$ the center $\mathfrak{c}(\mathfrak{g}_H)$ of $\mathfrak{g}_H$ satisfies the inclusions

$$H \in \mathfrak{c}(\mathfrak{g}_H) \subset \mathfrak{t} \subset \mathfrak{g}_H.$$

Then using the root decomposition we express their complexifications as follows:

$$\begin{aligned}
\mathfrak{g}^{\mathbb{C}} =& \mathfrak{g}_H^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}} \\
=& \left(\mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha} \right) + \sum_{\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha} \\
=& \left(\mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha} \right) + \left(\sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha} \right).
\end{aligned}$$

Here

$$\mathfrak{g}_H^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

$$T_{eG_H}(G/G_H)^{\mathbb{C}} \cong \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha}$$

Note that $\mathfrak{g}^{\alpha} = \overline{\mathfrak{g}^{-\alpha}}$. Then we see that

$$\sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{-\alpha} \text{ and } \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha}$$

are invariant under $\mathrm{Ad}G_H$, respectively. Thus we can define a G-invariant complex structure J_H on G/G_H such that

$$\begin{aligned}
T_{eG_H}(G/G_H)^{1,0} &\cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{-\alpha}, \\
T_{eG_H}(G/G_H)^{0,1} &\cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha}.
\end{aligned}$$

We observe that if $\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})$, then $-\theta\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})$ and $\theta(\mathfrak{g}^{-\alpha}) = \mathfrak{g}^{-\theta\alpha}$. Here note that $-\theta\alpha = \sigma\alpha > 0$ and $-\theta\alpha(H) = \alpha(H) > 0$. Hence we get

Lemma 2.2.

$$\theta \left(\sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})} \mathfrak{g}^{\alpha}.$$

By Lemma 2.2 we have

Proposition 2.5. *The involutive diffeomorphism $\hat{\theta}_H : G/G_H \to G/G_H$ is anti-holomorphic with respect to J_H, that is,*

$$J_H \circ d\hat{\theta}_H = -d\hat{\theta}_H \circ J_H.$$

Moreover ω_H becomes a (-1)times Kähler form with respect to the invariant complex structure J_H, because of $(H, \alpha) > 0$ for $\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_H(\mathfrak{g})$, and the corresponding G-invariant Kähler metric g_H on $M = G/G_H$ is defined by

$$\omega_H(X, Y) = (-1) g_H(J_H X, Y) \quad \text{for each } X, Y \in \mathfrak{m}.$$

or

$$g_H = \sum_{\alpha\in\Sigma^+(\mathfrak{g})\setminus\Sigma^+_H(\mathfrak{g})} (H,\alpha)\,\omega^{-\alpha}\cdot\overline{\omega^{-\alpha}}.$$

Since we compute $g_H(\theta(J_H(X)),\theta(Y)) = g_H(J_H X, Y)$ for each $X, Y \in \mathfrak{m}$, the diffeomorphism $\hat{\theta}_H : M \to M$ preserves the Kähler metric g_H. Hence we have

Proposition 2.6. *The diffeomorphism $\hat{\theta} : M \to M$ is an isometry of M with respect to g_H.*

Let

$$\Pi = \Pi(\mathfrak{g}) = \{\alpha_1, \cdots, \alpha_\ell\}$$

be the fundamental root system of $\mathfrak{g}$ with respect to the σ-order $<$. Set

$$\Pi_0 := \Pi(\mathfrak{g})_0 := \Pi(\mathfrak{g})_0 \cap \mathfrak{b}.$$

For the above H, set

$$\Pi_H := \Pi_H(\mathfrak{g}) := \{\alpha_i \in \Pi(G) \mid (\alpha_i, H) = 0\}.$$

Note that $\Pi_0 \subset \Pi_H$ and thus $\Pi\setminus\Pi_H \subset \Pi\setminus\Pi_0$.

Let

$$\{\Lambda_1, \cdots, \Lambda_l\} \subset \mathfrak{t}$$

be the fundamental weight system of $\mathfrak{g}$ corresponding to Π defined by

$$\frac{2(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij} \quad (i, j = 1, \cdots, l).$$

Then we have

$$\Sigma_H(\mathfrak{g}) = \Sigma(\mathfrak{g}) \cap (\bigoplus_{\alpha_i\in\Pi_H} \mathbb{Z}\alpha_i),$$

$$\Sigma^+_H(\mathfrak{g}) = \Sigma(\mathfrak{g})^+ \cap (\bigoplus_{\alpha_i\in\Pi_H} \mathbb{Z}^{\geq 0}\alpha_i),$$

where $\mathbb{Z}^{\geq 0}$ denotes the set of all nonnegative integers. Then we have

$$\Sigma(\mathfrak{g})\setminus\Sigma_H(\mathfrak{g}) = \{\alpha\in\Sigma(\mathfrak{g}) \mid (\alpha, H)\neq 0\},$$

$$\Sigma(\mathfrak{g})^+\setminus\Sigma_H(\mathfrak{g}) = \{\alpha\in\Sigma(\mathfrak{g})^+ \mid (\alpha, H) > 0\}.$$

Note that

$$\Sigma_0(\mathfrak{g}) \subset \Sigma_H(\mathfrak{g}), \quad \Sigma_0(\mathfrak{g}) \subset \Sigma_H(\mathfrak{g}),$$

$$\Sigma(\mathfrak{g})\setminus\Sigma_H(\mathfrak{g}) \subset \Sigma(\mathfrak{g})\setminus\Sigma_0(\mathfrak{g}), \quad \Sigma(\mathfrak{g})^+\setminus\Sigma_H(\mathfrak{g}) \subset \Sigma(\mathfrak{g})^+\setminus\Sigma_0(\mathfrak{g}).$$

Then we have

$$\mathfrak{g}^{\mathbb{C}}_H = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha\in\Sigma_H(\mathfrak{g})} \mathfrak{g}^\alpha \quad \text{and} \quad \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha\in\Sigma(\mathfrak{g})\setminus\Sigma(\mathfrak{g})_H} \mathfrak{g}^\alpha.$$

Now we set

$$\mathfrak{c}_H := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_H} \mathbb{R}\Lambda_i \subset \mathfrak{t} = \bigoplus_{\alpha_i\in\Pi} \mathbb{R}\Lambda_i,$$

$$Z_{\mathfrak{c}_H} := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_H} \mathbb{Z}\Lambda_i \subset Z := \bigoplus_{\alpha_i\in\Pi} \mathbb{Z}\Lambda_i \subset \mathfrak{t}.$$

Then $Z_{\mathfrak{c}_H} \subset \mathfrak{c}_H$ and $\mathfrak{c}_H$ coincides with the center $\mathfrak{c}(\mathfrak{g}_H)$ of $\mathfrak{g}_H$. Define

$$\mathfrak{c}_H^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_i \subset \mathfrak{c}(\mathfrak{g}_H) \subset \mathfrak{t},$$
$$Z_{\mathfrak{c}_H}^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_i \subset \mathfrak{c}^+ \subset \mathfrak{c}(\mathfrak{g}_H) \subset \mathfrak{t},$$

where $\mathbb{R}^{\pm}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote the sets of all nonzero real numbers, all positive real numbers and all positive integers, respectively. Note that $H \in \mathfrak{c}_H^+$.

For each $\xi \in \mathfrak{c}_H^+$, since $\Pi_\xi = \Pi_H$, $\Sigma_\xi(\mathfrak{g}) = \Sigma_H(\mathfrak{g})$, we have

$$\mathfrak{g}_\xi^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_\xi(\mathfrak{g})} \mathfrak{g}^\alpha = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_H(\mathfrak{g})} \mathfrak{g}^\alpha = \mathfrak{g}_H^{\mathbb{C}}$$

and thus $\mathfrak{g}_\xi = \mathfrak{g}_H$. By the connectedness of G_ξ and G_H, we obtain $G_\xi = G_H$ and $G/G_\xi = G/G_H = M$. In particular ω_ξ is a G-invariant symplectic form on $M = G/G_H = G/G_\xi$. However ξ and H define the same G-invariant complex structure $J_\xi = J_H$ on $M = G/G_H = G/G_\xi$.

From now we assume that G is semisimple. Let $\widetilde{G}$ be the universal covering group of G, that is, a connected simply connected compact Lie group with Lie algebra $\mathfrak{g}$, and $\phi : \widetilde{G} \to G$ be the natural covering map which is a surjective Lie group homomorphism. Set $\widetilde{G}_H := \phi^{-1}(G_H)$. Then we know that $\widetilde{G}_H$ is also a connected compact Lie subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}_H$ and we have a natural diffeomorphism $\widetilde{G}/\widetilde{G}_H = \widetilde{G}/\phi^{-1}(G_H) \cong G/G_H = M$. Let $\widetilde{K}$ be a connected compact Lie subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{k}$. Then $\widetilde{K}$ is the identity component of $\phi^{-1}(K)$ and we have natural covering maps $\phi : \widetilde{K} \subset \phi^{-1}(K) \longrightarrow K$ and $\widetilde{G}/\widetilde{K} \longrightarrow \widetilde{G}/\phi^{-1}(K) \cong G/K$. Set $\widetilde{K}_H := \widetilde{K} \cap \widetilde{G}_H = \widetilde{K} \cap \phi^{-1}(G_H) = (\phi|_{\widetilde{K}})^{-1}(K \cap G_H) = (\phi|_{\widetilde{K}})^{-1}(K_H)$. Then we have $\widetilde{K}/\widetilde{K}_H = \widetilde{K}/(\widetilde{K} \cap \phi^{-1}(G_H)) \cong K/K_H = L$. Let $\widetilde{T}$ be the maximal torus of $\widetilde{G}$ with Lie algebra $\mathfrak{t}$. Then we have $\widetilde{T} = \phi^{-1}(T)$.

We know the following diagram of linear isomorphisms and $\mathbb{Z}$-module isomorphisms:

$$\begin{tikzcd}
\frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_H & H^1(\widetilde{G}_H, \mathbb{R}) \arrow[l, "\iota^*"'] \arrow[r, "\tau"] & H^2(M, \mathbb{R}) \\
\frac{1}{2\pi\sqrt{-1}}Z_{\mathfrak{c}_H} \arrow[u] & H^1(\widetilde{G}_H, \mathbb{Z}) \arrow[l, "\iota^*"'] \arrow[r, "\tau"] \arrow[u] & H^2(M, \mathbb{Z}) \arrow[u]
\end{tikzcd}$$

Let $\mathcal{I}_G^2(M)$ denote the real vector space of all G-invariant closed 2-forms on $M = G/G_H$. Then we know that the natural linear map

$$\mathfrak{w} : \mathcal{I}_G^2(M) \ni \omega \longmapsto [\omega] \in H^2(M, \mathbb{R}).$$

is a linear isomorphism and there is a linear isomorphism

$$\omega : \frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_H \longrightarrow \mathcal{I}^2_G(M)$$

defined by

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) := -\frac{1}{2\pi\sqrt{-1}}\sum_{\alpha\in\Sigma^+_{\mathfrak{m}}}(\lambda,\alpha)\omega^{-\alpha}\wedge\bar{\omega}^{-\alpha}$$

or equivalently

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(X,Y) := -\frac{1}{2\pi}\langle[\lambda,X],Y\rangle \quad (X,Y\in\mathfrak{m})$$

for $\lambda\in\mathfrak{c}$. Moreover we know that the linear isomorphism

$$\tau = \mathfrak{w}\circ\omega : \frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_H \longrightarrow \mathcal{I}^2_G(M) \longrightarrow H^2(M,\mathbb{R})$$

is restricted to a $\mathbb{Z}$-module isomorphism

$$\mathfrak{w}\circ\omega : \frac{1}{2\pi\sqrt{-1}}Z_{\mathfrak{c}_H} \longrightarrow H^2(M,\mathbb{Z}).$$

For each $\lambda\in\mathfrak{c}_H$, define a G-invariant symmetric tensor field on $M=G/G_H$ by

$$g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) := \frac{1}{2\pi}\sum_{\alpha\in\Sigma^+(\mathfrak{g})\setminus\Sigma_H(\mathfrak{g})}(\lambda,\alpha)\omega^{-\alpha}\cdot\bar{\omega}^{-\alpha}.$$

Then it holds

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(X,Y) = g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(J_H X,Y).$$

If $\lambda\in\mathfrak{c}^+_H$, then $g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ is a G-invariant Kähler metric on a complex manifold $(M=G/G_H,J_H)$ whose Kähler form coincides with $\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$. Therefore the map

$$\mathfrak{c}^+_H\ni\lambda\longmapsto g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)\in\mathcal{I}^2_G(M)$$

parametrizes all G-invariant Kähler metrics on $M=G/G_H$ relative to the complex structure J_H.

For each $\lambda\in\mathfrak{c}^+_H\cap\mathfrak{a}$, the diffeomorphism $\hat{\theta}_H : M=G/G_H\to M=G/G_H$ preserves a G-invariant Kähler metric $g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ on M, that is, $\hat{\theta}_H : M = G/G_H\to M=G/G_H$ is an isometry with respect to $g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$.

For each $H'\in\mathfrak{c}^+_H\cap\mathfrak{a}$, since $G_{H'}=G_H$ and $G/G_{H'}=G/G_H$, we have $K_{H'}=K\cap G_{H'}=K\cap G_H=K_H$ and thus $K/K_{H'}=K/K_H=L$. Hence all $H'\in\mathfrak{c}^+_H\cap\mathfrak{a}$ correspond to the same R-space $L=K/K_H$ and the convex set $\mathfrak{c}^+_H\cap\mathfrak{a}$ parametrizes orbits of the same type K_H.

Next we discuss the characterization of a G-invariant Einstein-Kähler metric on $M = G/G_H$. Set

$$\delta_{\mathfrak{m}} := \frac{1}{2} \sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \alpha \in \mathfrak{t}.$$

We use the following results due to Borel-Hirzebruch and M. Takeuchi.

Lemma 2.3 ([2])**.**

$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \alpha \in Z^+_{\mathfrak{c}_H} = \bigoplus_{\alpha \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_\alpha. \tag{2.11}$$

and it corresponds to the first Chern class of the complex manifold (M, J_H)*:*

$$c_1(M) = \left[\omega \left(\frac{1}{2\pi\sqrt{-1}} 2\delta_{\mathfrak{m}} \right) \right] = \tau \left(\frac{1}{2\pi\sqrt{-1}} 2\delta_{\mathfrak{m}} \right).$$

Proposition 2.7 ([17])**.** *The G-invariant Kähler metric $g = g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ on M is Einstein if and only if $\lambda = b\,\delta_{\mathfrak{m}}$ for some $b > 0$.*

Since $\theta(\mathfrak{g}_H) = \mathfrak{g}_H$ and thus $\theta(\mathfrak{c}(\mathfrak{g}_H)) = \mathfrak{c}(\mathfrak{g}_H)$, note that we have a direct sum decomposition

$$\mathfrak{c}(\mathfrak{g}_H) = \mathfrak{c}_H = (\mathfrak{c}_H \cap \mathfrak{b}) + (\mathfrak{c}_H \cap \mathfrak{a}).$$

Then we show

Lemma 2.4.

$$2\delta_{\mathfrak{m}} \in \mathfrak{a}.$$

Proof. We compute

$$\begin{aligned}
\theta(2\delta_{\mathfrak{m}}) &= -\,\sigma(2\delta_{\mathfrak{m}}) \\
&= -\,\sigma\left(\sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \alpha \right) \\
&= - \sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \sigma\alpha \\
&= - \sum_{\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \alpha \\
&= -\,2\delta_{\mathfrak{m}}.
\end{aligned}$$

Because, $\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})$ if and only if $\sigma\alpha \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})$. $\square$

Therefore we obtain

Proposition 2.8. *The element*

$$H^{ein} := 2\delta_{\mathfrak{m}} \in Z^+_{\mathfrak{c}_H} \cap \mathfrak{a} \subset \mathfrak{c}^+_H \cap \mathfrak{a}$$

corresponds to the canonical embedding $\iota_{H^{ein}}$ of the same R-space $L = K/K_H$ into an Einstein-Kähler C-space $\left(M = G/G_H, \omega_{H^{ein}}, J_H, g\left(\frac{1}{2\pi\sqrt{-1}}H^{ein}\right)\right)$. Moreover, the element H^{ein} is such a unique element of $\mathfrak{c}^+_H \cap \mathfrak{a}$ up to the multiplication by a positive constant.

By the above argument we can choose $H = 2\delta_{\mathfrak{m}}$. Then $\iota_H : L = K/K_H \to M = G/G_H$ is the canonical embedding of an R-space into an Einstein-Kähler C-space.

Set

$$k_i(M) := \frac{2(2\delta_{\mathfrak{m}}, \alpha_i)}{(\alpha_i, \alpha_i)} = \sum_{\beta \in \Sigma(\mathfrak{g})^+ \setminus \Sigma_H(\mathfrak{g})} \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}^+$$

for $\alpha_i \in \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g})$. Let $\kappa(M)$ be the greatest common divisor of $\{k_i(M) \mid \alpha_i \in \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g})\}$ and set

$$\kappa_i(M) := \frac{k_i(M)}{\kappa(M)} \in \mathbb{Z}^+$$

for $\alpha_i \in \Pi \setminus \Pi_H$. Then $\{\kappa_i(M) \mid \alpha_i \in \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g})\}$ are relatively prime and we have expression

$$2\delta_{\mathfrak{m}} = \sum_{\alpha_i \in \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g})} k_\alpha(M) \Lambda_\alpha = \kappa(M) \sum_{\alpha \in \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g})} \kappa_\alpha(M) \Lambda_\alpha. \tag{2.12}$$

Then the invariant γ_{c_1} in Proposition 1.2 is given as follows:

Lemma 2.5.

$$\gamma_{c_1} = \kappa(M). \tag{2.13}$$

Proof. For each

$$A = \sum_{\alpha_i \in \Pi \setminus \Pi_H} m_i \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \in H_2(M, \mathbb{Z}) \cong H_1(\widetilde{G}_H, \mathbb{Z}) \cong \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z} \frac{2\alpha_i}{(\alpha_i, \alpha_i)},$$

we have

$$c_1(M)(A) = \kappa(M) \sum_{\alpha_i \in \Pi \setminus \Pi_H} \kappa_i \, m_i.$$

Since $\{\kappa_i\}$ are relatively prime, it attains $\sum_{\alpha_i \in \Pi \setminus \Pi_H} \kappa_i \, m_i = 1$ for some integers $\{m_i\}$. Hence the positive minimum γ_{c_1} of $c_1(M)(A)$ is equal to $\kappa(M)$. □

3. Minimal Maslov number of R-spaces

Suppose that $H = H^{ein} = 2\delta_{\mathfrak{m}}$. Then, as discussed in the last section, the corresponding canonical embedding of an R-space

$$\iota = \iota_H : L = \widetilde{K}/\widetilde{K}_H \longrightarrow (M = \widetilde{G}/\widetilde{G}_H, \omega\left(\frac{1}{2\pi\sqrt{-1}} 2\delta_{\mathfrak{m}}\right))$$

is a compact totally geodesic Lagrangian submanifold embedded in an Einstein-Kähler C-space and thus it is monotone by Proposition 1.1. By means of the formula (1.1) in Proposition 1.2, we shall calculate the minimal Maslov number Σ_L of such an R-space.

We take an orthogonal direct sum decomposition of $\mathfrak{g}_H$ into ideals as follows:

$$\mathfrak{g}_H = \mathbb{R}H \oplus \mathfrak{g}'_H.$$

Let $\widetilde{G}'_H$ be a connected compact Lie subgroup of $\widetilde{G}_H$ with Lie algebra $\mathfrak{g}'_H$. Then $\widetilde{G}/\widetilde{G}'_H$ is a simply connected compact homogeneous space with the natural projection

$$\pi : \widetilde{G}/\widetilde{G}'_H \longrightarrow \widetilde{G}/\widetilde{G}_H.$$

It is a $\widetilde{G}$-homogeneous principal fiber bundle $P = \widetilde{G}/\widetilde{G}'_H$ over $M = \widetilde{G}/\widetilde{G}_H$ with structure group $\widetilde{G}_H/\widetilde{G}'_H \cong U(1)$ such that the curvature form of the standard $U(1)$-connection is equal to $2\pi\sqrt{-1}\omega_H = 2\pi\sqrt{-1}\omega_{2\delta_{\mathfrak{m}}}$. It is known that there is a homogeneous Einstein-Sasakian contact structure on $\widetilde{G}/\widetilde{G}'_H$ induced from the Einstein-Kähler structure $\omega_{2\delta_{\mathfrak{m}}}$ on $\widetilde{G}/\widetilde{G}_H = M$.

Set $\widetilde{K}'_H := \widetilde{K} \cap \widetilde{G}'_H$ and define a compact homogeneous space $\hat{L} := \widetilde{K}/\widetilde{K}'_H$. Then we have the following diagram of the natural inclusions and projections of those compact homogeneous spaces:

$$\begin{tikzcd}
\hat{L} = \widetilde{K}/\widetilde{K}'_H \arrow[r, "\hat{\iota}_H"] \arrow[d, "\pi_{\hat{L}}"', "\widetilde{K}_H/\widetilde{K}'_H"] & \widetilde{G}/\widetilde{G}'_H = P \arrow[d, "\pi_P"', "U(1)"] \\
L = \widetilde{K}/\widetilde{K}_H \arrow[r, "\iota_H"] & \widetilde{G}/\widetilde{G}_H = M
\end{tikzcd}$$

Let E be the complex line bundle over M dual to the associated bundle $P \times_{G_H/G'_H} \mathbb{C}v_\Lambda$, where v_Λ denotes a (nonzero) highest weight vector of the representation space of $\widetilde{G}$ corresponding to $2\delta_{\mathfrak{m}} \in Z^+_{\mathfrak{c}_H}$. Then $c_1(E) = \tau(\frac{1}{2\pi\sqrt{-1}}2\delta_{\mathfrak{m}}) = c_1(M)$ and the pull-back bundle $\pi_P^{-1}E$ is trivial as a complex line bundle over P.

$$\begin{tikzcd}
P = \widetilde{G}/\widetilde{G}'_H \arrow[d, "\pi_P"'] & \pi_P^{-1}E \arrow[l] \arrow[d] \\
M = \widetilde{G}/\widetilde{G}_H & E \arrow[l]
\end{tikzcd}$$

The Lagrangian property of $\iota : L \to M$ is equivalent to the flatness of the pull-back connection of the pull-back principal bundle $\iota^{-1}P$ by $\iota : L \to M$. $\hat{L} = \widetilde{K}/\widetilde{K}'_H \subset \widetilde{G}/\widetilde{G}'_H = P$ is the horizontal lift of L to $\iota^{-1}P$ with respect to the flat connection. The image of the holonomy homomorphism $\rho : \pi_1(L) \to U(1) \cong \widetilde{G}_H/\widetilde{G}'_H$ of the flat connection is isomorphic to $\widetilde{K}_H/\widetilde{K}'_H$, which must be a cyclic group of finite order $\sharp(\widetilde{K}_H/\widetilde{K}'_H)$ and the pull-back flat connection of the pull-back principal bundle of $\iota^{-1}P$ over through the covering map $\hat{L} \to L$ is trivial. Therefore, since $\iota^{-1}E$ has the holonomy group equal to a cyclic group of order $\sharp(\widetilde{K}_H/\widetilde{K}'_H)$, we obtain

$$n_L = \sharp(\widetilde{K}_H/\widetilde{K}'_H). \tag{3.1}$$

We also observe that $\hat{L} = \widetilde{K}/\widetilde{K}'_H \longrightarrow \widetilde{G}/\widetilde{G}'_H = P$ is a compact totally geodesic Legendrian submanifold embedded in a Sasakian contact manifold $\widetilde{G}/\widetilde{G}'_H = P$.

Therefore by (2.13) and (3.1) we obtain

Theorem 3.1. *The minimal Maslov number Σ_L of an R-space L canonically embedded in an Einstein-Kähler C-space M is given by the formula*

$$\Sigma_L = \frac{2\,\kappa(M)}{\sharp(\widetilde{K}_H/\widetilde{K}'_H)}. \tag{3.2}$$

4. Some examples

In this section we use some notations from the table of root systems in [3].

4.1. $\tilde{G} = G = SU(n+1)$, $\tilde{K} = K = SO(n+1)$, $\theta(A) = \bar{A}$ $(A \in SU(n+1))$. In this case, $\mathfrak{g} = \mathfrak{su}(n+1)$, $\mathfrak{k} = \mathfrak{o}(n+1)$, $\mathfrak{p} = \sqrt{-1}\mathrm{Sym}_0(\mathbb{R}^{n+1})$,

$$\mathfrak{t} = \mathfrak{a} = \left\{ \sqrt{-1} \begin{bmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \\ 0 & 0 & \xi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_{n+1} \end{bmatrix} \mid \xi_1, \cdots, \xi_{n+1} \in \mathbb{R},\ \sum_{i=1}^{n+1} \xi_i = 0 \right\}.$$

$$\Pi(\mathfrak{g}) = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_n = \varepsilon_n - \varepsilon_{n+1}\},$$
$$\Sigma^+(\mathfrak{g}) = \{\varepsilon_i - \varepsilon_j = \sum_{i \le k < j} \alpha_k \mid 1 \le i < j \le n+1\}.$$

4.1.1. *The case when $L = \mathbb{R}P^n$ and $M = \mathbb{C}P^n$.* For

$$H = \sqrt{-1} \begin{bmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \\ 0 & 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_2 \end{bmatrix} \in \mathfrak{a} = \mathfrak{t}.$$

with $\xi_1 > \xi_2$, we have

$$\Pi_H(\mathfrak{g}) = \{\alpha_2, \cdots, \alpha_n\}, \quad \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g}) = \{\alpha_1\},$$
$$\Sigma^+_H(\mathfrak{g}) = \left\{ \varepsilon_i - \varepsilon_j = \sum_{i \le k < j} \alpha_k \mid 2 \le i < j \le n+1 \right\},$$
$$\Sigma^+(\mathfrak{g}) \setminus \Sigma^+_H(\mathfrak{g}) = \left\{ \varepsilon_1 - \varepsilon_j = \sum_{1 \le k < j} \alpha_k \mid 1 < j \le n+1 \right\}$$
$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Sigma^+_{\mathfrak{m}}} \alpha = (n+1)\left(\varepsilon_1 - \frac{1}{n+1} \sum_{j=1}^{n+1} \varepsilon_j \right) = \kappa(M)\Lambda_1.$$

Thus we have $\kappa(M) = n+1$. Choose

$$H^{ein} = 2\delta_{\mathfrak{m}} = \frac{\sqrt{-1}}{n+1} \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \in \mathfrak{a} = \mathfrak{t}.$$

Then

$$
\begin{aligned}
&G_H = S(U(1) \times U(n)),\\
&M = G/G_H = SU(n+1)/S(U(1) \times U(n)) = \mathbb{C}P^{n+1},\\
&K_H = S(O(1) \times O(n)),\\
&L = K/K_H = SO(n+1)/S(O(1) \times O(n)) = \mathbb{R}P^{n+1}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\mathfrak{c}(\mathfrak{g}_H) = \mathbb{R}\Lambda_1 = \mathbb{R}H,\ \mathfrak{g}'_H = \{0\} \oplus \mathfrak{su}(n),\\
&G'_H = \{1\} \times SU(n),\ G/G'_H = SU(n+1)/(\{1\} \times SU(n)) \cong S^{2n+1},\\
&K'_H = K \cap G'_H = \{1\} \times SO(n),\\
&K/K'_H = SO(n+1)/(\{1\} \times SO(n)) \cong S^n.
\end{aligned}
$$

Thus

$$
K_H/K'_H = S(O(1) \times O(n))/(\{1\} \times SO(n)) \cong \mathbb{Z}_2
$$

and hence $\sharp(\widetilde{K}_H/\widetilde{K}'_H) = \sharp(K_H/K'_H) = 2$. Therefore by formula (3.2) we obtain $\Sigma_L = \frac{2(n+1)}{2} = n+1$.

4.1.2. *The case when L is a regular R-space.* For a regular element

$$
H = \sqrt{-1}\begin{bmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \\ 0 & 0 & \xi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_{n+1} \end{bmatrix} \in \mathfrak{a} = \mathfrak{t}
$$

with $\xi_1 > \cdots > \xi_{n+1}$,

$$
G_H = \left\{ \begin{bmatrix} e^{\sqrt{-1}\eta_1} & 0 & \cdots & 0 \\ 0 & e^{\sqrt{-1}\eta_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\sqrt{-1}\eta_{n+1}} \end{bmatrix} \mid \eta_i \in \mathbb{R},\ \sum_{i=1}^{n+1} \eta_i = 0 \right\} \cong T^n
$$

$$
K_H = \left\{ \begin{bmatrix} e^{\sqrt{-1}\pi l_1} & 0 & \cdots & 0 \\ 0 & e^{\sqrt{-1}\pi l_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\sqrt{-1}\pi l_{n+1}} \end{bmatrix} \mid l_i \in \mathbb{Z},\ \sum_{i=1}^{n+1} l_i = 0 \right\}
$$

and the corresponding canonical embedding of an R-space is

$$
\begin{aligned}
\iota_H : L = &\frac{SO(n+1)}{S(O(1) \times \cdots \times O(1))} =: F_{1,\cdots,1}(\mathbb{R}^{n+1})\\
&\longrightarrow M = \frac{SU(n+1)}{S(U(1) \times \cdots \times U(1))} =: F_{1,\cdots,1}(\mathbb{C}^{n+1}).
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
&\Pi_H(\mathfrak{g}) = \emptyset, \quad \Pi(\mathfrak{g}) \setminus \Pi_H(\mathfrak{g}) = \Pi(\mathfrak{g}),\\
&\Sigma_H^+(\mathfrak{g}) = \emptyset, \quad \Sigma^+(\mathfrak{g}) \setminus \Sigma_H^+(\mathfrak{g}) = \Sigma^+(\mathfrak{g})
\end{aligned}
$$

and

$$2\delta_{\mathfrak{m}} = \sum_{\alpha\in\Sigma^+\setminus\Sigma_H} \alpha = \sum_{\alpha\in\Sigma^+} \alpha = \sum_{i=1}^{n+1}(n-2i+2)\varepsilon_i = \sum_{i=1}^{n} 2\Lambda_i$$
$$=\sqrt{-1}\begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -n \end{bmatrix} \in Z_{\mathfrak{c}}^+.$$

Thus we have $\kappa(M) = 2$.

Choose

$$H = H^{ein} = 2\delta_{\mathfrak{m}} = \sqrt{-1}\begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -n \end{bmatrix} \in \mathfrak{a} \subset \mathfrak{p}.$$

which is also a regular element of $\mathfrak{a}$. Then

$$G'_H = \left\{ \begin{bmatrix} e^{\sqrt{-1}\eta_1} & 0 & \cdots & 0 \\ 0 & e^{\sqrt{-1}\eta_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\sqrt{-1}\eta_{n+1}} \end{bmatrix} \mid \eta_i \in \mathbb{R},\ \sum_{i=1}^{n+1}\eta_i = 0,\ \sum_{i=1}^{n+1}(n-2i+2)\eta_i = 0 \right\},$$
$$K'_H = K_H \cap G'_H$$
$$= \left\{ \begin{bmatrix} e^{\sqrt{-1}\pi l_1} & 0 & \cdots & 0 \\ 0 & e^{\sqrt{-1}\pi l_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\sqrt{-1}\pi l_{n+1}} \end{bmatrix} \mid l_i \in \mathbb{Z},\ \sum_{i=1}^{n+1} l_i = 0,\ \sum_{i=1}^{n+1} i l_i = 0 \right\}$$

Then we have

$$K_H/K'_H \cong \mathbb{Z}_2$$

and thus $\sharp(\widetilde{K}_H/\widetilde{K}'_H) = \sharp(K_H/K'_H) = 2$. Therefore by formula (3.2) we obtain $\Sigma_L = \dfrac{2\cdot 2}{2} = 2$.

4.2. **The case when maximal flag manifolds $L = K/F$.** Let K be a connected compact *semisimple* Lie group and F be a maximal torus of K. In this case $G = K \times K$, $K = \Delta K$. We equip a maximal flag manifold $L = K/F$ with an K-invariant Einstein-Kähler metric. The canonical embedding of $L = K/F$ as an R-space is given by

$$\iota_H : L = K/F \longrightarrow M = K/F \times \overline{K/F},$$

where $\overline{K/F}$ denotes the conjugate manifold of K/F. Then $\kappa(M) = 2$ by the root system computation and $\sharp(\widetilde{K}_H/\widetilde{K}'_H) = 1$ by the simply connectedness of $L = K/F$. Hence by formula (3.2) we obtain $\Sigma_L = 4$.

4.3. **The case when L is a symmetric R-space.** By the formula (3.2) we can compute the minimal Maslov number for each irreducible symmetric R-spaces L canonically embedded in a symmetric Einstein-Kähler C-space M. An Irreducible symmetric R-space means a symmetric R-space L with simple G. Symmetric Einstein-Kähler C-spaces are nothing but irreducible Hermitian symmetric space of compact type. The number γ_{c_1} for each irreducible Hermitian symmetric space M of compact type is given in [2, p.521].

$M = G/G_H$	$L = K/K_H$	$\dim L$	γ_{c_1}	n_L	Σ_L
$G_{p,q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{R})$	pq	$p+q$	2	$p+q$
$G_{2p,2q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{H})$	$4pq$	$2p+2q$	1	$4(p+q)$
$G_{m,m}(\mathbf{C})$	$U(m)$	m^2	$2m$	2	$2m$
$\dfrac{SO(2m)}{U(m)}$	$SO(m), m \geq 5$	$\dfrac{m(m-1)}{2}$	$2m-2$	2	$2(m-1)$
$\dfrac{SO(4m)}{U(2m)}, m \geq 3$	$\dfrac{U(2m)}{Sp(m)}$	$m(2m-1)$	$2(2m-1)$	2	$2(2m-1)$
$\dfrac{Sp(2m)}{U(2m)}$	$Sp(m), m \geq 2$	$m(2m+1)$	$2m+1$	1	$2(2m+1)$
$\dfrac{Sp(m)}{U(m)}$	$\dfrac{U(m)}{O(m)}$	$\dfrac{m(m+1)}{2}$	$m+1$	2	$m+1$
$Q_{p+q-2}(\mathbf{C})$	$Q_{p,q}(\mathbf{R})$, $p \geq 2$	$p+q-2$	$p+q-2$	2	$p+q-2$
$Q_{q-1}(\mathbf{C})$, $q \geq 3$	$Q_{1,q}(\mathbf{R})$	$q-1$	$q-1$	1	$2(q-1)$
$\dfrac{E_6}{T \cdot Spin(10)}$	$P_2(\mathbf{K})$	16	12	1	24
$\dfrac{E_6}{T \cdot Spin(10)}$	$G_{2,2}(\mathbf{H})/\mathbf{Z}_2$	16	12	2	12
$\dfrac{E_7}{T \cdot E_6}$	$\dfrac{SU(8)}{Sp(4)\mathbf{Z}_2}$	27	18	2	18
$\dfrac{E_7}{T \cdot E_6}$	$\dfrac{T \cdot E_6}{F_4}$	27	18	1	36

where $G_{p,q}(\mathbf{F})$: Grassmanian manifold of all p-dimensional subspaces of $\mathbf{F}^{p+q}$, for each $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. $P_2(\mathbf{K})$: Cayley projective plane. $Q_n(\mathbf{C})$: complex hyperquadric of complex dimension n.

Acknowledgements. This paper is contributed to the proceedings of the 5th workshop "Complex Geometry and Lie Groups" at Florence, Italy, in June, 2018. The author sincerely would like to thank the organizers, especially Professors A. Fino, F. Podestà, K. Hasegawa and R. Goto for kind invitation and warm hospitality. He also would like to thank Professor R. Miyaoka for her kind support at the workshop, and Professors H. Iriyeh, T. Sakai and T. Okuda for valuable suggestions and interests in this work.

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